Notes on Nodal Analysis, Prof. Mack Grady, June 4, 2007

Definitions

Node: A point or set of points at the same potential that have at least two branches connected to them.

Branch: A circuit element that connects nodes.

Major Node: A node with three or more branches connected to it.

Super Node: Two major nodes with an ideal voltage source between them.

Reference Node: The node to which all other node potentials are referenced. The relative voltage of the reference node is zero.

Solution Procedure

1. Draw a neat circuit diagram and try to eliminate as many branch crossings as possible.

2. Choose a reference node. All other node voltages will be referenced to it. Ideally, it should be the node with the most branches connected to it, so that the number of terms in the admittance matrix is minimal.

3. If the circuit contains voltage sources, do either of the following:

   - Convert them to current sources (if they have series impedances)
   - Create super nodes by encircling the corresponding end nodes of each voltage source.

4. Assign a number to every major node (except the reference node) that is not part of a super node (N1 of these).

5. Assign a number to either end (but not both ends) of every super node that does not touch the reference node (N2 of these)

6. Apply KCL to every numbered node from Step 4 (N1 equations)

7. Apply KCL to every numbered super node from Step 5 (N2 equations)

8. The dimension of the problem is now N1 + N2. Solve the set of linear equations for the node voltages. At this point, the circuit has been "solved."

9. Using your results, check KCL for at least one node to make sure that your currents sum to zero.

9. Use Ohm's Law, KCL, and the voltage divider principle to find other node voltages, branch currents, and powers as needed.

Building the Admittance Matrix

Most power system networks are analyzed by first forming the admittance matrix. The admittance matrix is based upon Kirchhoff's current law (KCL), and it is easily formed and very sparse for most large networks.

Consider the three-bus network shown in Figure that has five branch impedances and one current source.

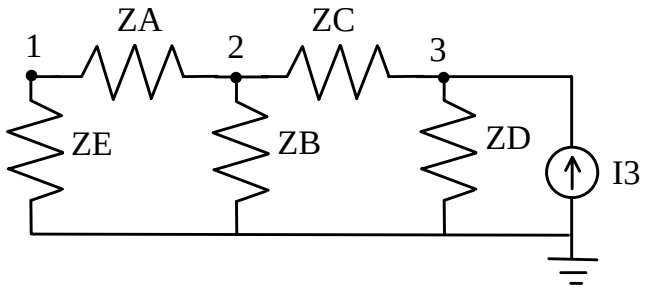


Figure 1. Three-Bus Network

Applying KCL at the three independent nodes yields the following equations for the bus voltages (with respect to ground):

At bus 1, $\frac{V_1}{Z_E}+\frac{V_1-V_2}{Z_A}=0$,

At bus 2, $\frac{V_2}{Z_B}+\frac{V_2-V_1}{Z_A}+\frac{V_2-V_3}{Z_C}=0$,

At bus 3, $\frac{V_3}{Z_D}+\frac{V_3-V_2}{Z_C}=I_3$.

Collecting terms and writing the equations in matrix form yields

$$\begin{bmatrix} \frac{1}{Z_E}+\frac{1}{Z_A} & -\frac{1}{Z_A} & 0 \\ -\frac{1}{Z_A} & \frac{1}{Z_A}+\frac{1}{Z_B}+\frac{1}{Z_C} & -\frac{1}{Z_C} \\ 0 & -\frac{1}{Z_C} & \frac{1}{Z_C}+\frac{1}{Z_D} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_3 \end{bmatrix},$$

or in matrix form,

$$YV = I \ ,$$

where *Y* is the admittance matrix, *V* is a vector of bus voltages (with respect to ground), and *I* is a vector of current injections.

Voltage sources, if present, can be converted to current sources using the usual network rules. If a bus has a zero-impedance voltage source attached to it, then the bus voltage is already known, and the dimension of the problem is reduced by one.

A simple observation of the structure of the above admittance matrix leads to the following rule for building *Y*:

1. The diagonal terms of Y contain the sum of all branch admittances connected directly to the corresponding bus.

2. The off-diagonal elements of *Y* contain the negative sum of all branch admittances connected directly between the corresponding busses.

These rules make *Y* very simple to build using a computer program. For example, assume that the impedance data for the above network has the following form, one data input line per branch:

From Bus	To Bus	Branch Impedance (Entered as Complex Numbers)
1	0	ZE
1	2	ZA
2	0	ZB
2	3	ZC
3	0	ZD

The following FORTRAN instructions would automatically build *Y*, without the need of manually writing the KCL equations beforehand:

```
      COMPLEX Y(3,3),ZB,YB
      DATA Y/9 * 0.0/
    1 READ(1,*,END=2) NF,NT,ZB
      YB = 1.0 / ZB
C  MODIFY THE DIAGONAL TERMS
      IF(NF .NE. 0) Y(NF,NF) = Y(NF,NF) + YB
      IF(NT .NE. 0) Y(NT,NT) = Y(NT,NT) + YB
      IF(NF .NE. 0 .AND. NT .NE. 0) THEN
C  MODIFY THE OFF-DIAGONAL TERMS
        Y(NF,NT) = Y(NF,NT) - YB
        Y(NT,NF) = Y(NT,NF) - YB
      ENDIF
      GO TO 1
    2 STOP
      END
```

Of course, error checking is needed in an actual computer program to detect data errors and dimension overruns. Also, if bus numbers are not compressed (i.e. bus 1 through bus *N*), then additional logic is needed to internally compress the busses, maintaining separate internal and external (i.e. user) bus numbers.

Note that the Y matrix is symmetric unless there are branches whose admittance is direction-dependent. In AC power system applications, only phase-shifting transformers have this asymmetric property. The normal 30^{o} phase shift in wye-delta transformers creates asymmetry.

SUM THE CURRENTS LEAVING THE NODES

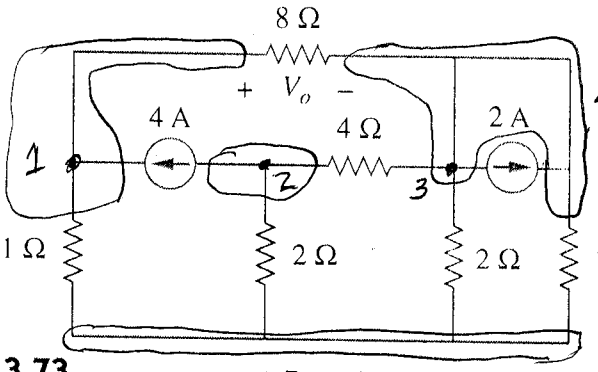


Figure 3.73
For Prob. 3.24.

REF, V = 0

← The "Node" is the point ⊕ the wires

CLASSIC NODE PROBLEM - HAS ONLY INDEPENDENT CURRENT SOURCES

KCL #1, $\frac{V_1-0}{1}+\frac{V_1-V_3}{8}-4=0$

#2, $\frac{V_2-0}{2}+4+\frac{V_2-V_3}{4}=0$

#3, $\frac{V_3-0}{2}+\frac{V_3-V_2}{4}+\frac{V_3-V_1}{8}+\frac{V_3-0}{1}-2+2=0$

Gather terms into A matrix

$$\begin{bmatrix} \frac{1}{1}+\frac{1}{8} & 0 & -\frac{1}{8} \\ 0 & \frac{1}{2}+\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{8} & -\frac{1}{4} & \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{1} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2-2 \end{bmatrix}$$

Y: Admittance Matrix
Units Mho, Siemens
℧ S

$YV = I$ ← current

- Sanity Check #1
  It is symmetric when All the sources are independent

Let's eliminate the fractions - multiply by 8

$$\begin{bmatrix} 9 & 0 & -1 \\ 0 & 6 & -2 \\ -1 & -2 & 15 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 32 \\ -32 \\ 0 \end{bmatrix}$$ } Standard Matrix Form

- Sanity Check #2 (for DC circuits)
  The diagonal terms are the largest magnitudes in their rows and are positive

- Sanity Check #3 (for DC circuits)
  The off-diagonal terms are zero when there is no conductance between them, and negative otherwise.

- Sanity Check #4
  The current injections are the proper sum for each node and, all together, sum to zero (to enforce KCL at the ref node)

Solve this one using Kramer's Rule:

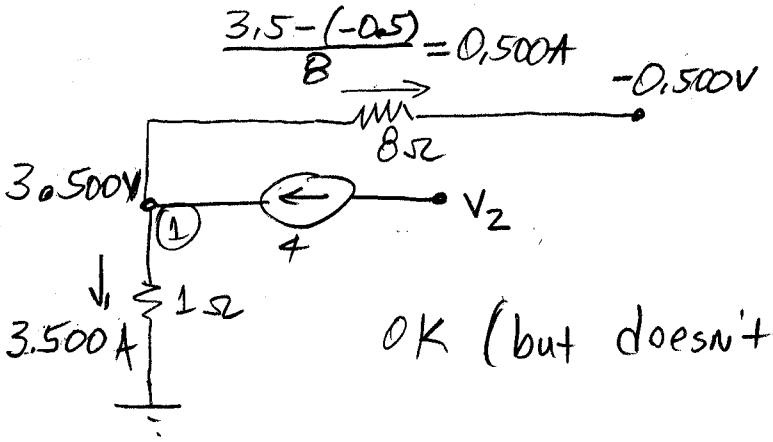
$$\Delta = 9\begin{vmatrix} 6 & -2 \\ -2 & 15 \end{vmatrix} - 0\begin{vmatrix} 0 & -1 \\ -2 & 15 \end{vmatrix} - 1\begin{vmatrix} 0 & -1 \\ 6 & -2 \end{vmatrix}$$

$$= 9(90-4) - 1(+6) = 774 - 6 = 768$$

$$V_1 = \frac{\begin{vmatrix} 32 & 0 & -1 \\ -32 & 6 & -2 \\ 0 & -2 & 15 \end{vmatrix}}{\Delta} = \frac{32(90-4)+32(0-2)}{768} = 3.500\,V$$

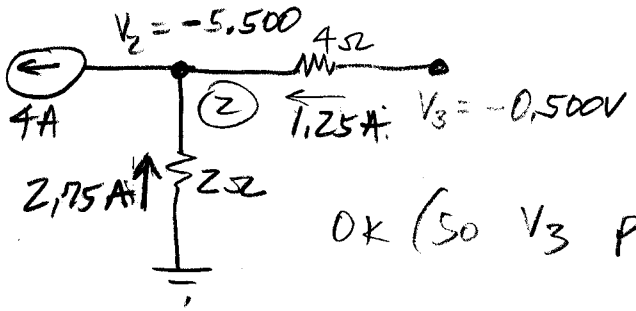
$$V_2 = \frac{\begin{vmatrix} 9 & 32 & -1 \\ 0 & -32 & -2 \\ -1 & 0 & 15 \end{vmatrix}}{\Delta} = \frac{9(-480)-1(-64-32)}{768} = -5.500\,V$$

$$V_3 = \frac{\begin{vmatrix} 9 & 0 & 32 \\ 0 & 6 & -32 \\ -1 & -2 & 0 \end{vmatrix}}{\Delta} = \frac{9(-64)-1(-192)}{768} = -0.500\,V$$

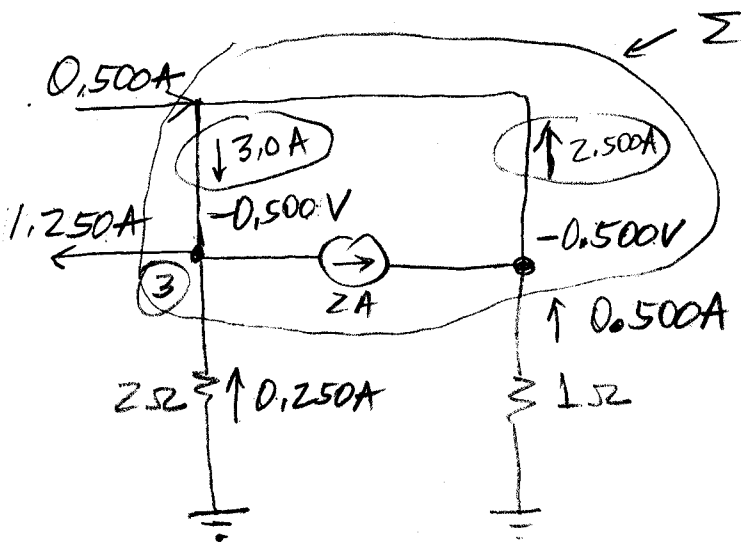


Now, go back to the circuit - did we solve the CORRECT set of equations!

OK (but doesn't check V_2)



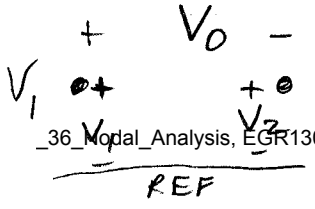
OK (So V_3 probably OK)



$-0,250 + 1,250 - 0,500 - 0,500 = 0$ OK

The (3.0A) and (2,500A) currents are obtained using KCL After the other currents are known. They are not directly governed by Ohm's Law

So, what is V_0?



$-V_1 + V_0 + V_2 = 0$

$V_0 = V_1 - V_2$

Automatic Method for circuits
where sources are independent current sources
(includes cases where independent voltage sources with Resistances are converted to current sources)

y_{ii} = $\sum$ Admittances connected directly to node i

$y_{ij} = -\sum$ " " " between nodes i and j

symmetry

$$\begin{bmatrix} \frac{1}{1}+\frac{1}{8} & 0 & -\frac{1}{8} \\ 0 & +\frac{1}{2}+\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{8} & -\frac{1}{4} & \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{1} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2-2 \end{bmatrix}$$

Y V I

(current injections)

Demonstrate Gaussian Elimination & Backward Substitution

$$\text{OK} \begin{bmatrix} 9 & 0 & -1 \\ \textcircled{0} & 6 & -2 \\ -1 & -2 & 15 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 32 \\ -32 \\ 0 \end{bmatrix}$$

Row 3 + $\frac{1}{9}$ Row 1

$$\begin{bmatrix} 9 & 0 & -1 \\ 0 & 6 & -2 \\ \textcircled{0} & -2 & 15-\frac{1}{9} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 32 \\ -32 \\ 32/9 \end{bmatrix}$$

Row 3 + $\frac{1}{3}$ Row 2

$$\left[\begin{array}{c|c|c} 9 & 0 & -1 \\ \hline 0 & 6 & -2 \\ \hline 0 & -2+2=0 & 15-\frac{1}{9}-\frac{2}{3}=\frac{128}{9} \end{array}\right] \begin{bmatrix} 32 \\ -32 \\ \frac{32}{9} \quad \frac{32}{3} \quad \frac{32-96}{9} \quad \frac{-64}{9} \end{bmatrix}$$