TRANSMISSION AND DISTRIBUTION OF ELECTRICAL ENERGY

Walter L. Weeks
Purdue University

HARPER & ROW, PUBLISHERS, New York
Cambridge, Philadelphia, San Francisco,
London, Mexico City, São Paulo, Sydney
1817

Chapter 2

Transmission Line Equations, Parameters, and Solutions

The lines and cables that deliver electrical energy have characteristic values of circuit parameters determined by their sizes, shapes, and material constitution. As suggested in the last chapter, for many diverse reasons it is important to know the specific values of these parameters. In this chapter, the calculation of these parameters is presented in probably the easiest and shortest correct method. The inherent relationship between the several circuit parameters is brought out by a consideration of the theory from both a distributed circuit and an electromagnetic field approach. In this way the labor of calculating the "external" line parameters is cut almost in half.

2.1 LINE EQUATIONS AND SOLUTIONS FROM A DISTRIBUTED CIRCUITS VIEWPOINT

First consider only single-phase lines. The physical arrangement of two types of such lines is suggested in Fig. 2.1(a). The circuit model for an incremental length, Δz, of such lines is suggested in Fig. 2.1(b). The transmission line (or telegrapher's) equations may be obtained by applying Kirchoff's voltage and current laws to the circuit, so as to obtain, in the

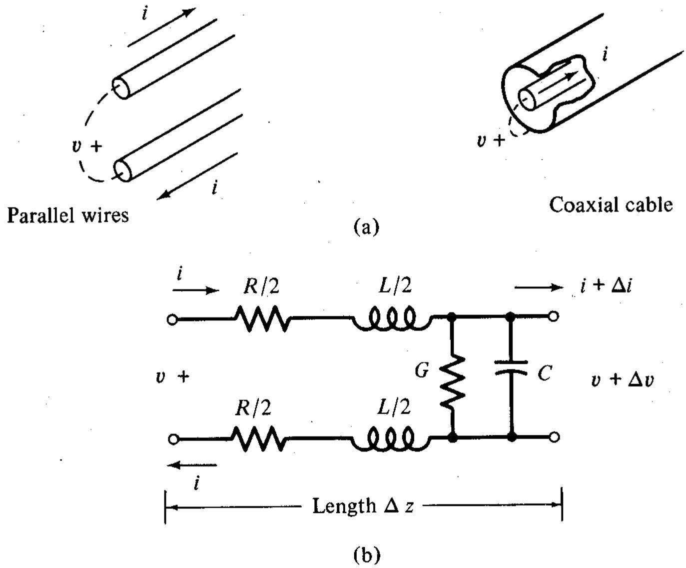

Figure 2.1 Some typical transmission lines and a circuit model for an incremental length. (a) Structures. (b) Distributed circuit model.

limit as $\Delta z \to 0$, the equations

$$\frac{\partial v}{\partial z} = -\mathcal{L}\frac{\partial i}{\partial t} - \mathcal{R} i$$

$$\frac{\partial i}{\partial z} = -\mathcal{C}\frac{\partial v}{\partial t} - \mathcal{G} v \tag{2.1}$$

In these equations $\mathcal{L}, \mathcal{R}, \mathcal{C}, \mathcal{G}$, are the per unit length values of inductance, resistance, capacitance, and conductance.

These equations can be solved with the aid of Laplace transforms to eliminate the time, and we will do this later. For now, we will simplify matters by assuming that $\mathcal{R}$ and $\mathcal{G}$ are zero. Then, combining the simplified equations by eliminating one or the other of the two variables, we obtain the equations

$$\frac{\partial^2 v}{\partial z^2} = \mathcal{L}\mathcal{C}\frac{\partial^2 v}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 i}{\partial z^2} = \mathcal{L}\mathcal{C}\frac{\partial^2 i}{\partial t^2} \tag{2.2}$$

Specific solutions to these equations depend on boundary, initial, and source conditions, but it is easy to show by trial that *any* function of the form

$$v = f(t \pm \sqrt{\mathcal{L}\mathcal{C}}\, z) \quad \text{or} \quad v = f\left(z \pm \frac{t}{\sqrt{\mathcal{L}\mathcal{C}}}\right)$$

satisfies the equations. Equations 2.2 are called *wave equations* and the solutions are traveling waves. The "plus" sign option in the solutions gives waves traveling in the negative z direction, the "minus" sign option gives waves traveling in the positive z direction. Each solution option describes a voltage wave that travels with a speed $c=1/\sqrt{\mathcal{L}\mathcal{C}}$. To repeat, $f(t\pm z/c)$ represents an arbitrary function; which particular function applies in a given situation is determined by the excitation.

Suppose we have a voltage function $v=f(t-z/c)$. The current function is then determined by the simplified line equations since

$$\frac{\partial i}{\partial t}=-\frac{1}{\mathcal{L}}\frac{\partial v}{\partial z}=-\frac{1}{\mathcal{L}}\left(-\frac{1}{c}\right)f'\left(t-\frac{z}{c}\right)$$

and integrating* the time function $i=(1/\mathcal{L}c)f(t-z/c)$. Note then that the ratio of v to i is a constant,

$$\frac{v}{i}=\frac{f(t-z/c)}{\frac{1}{\mathcal{L}c}f(t-z/c)}=\mathcal{L}c=\mathcal{L}\frac{1}{\sqrt{\mathcal{L}\mathcal{C}}}=\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\equiv Z_0$$

this constant ratio being called the surge impedance (or sometimes the *characteristic* impedance). Similarly, if the current is a known function,

$$i=f_1\left(t-\frac{z}{c}\right),\qquad \frac{\partial v}{\partial t}=\frac{1}{\mathcal{C}c}f_1'\left(t-\frac{z}{c}\right),\qquad v=\frac{1}{\mathcal{C}c}f_1\left(t-\frac{z}{c}\right)$$

$$\frac{i}{v}=\mathcal{C}c=\mathcal{C}\frac{1}{\sqrt{\mathcal{L}\mathcal{C}}}=\sqrt{\frac{\mathcal{C}}{\mathcal{L}}}=\frac{1}{Z_0}\equiv Y_0$$

the latter being known as the surge admittance.

Example 2.1

Suppose a current $I_g=Af(t)$ is injected into an infinitely long transmission line at its center, $z=0$, as in Fig. 2.2. The function may be thought of as a pulse of any shape. The current splits so that

$$i(t,0+)=\frac{A}{2}f(t),\qquad i(t,0-)=\frac{-A}{2}f(t)$$

Thus, for points $z>0$, the solutions to Eqs. 2.2 are

$$i=\frac{A}{2}f\left(t-\frac{z}{c}\right),\qquad v=Z_0\frac{A}{2}f\left(t-\frac{z}{c}\right)$$

This is a wave traveling to the right (i.e., the pulse propagates). For $z<0$, the solutions are

$$i=\frac{-A}{2}f\left(t+\frac{z}{c}\right),\qquad v=Z_0\frac{A}{2}f\left(t+\frac{z}{c}\right)$$

*We can ignore the arbitrary constant associated with the integration; that constant simply allows for the possibility of a steady dc current to be present along with the time varying current that is of main interest to us.

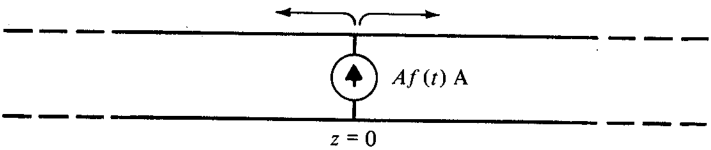

Figure 2.2 A current source injects a current into a line assumed to be infinitely long in both directions. The current splits and waves start out to the left and right as indicated.

This is a wave traveling to the left (i.e., toward negative values).

• ***Note in Summary***

$$\mathcal{L}c = Z_0, \qquad \mathcal{C}c = Y_0 = \frac{1}{Z_0}, \qquad \mathcal{L}\mathcal{C}c^2 = 1$$

The boxed relations above are quite helpful because the quantity c, the velocity of the waves, turns out to be a constant depending only on the line material—in particular $c = 1/\sqrt{\mu\varepsilon}$ for lossless lines, as will be demonstrated next.

2.2 TRANSMISSION LINE THEORY FROM THE POINT OF VIEW OF ELECTROMAGNETIC FIELDS

There are several important transmission line properties that hold for idealized transmission structures independent of the actual type of structure. For example, the velocity of propagation is determined primarily by the properties (dielectric constant and permeability) of the insulating medium about the conductors. This fact is shown readily by a study of the field equations. We can make the connection between the transmission line variables v and i and the field variables E and H as follows: The line voltage is the (line) integral of the electric field, E, from one conductor to the other in a z-plane, that is,

$$v = \int_1^2 \mathbf{E} \cdot \mathbf{dl}_1$$

The current is the closed line integral of the magnetic field, H, around the conductor, in a plane of z, that is,

$$i = \oint \mathbf{H} \cdot \mathbf{dl}_2$$

It follows from these equations that both the z and the t variations of the circuit variable, v, and the field variable, E, must be the same and that both the z and t variations of i and H must also be the same. We will therefore study the field equations to see how E and H vary with z and t.

Consider Maxwell's equations as they apply to the space around the transmission lines, since these are among the most reliable equations known.

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \qquad \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

Now recall that the magnetic fields generated by the currents in most idealized transmission line structures have no component along the direction of propagation. Moreover, the conductivity of most line conductors is quite high, which means that typically the component of the electric field in the direction of propagation is very small. Thus, taking z to be the direction of propagation as before, we will assume that $H_z = 0 = E_z$. Then, if we expand $\nabla \times \mathbf{H}$ in Maxwell's equation

$$\nabla \times \mathbf{H} = \begin{vmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & 0 \end{vmatrix} = \hat{\mathbf{a}}_y \frac{\partial H_x}{\partial z} + \hat{\mathbf{a}}_z \frac{\partial H_y}{\partial x} - \hat{\mathbf{a}}_z \frac{\partial H_x}{\partial y} - \hat{\mathbf{a}}_x \frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

we note that since $\partial \mathbf{E}/\partial t$ has no z-component, it follows that

$$\frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial x}$$

and more importantly,

$$-\hat{\mathbf{a}}_x \frac{\partial H_y}{\partial z} + \hat{\mathbf{a}}_y \frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{2.3}$$

or equating corresponding components

$$\frac{\partial H_y}{\partial z} = -\varepsilon \frac{\partial E_x}{\partial t}, \qquad \frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial E_y}{\partial t} \tag{2.4}$$

In similar fashion from the second Maxwell equation, it follows that

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}, \qquad \frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t} \tag{2.5}$$

That is, inspection of these equations shows that the components of E and H satisfy transmission line-like equations. Taking the first and second members of Eqs. 2.4 and 2.5 in pairs, it can be seen that the components E_x and H_y are analogous to v and i in the lossless version of the transmission line equations, (Eqs. 2.1), while the components E_y and H_x are also analogous to v and i in the same way. The parameters ε (permittivity) and μ (permeability) are analogous to $\mathcal{C}$ and $\mathcal{L}$ in the lossless transmission line equations.

With these analogies in mind, we can combine the equations by eliminating one or the other of E or H and so obtain wave equations similar to Eqs. 2.2. Eliminating E_x first by the step of differentiating the first of Eq. 2.4 with respect to z and the first of 2.5 with respect to t, we have

$$\frac{\partial^2 H_y}{\partial z^2} = -\varepsilon \frac{\partial^2 E_x}{\partial z\,\partial t} = -\varepsilon\left(-\mu \frac{\partial^2 H_y}{\partial t^2}\right) = \mu\varepsilon \frac{\partial^2 H_y}{\partial t^2}$$

Similarly with the other components

$$\frac{\partial^2 H_x}{\partial z^2} = \varepsilon\left(\frac{\partial^2 E_y}{\partial z\,\partial t}\right) = \mu\varepsilon \frac{\partial^2 H_x}{\partial t^2}$$

Both components of H thus satisfy the wave equations. Consequently, we can write a wave equation for the vector **H** itself.

$$\frac{\partial^2 \mathbf{H}}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

In a similar fashion, we may obtain a wave equation for both components of the vector **E**

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

All of the components of E and H thus satisfy equations similar to the wave equations for v and i written as Eqs. 2.2. The solutions therefore have the delayed functional form found above for v and i. For example, the function $E_y = f(t \pm z/c)$, where $f(t)$ is any function, satisfies the wave equation for E and by analogy $c = 1/\sqrt{\mu\varepsilon}$; then from Eqs. 2.4 and 2.5 it follows that H_x must have the form $H_x = \pm\sqrt{\varepsilon/\mu}\; f(t \pm z/c)$ since

$$\frac{\partial E_y}{\partial z} = \pm\frac{1}{c} f'\left(t \pm \frac{z}{c}\right) = \mu \frac{\partial H_x}{\partial t}$$

$$\text{or} \quad H_x = \pm\frac{1}{\mu c} f\left(t \pm \frac{z}{c}\right) = \pm\sqrt{\frac{\varepsilon}{\mu}}\; f\left(t \pm \frac{z}{c}\right)$$

It is clear then that both E and H are waves that travel at the same speed* $c = 1/\sqrt{\mu\varepsilon}$ and they have the same functional form. But as pointed out above, the circuit variables v and i must have the same z (spatial) and t (time) variation as the field variables E and H, respectively; it follows then that the waves for v and i must travel at this same speed, $c = 1/\sqrt{\mu\varepsilon}$. In the MKS system of units, free space has the following parameter values.

$$\varepsilon_0 = \frac{1}{36\pi} \times 10^{-9} \text{ farad (F)/m}, \qquad \mu_0 = 4\pi \times 10^{-7} \text{ H/m},$$

$$\text{or} \quad \mu_0\varepsilon_0 = \tfrac{1}{9} \times 10^{-16}, \qquad (\mu_0\varepsilon_0)^{-1/2} = 3 \times 10^8 \text{ m/s}$$

This is what we wanted to show.

*We have adopted the symbol c here for the speed, rather than v, in order to avoid confusion with our symbol for the time varying voltage. Here the speed c is not always equal to the speed of light in free space, that is, it depends upon μ and ε.

According to the transmission line equations presented earlier, (Eqs. 2.1), the speed of propagation on a line is $c=1/\sqrt{\mathcal{L}\mathcal{C}}$. It must be then that $\mathcal{L}\mathcal{C}=\mu\varepsilon$. The finite conductivity of actual conductors is responsible for the fact that the velocity of propagation on actual lines is slightly less than the value given above, as we shall see later.

Summarizing once again, the voltage v and current i on a lossless transmission line must propagate with speed $c=1/\sqrt{\mu\varepsilon}$ and also $c=1/\sqrt{\mathcal{L}\mathcal{C}}$. It follows that

$$\boxed{\mathcal{L}_{ext}\mathcal{C}_{ext}=\mu_{ext}\varepsilon_{ext}}$$

which will prove to be a very useful relationship. The subscript "ext" has been attached to remind us that μ and ε are the permeability and permittivity values for the space outside or "external" to the conductors. $\mathcal{L}_{ext}$ is associated with the magnetic flux *outside* the conductors and $\mathcal{C}_{ext}$ is the capacitance arising from the charges on the surface of and the electric field between conductors. Also, we have for the surge impedance $Z_{0(ext)}=\mathcal{L}_{ext}c=\sqrt{\mathcal{L}_{ext}/\mathcal{C}_{ext}}$, and similarly

$$Y_{0(ext)}=\mathcal{C}_{ext}c=\sqrt{\mathcal{C}_{ext}/\mathcal{L}_{ext}}$$

Note that with μ, ε, and $\mathcal{C}$ given, $\mathcal{L}_{ext}$ is determined; no separate calculation is required to find the external inductance (or if $\mathcal{L}_{ext}$ is given, $\mathcal{C}_{ext}$ is determined).

The line parameters are important in that they influence stability, energy storage, and the specific values of v and i in surges. As we have seen, in a surge traveling in one direction, if the current is $i=f_1(t-z/c)$ then the voltage is $v=Z_0f_1(t-z/c)$ or if $v=f(t-z/c)$ then $i=Y_0f(t-z/c)$. Surges travel more or less undisturbed until they encounter a discontinuity in the line. As we will see later, they are in general reflected from discontinuities.

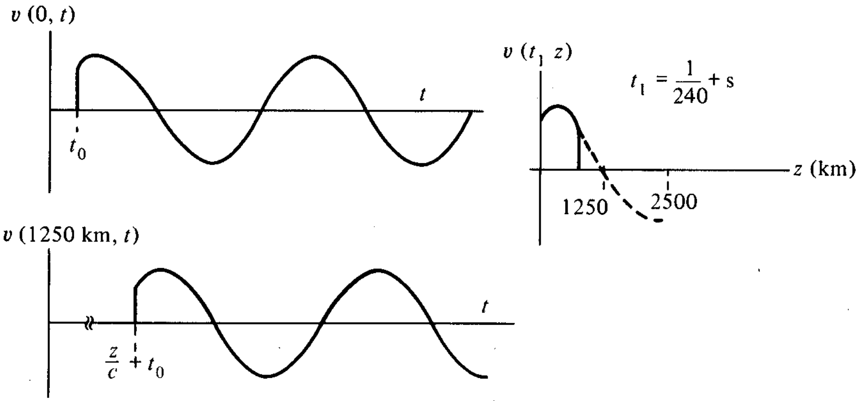

Figure 2.3 Waveforms for the example of the 60-Hz generator that is switched on. $t_0=\frac{1}{8}T=\frac{1}{480}$ s.

As an example, suppose we have an ideal voltage generator at $z=0$ on a line having $Z_0=400\ \Omega$. If the generator is switched on at time $t=t_0$, to a 60-cycle, rms voltage of 138 kV, we might have, ideally, $v(t,0)=\sqrt{2}\ 138{,}000u(t-t_0)\sin\omega t$, $\omega=377$. Then, until the wave meets something that forces v/i to be different from 400 Ω, the voltage and current will be $v(t,z)=\sqrt{2}\times 138{,}000u(t-z/c-t_0)\sin\omega(t-z/c)V$, $i(z,t)=\sqrt{2}\times 345u(t-z/c-t_0)\sin\omega(t-z/c)$, $c=\sqrt{1/\mu_0\varepsilon_0}$ =just under 3×10^5 km/s. The waveform as a function of both z and t is suggested in Fig. 2.3.

2.3 SURGE IMPEDANCE LOADING

If a line is terminated in a resistance $R_T=Z_0$, the pulse or surge is not reflected since the ratio v/i is the same at the resistor terminals as on the line with the surge. We will consider the question of reflections in detail in the next chapter. For now, let us simply agree that for certain terminations, a traveling wave will not be reflected. When a line has a wave that travels in one direction only, it is easy to find the power being transferred. Thus, in preliminary design consideration, planners often assume "*surge impedance loading*" (SIL),* or at least reference the transmitted power to that handled with SIL. With SIL, since we have a wave that goes in one direction only, $v=f(t-z/c)$, and $i=v/Z_0$, the power is $p=vi=v^2/Z_0$, so the power capability of different voltage classes is readily estimated. For example, with the voltage generator and line described in the previous paragraph, $p=(\sqrt{2}\times 138{,}000)^2/Z_0u(t-z/c-t_0)\sin^2\omega(t-z/c)$, and, at any location $z<ct$, the average power passing after the first half-cycle goes by is $P_{av}=(138{,}000)^2/Z_0=47.6$ MW. As before, the current implied by the surge impedance loading is about 345 amperes (A), rms.

The questions that remain are: "Is 400 Ω a reasonable value for surge impedance?" "How large a conductor is required to handle 345 A (or more generally, the current implied by surge impedance loading a given voltage class)?" To answer these questions, we should know how to calculate the line parameters; this we shall do next. We will calculate the line capacitance first, since many other parameters can be found once it is known.

PROBLEMS

2.1 An approximation for the effect of lightning striking a transmission line is the insertion of a current source across the line, with the current having the form

$$I=I_0\frac{t}{t_1}\exp\left(\frac{-t}{t_2}\right), \qquad t>0$$

$$I=0, \qquad t<0$$

*Even though lines are not usually loaded at this special level at the 60-Hz operating frequency.

Chapter 3

Transients on Transmission Lines

Having learned to calculate the parameters of single-phase lines, we now return to the subject of transient voltages and currents. The important transients in power systems are those associated with various switching operations, with faults, and with lightning (plus those associated with information carried at high frequencies along the wires).

3.1 LAPLACE TRANSFORM SOLUTION OF TRANSMISSION LINE EQUATIONS

To handle general transients on general lines, it is economical to employ a transform (Laplace or Fourier) method to eliminate the time from the transmission line equations. Recall the definition of the Laplace transform.

$$F(s)=\int_0^{\infty} e^{-st}f(t)\,dt$$

Applying this transform to the line equations and boundary conditions we obtain

TM Line Equations		*Transformed Equations*	
$\frac{\partial v}{\partial z}=-\mathfrak{L}\frac{\partial i}{\partial t}-\mathfrak{R}i$	$\Rightarrow$	$\frac{dV}{dz}=-(s\mathfrak{L}+\mathfrak{R})I$	(3.1)
$\frac{\partial i}{\partial z}=-\mathfrak{C}\frac{\partial v}{\partial t}-\mathfrak{G}v$	$\Rightarrow$	$\frac{dI}{dz}=-(s\mathfrak{C}+\mathfrak{G})V$	(3.2)
$v(0,t)=f(t)$	$\Rightarrow$	$V(0,s)=F(s)$	
$v(z,t)\Rightarrow 0$ (infinite line) as $z\rightarrow\infty$	$\Rightarrow$	$V(z,s)\Rightarrow 0$ as $z\rightarrow\infty$	
$v(L,t)\Rightarrow i(L,t)R_L$	$\Rightarrow$	$V(L,s)=I(L,s)R_L$	

In the latter equation, it has been assumed that the line is terminated in a resistance, R_L.

In the transformed operations, it has been assumed that the quantities $i(z,0+)$, and $v(z,0+)$, the initial values of current and voltage, are zero. For certain problems it may be advantageous to insert appropriate values for such initial conditions when they are known.

The next step is to solve the transformed equations in the space variable. The actual voltages and currents are then obtained by the inverse transform operation.

For example, eliminating I between Eqs. 3.1 and 3.2

$$\frac{d^2V}{dz^2}=(\mathfrak{R}+s\mathfrak{L})(\mathfrak{G}+s\mathfrak{C})V\equiv\gamma^2V, \qquad \gamma^2\equiv(\mathfrak{R}+s\mathfrak{L})(\mathfrak{G}+s\mathfrak{C})$$

The solution to this second-order differential equation for V can be written in terms of sines, hyperbolic functions, or exponentials.

We will choose the exponential form. Two arbitrary constants are necessary for the solution of the second-order differential equation.

$$V(z,s)=Ae^{\gamma z}+Be^{-\gamma z} \tag{3.3}$$

where A and B are the constants to be determined. From Eq. 3.1

$$I=-\frac{1}{(\mathfrak{R}+s\mathfrak{L})}\frac{dV}{dz}=-\frac{1}{(\mathfrak{R}+s\mathfrak{L})}\gamma[Ae^{\gamma z}-Be^{-\gamma z}]$$

Define

$$Y_0=\frac{1}{Z_0}=\frac{\gamma}{(\mathfrak{R}+s\mathfrak{L})}=\sqrt{\frac{\mathfrak{G}+s\mathfrak{C}}{\mathfrak{R}+s\mathfrak{L}}} \quad \text{(characteristic or surge admittance)} \tag{3.4}$$

$$I=-Y_0[Ae^{\gamma z}-Be^{-\gamma z}] \tag{3.5}$$

If we have an ideal voltage generator connected to the line at $z=0$, we have the conditions listed below Eq. 3.2, $v(0,t)=f(t)$ and $V(0,s)=F(s)$ and there is a source or excitation condition that gets things going. This also provides a condition on the constants: evaluating 3.3 at $z=0$.

$$F(s)=A+B \tag{3.6}$$

One more condition is needed to determine A and B. If the line is infinite in length, $V_{z\to\infty}(z,s)\Rightarrow 0$, so from 3.3, $A=0$, and

$$V(z,s)=F(s)e^{-\gamma z}, \qquad I(z,s)=Y_0F(s)e^{-\gamma z}$$

If the line is terminated at $z=L$ by a resistor R_L,

$$\frac{V(L,s)}{I(L,s)}=R_L=\frac{Ae^{\gamma L}+Be^{-\gamma L}}{-Y_0(Ae^{\gamma L}-Be^{-\gamma L})} \tag{3.7}$$

Define a reflection coefficient $\rho(L)=(A/B)e^{2\gamma L}$; this definition, inserted into Eq. 3.7 gives

$$\frac{R_L}{Z_0}=\frac{1+\rho(L)}{1-\rho(L)}$$

or solving for $\rho(L)$,

$$\rho(L)=\frac{R_L/Z_0-1}{R_L/Z_0+1}=\frac{R_L-Z_0}{R_L+Z_0}$$

Then from Eq. 3.6, $B=F(s)-A=F(s)-B\rho(L)e^{-2\gamma L}$ or

$$B=\frac{F(s)}{1+\rho(L)e^{-2\gamma L}} \quad \text{and then} \quad A=\frac{F(s)\rho(L)e^{-2\gamma L}}{1+\rho(L)e^{-2\gamma L}} \tag{3.8}$$

3.1.1 Special Cases

- *A. Matched Load.* $R_L=Z_0\Rightarrow\rho(L)=0\Rightarrow A=0$. This is surge impedance (or "natural") loading and $V(z,s)=F(s)e^{-\gamma z}$ as in the case of the infinite line above.

- *B. Open Circuit.* $R_L=\infty\Rightarrow\rho(L)=1\Rightarrow B=F(s)/1+e^{-2\gamma L}$. From Eqs. 3.3 and 3.8,

$$V(z)=\frac{F(s)}{1+e^{-2\gamma L}}\left[e^{-\gamma z}+e^{\gamma(z-2L)}\right] \tag{3.9}$$

- *C. Short Circuit.* $R_L=0\Rightarrow\rho(L)=-1$, $B=F(s)/1-e^{-2\gamma L}$. Similarly,

$$V(z)=\frac{F(s)}{1-e^{-2\gamma L}}\left[e^{-\gamma z}-e^{\gamma(z-2L)}\right] \tag{3.10}$$

To find the time variation of voltage, and current, we must obtain the inverse transforms of these voltages: that is, $v(t,z)=L^{-1}[V(s,z)]$.

3.2 THE PROPAGATION CONSTANT AND SIMPLE INVERSIONS

The quantity, γ, contains the s-variable

$$\gamma=\sqrt{(\mathfrak{R}+s\mathfrak{L})(\mathfrak{G}+s\mathfrak{C})} \tag{3.11}$$

and therefore it strongly influences the exact form of the inverse transformation. In general, with the square root function indicated, the inverse transform can only be done numerically. But under certain conditions, the transform can be obtained analytically to a good approximation. With one exception, the "certain conditions" are that $\mathfrak{R}$ and $\mathfrak{G}$ be very small, which is often a good approximation. We will examine a few common special cases.

Special Case 1

$\mathfrak{G} \doteq 0$, $\mathfrak{R} \ll s\mathfrak{L}$. In this case

$$\gamma=\sqrt{(\mathfrak{R}+s\mathfrak{L})s\mathfrak{C}}=s\sqrt{\mathfrak{L}\mathfrak{C}}\left(1+\frac{\mathfrak{R}}{s\mathfrak{L}}\right)^{1/2}\doteq s\sqrt{\mathfrak{L}\mathfrak{C}}\left(1+\frac{\mathfrak{R}}{2s\mathfrak{L}}\right)$$

or

$$\gamma \doteq s\sqrt{\mathfrak{L}\mathfrak{C}}+\frac{1}{2}\mathfrak{R}\sqrt{\frac{\mathfrak{C}}{\mathfrak{L}}}$$

Let us utilize this result for special case 1, surge impedance loading. In this case

$$V(s,z)=F(s)e^{-\gamma z}=F(s)e^{-s\sqrt{\mathfrak{L}\mathfrak{C}}\,z}e^{(-1/2)\mathfrak{R}\sqrt{\mathfrak{C}/\mathfrak{L}}\,z}$$

or

$$v(t,z)=L^{-1}[V(s,z)]=e^{(-1/2)\mathfrak{R}\sqrt{\mathfrak{C}/\mathfrak{L}}\,z}L^{-1}\left[F(s)e^{-s\sqrt{\mathfrak{L}\mathfrak{C}}\,z}\right]$$

$$=e^{-(\mathfrak{R}/2)\sqrt{\mathfrak{C}/\mathfrak{L}}\,z}f(t-\sqrt{\mathfrak{L}\mathfrak{C}}\,z)u(t-\sqrt{\mathfrak{L}\mathfrak{C}}\,z)$$

where $u(t)$ is the familiar step function. This is an attenuated wave traveling at a speed $=1/\sqrt{\mathfrak{L}\mathfrak{C}}$, having value zero until $t>(\mathfrak{L}\mathfrak{C})^{1/2}z$. Notice that the form of this solution is the same as that we encountered in Chapter 2, for lossless lines ($\mathfrak{R}=\mathfrak{G}=0$), except that in the present case there is attenuation. The wave maintains its shape as it moves along at speed $v=1/\sqrt{\mathfrak{L}\mathfrak{C}}$ but it shrinks in amplitude.

The other common cases give similar solutions but differ in the detailed formula for the attenuation constant.

Special Case 2

$\mathfrak{R}\ll s\mathfrak{L}$, $\mathfrak{G}\ll s\mathfrak{C}$, $\mathfrak{R}\mathfrak{G}\ll s^2\mathfrak{L}\mathfrak{C}$. Then

$$\gamma=\sqrt{(\mathfrak{R}+s\mathfrak{L})(\mathfrak{G}+s\mathfrak{C})}\doteq\sqrt{0+s(\mathfrak{R}\mathfrak{C}+\mathfrak{G}\mathfrak{L})+s^2\mathfrak{L}\mathfrak{C}}$$

$$\doteq s\sqrt{\mathfrak{L}\mathfrak{C}}\left[1+\frac{(\mathfrak{R}\mathfrak{C}+\mathfrak{G}\mathfrak{L})}{s\mathfrak{L}\mathfrak{C}}\right]^{1/2}$$

$$\doteq s\sqrt{\mathfrak{L}\mathfrak{C}}\left\{1+\frac{1}{2}\frac{(\mathfrak{R}\mathfrak{C}+\mathfrak{G}\mathfrak{L})}{s\mathfrak{L}\mathfrak{C}}\right\}$$

$$=s\sqrt{\mathfrak{L}\mathfrak{C}}+\frac{1}{2}\mathfrak{R}\sqrt{\frac{\mathfrak{C}}{\mathfrak{L}}}+\frac{1}{2}\mathfrak{G}\sqrt{\frac{\mathfrak{L}}{\mathfrak{C}}}$$

To save writing, define

$$\alpha \equiv \frac{1}{2}\mathcal{R}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}} + \frac{1}{2}\mathcal{G}\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}$$

an attenuation constant. Then for surge impedance loading

$$V(s,z) = F(s)e^{-\gamma z} = F(s)e^{-\alpha z}e^{-s\sqrt{\mathcal{L}\mathcal{C}}\,z}$$

and performing the inverse transformation,

$$v(t,z) = e^{-\alpha z}f(t-\sqrt{\mathcal{L}\mathcal{C}}\,z)u(t-\sqrt{\mathcal{L}\mathcal{C}}\,z)$$

which is again an attenuated but undistorted wave.

In neither of the two cases is the approximation valid at dc or near zero frequency.

Special Case 3

A line may be *designed* such that $\mathcal{R}/\mathcal{L} = \mathcal{G}/\mathcal{C}$, with no other restriction on the magnitude of $\mathcal{R}$ and $\mathcal{G}$. In this case the propagation constant has a simple form as follows.

$$\gamma^2 = s^2\mathcal{L}\mathcal{C} + s(\mathcal{R}\mathcal{C} + \mathcal{G}\mathcal{L}) + \mathcal{R}\mathcal{G} = s^2\mathcal{L}\mathcal{C}\left[1 + s\frac{2\mathcal{R}\mathcal{C}}{s^2\mathcal{L}\mathcal{C}} + \frac{\mathcal{R}\mathcal{G}}{s^2\mathcal{L}\mathcal{C}}\right]$$

$$= s^2\mathcal{L}\mathcal{C}\left[1 + \frac{2\mathcal{R}}{s\mathcal{L}} + \frac{\mathcal{R}^2}{s^2\mathcal{L}^2}\right] = s^2\mathcal{L}\mathcal{C}\left[1 + \frac{\mathcal{R}}{s\mathcal{L}}\right]^2$$

or

$$\gamma = s\sqrt{\mathcal{L}\mathcal{C}}\left[1 + \frac{\mathcal{R}}{s\mathcal{L}}\right] = s\sqrt{\mathcal{L}\mathcal{C}} + \mathcal{R}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}}$$

This result is of the same form as the other two special cases except for a different attenuation rate. For a line that is surge impedance loaded, $v(t,z)$ is an attenuated but undistorted traveling wave. A line having parameters so designed is called a "distortionless" line or "heaviside" line.

3.3 INVERTING THE LAPLACE TRANSFORM SOLUTION TO GIVE THE TIME DOMAIN VALUES

Except for these special cases, the line introduces distortions that are complicated and difficult to discuss. Generally speaking, the effects are the rounding of the corners of square waves and the "stretching" of pulses.

Returning to the special cases of open circuit and short circuit termination, we will let the propagation constant be $\gamma = \alpha + s\sqrt{\mathcal{L}\mathcal{C}}$, with α not a function of s. In Sec. 3.1 we found the following result for the case of the open circuited line.

$$V(z) = \frac{F(s)}{1+e^{-2\gamma L}}\left[e^{-\gamma z} + e^{+\gamma(z-2L)}\right] = V(s,z)$$

The easiest way to inverse transform this function is to do a little algebra first, to get a series representation of the factor $1/1+e^{-2\gamma L}$; this can be done, for example, by synthetic division.

$$
\begin{array}{r|l}
 & 1-e^{-2\gamma L}+e^{-4\gamma L}-e^{-6\gamma L}+\cdots \\
\hline
1+e^{-2\gamma L} & 1 \\
 & \underline{1+e^{-2\gamma L}} \\
 & -e^{-2\gamma L} \\
 & \underline{-e^{-2\gamma L}-e^{-4\gamma L}} \\
 & \qquad e^{-4\gamma L} \\
 & \qquad \underline{e^{-4\gamma L}+e^{-6\gamma L}} \\
 & \qquad\qquad -e^{-6\gamma L}
\end{array}
$$

That is, from the steps shown above

$$\frac{1}{1+e^{-2\gamma L}}=1-e^{-2\gamma L}+e^{-4\gamma L}-e^{-6\gamma L}+\cdots$$

The advantage of this may not be evident at first, but note

$$V(s,z)=F(s)\left[e^{-\gamma z}+e^{+\gamma(z-2L)}\right](1-e^{-2\gamma L}+e^{-4\gamma L}-e^{-6\gamma L}+\cdots)$$

or

$$V(s,z)=F(s)e^{-\gamma z}+F(s)e^{+\gamma(z-2L)}-F(s)e^{-\gamma(z+2L)}$$
$$-F(s)e^{+\gamma(z-4L)}+F(s)e^{-\gamma(z+4L)}+\cdots$$

Now we can inverse transform term by term ($c\equiv 1/\sqrt{\mathcal{L}\mathcal{C}}$) to obtain the voltage on the open circuited line.

$$v(t,z)=e^{-\alpha z}f\left(t-\frac{z}{c}\right)u\left(t-\frac{z}{c}\right)+e^{-\alpha(2L-z)}f\left(t+\frac{(z-2L)}{c}\right)$$
$$\times u\left(t+\frac{(z-2L)}{c}\right)-e^{-\alpha(z+2L)}f\left(t-\frac{(z+2L)}{c}\right)u\left(t-\frac{(2-zL)}{c}\right)+\cdots$$

This is most easily interpreted with the aid of a *bounce diagram*.

However, before discussing this bounce diagram, we should pause to point out a few more helpful details about the voltages and currents on lines. If the voltage generator at the input is not ideal but has some internal impedance Z_g, the effect is two-fold. *First*, the voltage that starts down the line is less than the ideal value and in fact, by voltage division, the voltage $(Z_0/Z_0+Z_g)f(t)$ actually starts down the line (Fig. 3.1).

Second, any wave traveling to the left and incident on the input end is reflected with a voltage reflection coefficient,

$$\rho_g=\frac{Z_g-Z_0}{Z_g+Z_0}$$

with $\rho_g=-1$ corresponding to the ideal voltage source discussed above. That is, the input end behaves like a load for those waves impinging on it from the right.

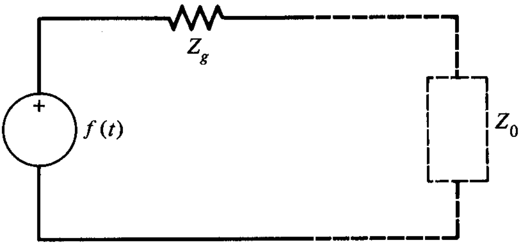

Figure 3.1 Voltage division between source and surge impedance.

We can then go back to Sec. 3.1 and work out the constants A and B that hold for the imperfect generator on a line terminated with a load R_L (reflection coefficient $\rho(L)=(R_L-Z_0)/(R_L+Z_0)$. The transformed voltage is found in this way to be

$$V(z,s)=\frac{F(s)Z_0}{Z_g+Z_0}\cdot\frac{1}{1-\rho_g\rho(L)e^{-2\gamma L}}\left[e^{-\gamma z}+\rho(L)e^{\gamma(z-2L)}\right]$$

The synthetic division discussed above can be done on the factor $1/(1-\rho_g\rho(L)e^{-2\gamma L})$ to get a series, from which $V(z,s)$ can be written

$$V(z,s)=F(s)\frac{Z_0}{Z_g+Z_0}\left[e^{-\gamma z}+\rho(L)e^{\gamma(z-2L)}\right.$$
$$\left.+\rho_g\rho(L)e^{-\gamma(z+2L)}+\rho_g\rho^2(L)e^{-\gamma(z-4L)}+\cdots\right] \tag{3.12}$$

Now this expression can be inverse transformed term by term to obtain $(c=1/\sqrt{\mathcal{L}\mathcal{C}})$

$$v(z,t)=\frac{Z_0}{Z_g+Z_0}\left[e^{-\alpha z}f\left(t-\frac{z}{c}\right)u\left(t-\frac{z}{c}\right)+\rho(L)e^{-\alpha(2L-z)}f\left(t+\frac{z-2L}{c}\right)\right.$$
$$\left.\times u\left(t-\frac{(2L-z)}{c}\right)+\rho_g\rho(L)e^{-\alpha(2L+z)}f\left(t-\frac{(2L+z)}{c}\right)u\left(t-\frac{2L+z}{c}\right)+\cdots\right]$$

and so on with the terms best interpreted with the aid of the *bounce diagram*.

Finally, note that we could have done the entire analysis from the start with current, $i(z,t)$, rather than voltage (by eliminating the voltage), with similar results. The current reflection coefficient would have been

$$\rho^I(L)=\frac{Y_L-Y_0}{Y_L+Y_0}=-\rho^V(L)$$

Note: *current and voltage reflection coefficients are negatives of each other.*

3.4 BOUNCE OR LATTICE DIAGRAMS

The device known as the bounce diagram is most helpful in keeping track of *one point* on the time function representing the wave as it reflects back and forth from the ends of the line. The time function may be either voltage or current, providing the appropriate reflection coefficient is used, that is, ρ^V for voltage diagrams, ρ^I for current diagrams.

To construct the bounce diagram, Fig. 3.2, one first draws a pair of parallel lines (those running up and down on the page); a horizontal scale between these lines represents distance along the line section lengths. A vertical scale represents time elapsed, and this scale is often measured in units of transit time, that is, the time for a wave to travel the length of a line section. A diagonal zig-zag line represents the wave as it bounces back and forth between the ends or between discontinuities. The position of a given point on the wave, say the wavefront, is then easily found given the time by placing a straight edge across the page at the designated time and finding the intersection with the zig-zag; or the time of arrival at a given position is found by orienting a straight edge up and down at the given position, and noting the intersections with the zig-zag line. The reflection coefficients are indicated under the parallel lines representing the ends or discontinuities. The amplitude on successive bounces is conveniently indicated on each bounce.

The leading and trailing edge of a pulse may be tracked by thinking of the leading edge as the beginning of a "string" of length proportional to the velocity times the time difference between leading and trailing edge. The total voltage or current at any position in space and time is the superposition of all "bounces" above that point on the diagram provided the trailing edge still exists on the "bounce" line at the position under consideration.

For example, the position of leading edge and trailing edge at a time 5.75τ after the leading edge left the generator end is indicated in Fig. 3.2. At this time, at the point $\frac{1}{8}L$ distant from the generator the "field" (voltage or current) is the sum of $A\rho(L)+A\rho(L)\rho_g+A\rho^2(L)\rho_g+A\rho^2(L)\rho_g^2$, while at a point $\frac{7}{8}L$ distant from the left end, it is the sum of $A\rho(L)\rho_g+A\rho^2(L)\rho_g+A\rho^2(L)\rho_g^2+A\rho^3(L)\rho_g^2$.

If the attenuation on the line is not negligible, the amplitude following each bounce should be decreased by a factor $e^{-\alpha L}$ since the wave would be attenuated by that amount on each transit.

The bounce diagram is a very useful conceptual tool. However, in power system practice, its digital equivalent is programmed for use on a large-scale digital computer. Such programs include techniques for studying all the cases we will consider, and many others. One of the more widely known and used of these is the "Bonneville Transient Analysis Program" (from BPA).

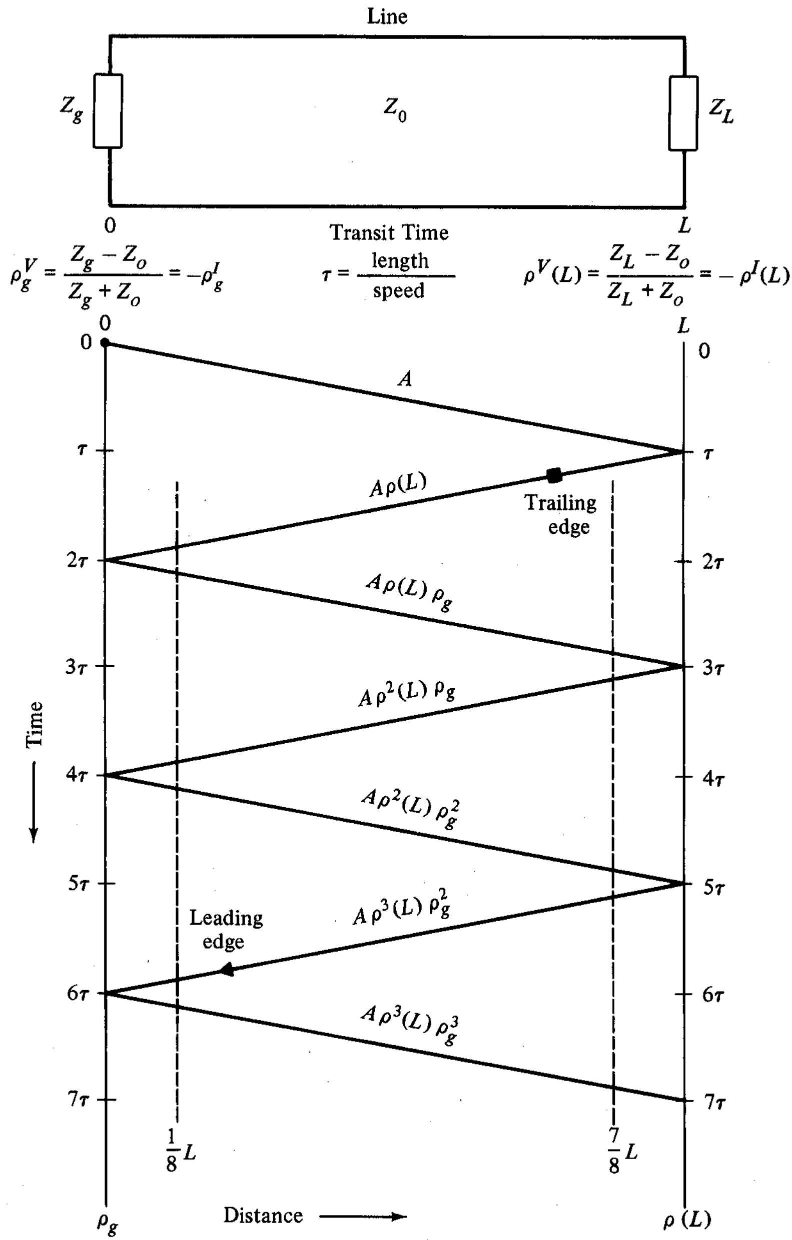

Figure 3.2 Simple bounce diagram for line shown with pulse excitation (not shown) at $z=0$ and $t=0$.

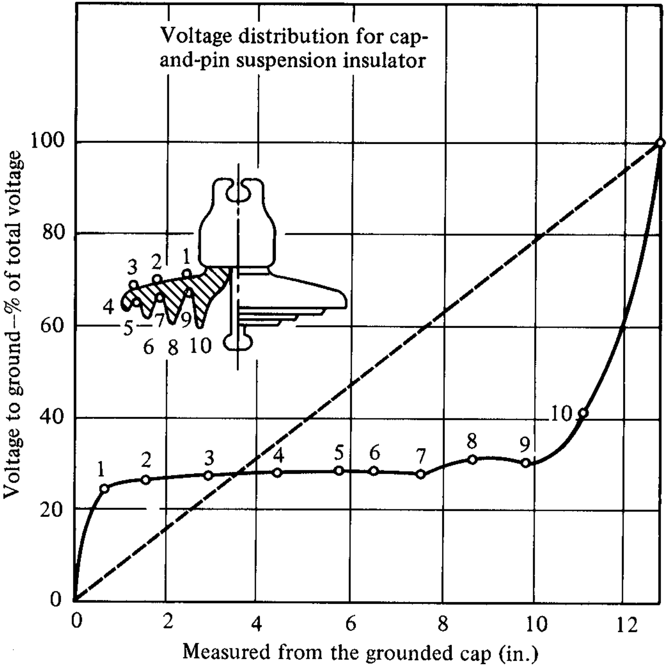

Figure 6.34 Voltage distribution along the surface of a single clean cap-and-pin insulator unit. [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

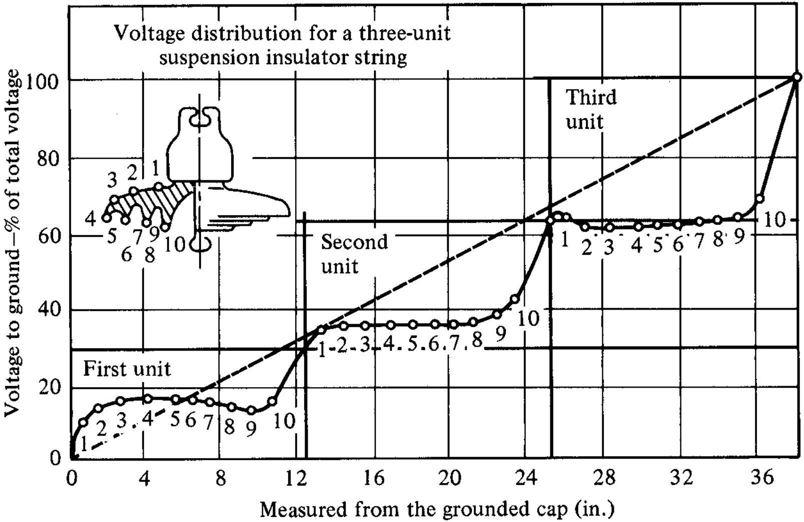

Figure 6.35 Typical voltage distribution on the surface of three clean cap-and-pin insulator units in series. [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

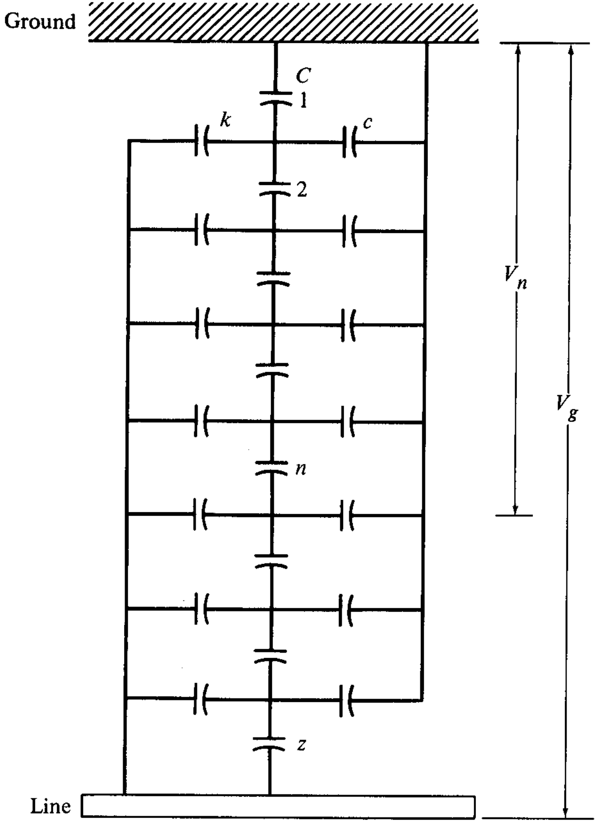

Figure 6.36 Equivalent circuit for the voltage distribution along a clean eight-unit insulator string.

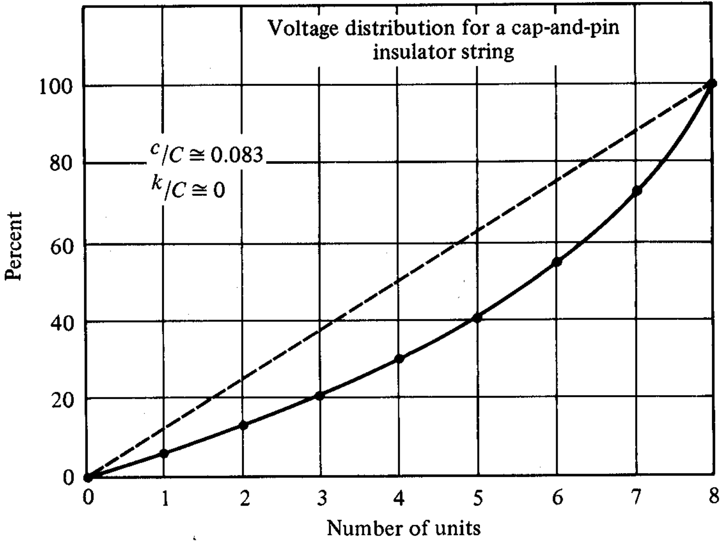

Figure 6.37 Voltage distribution along a clean eight-unit cap-and-pin insulator string. [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

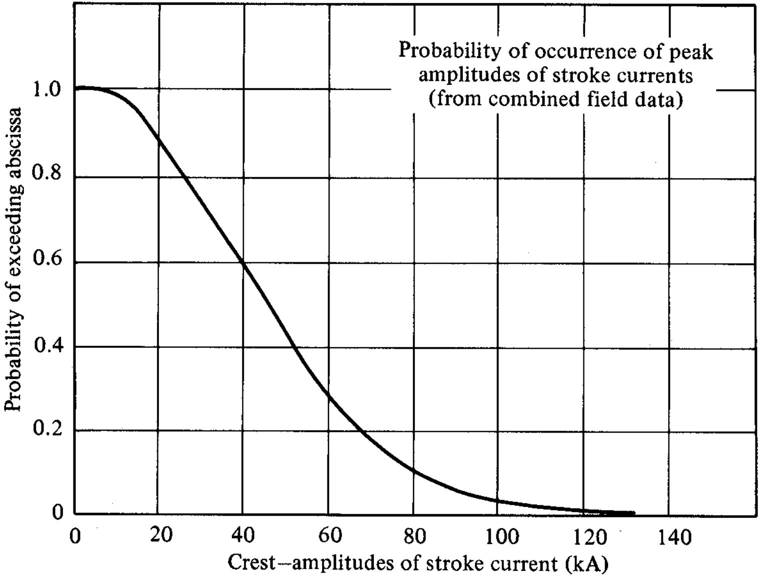

Figure 6.38 Probability of occurrence of peak amplitudes of stroke currents (from combined field data). [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

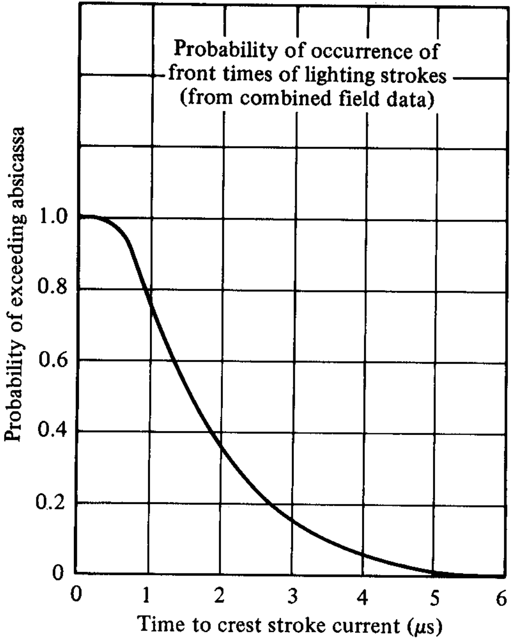

Figure 6.39 Probability of occurrence of front times of lightning strokes (from combined data). [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

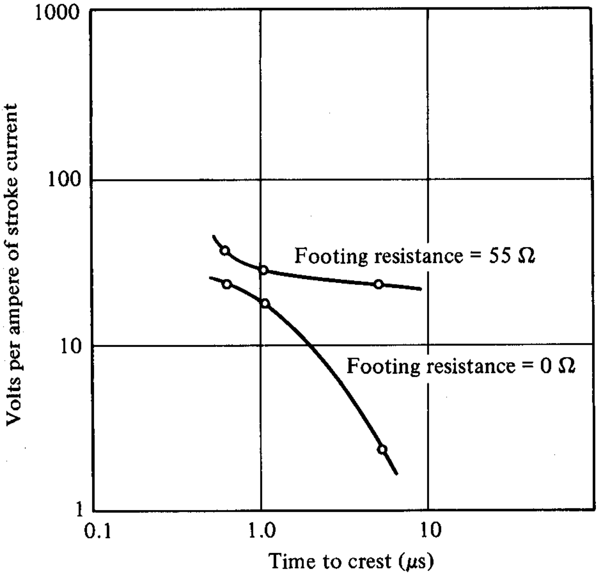

Figure 6.40 Representative measured values of the peak voltage generated across an insulator string by a stroke. [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

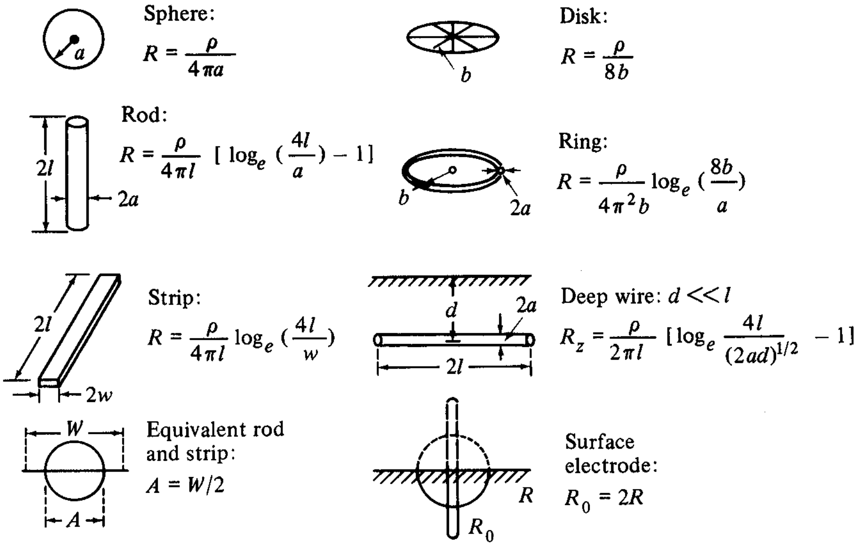

Figure 6.41 Ground resistance of elementary electrodes, buried deeply in earth.
[Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

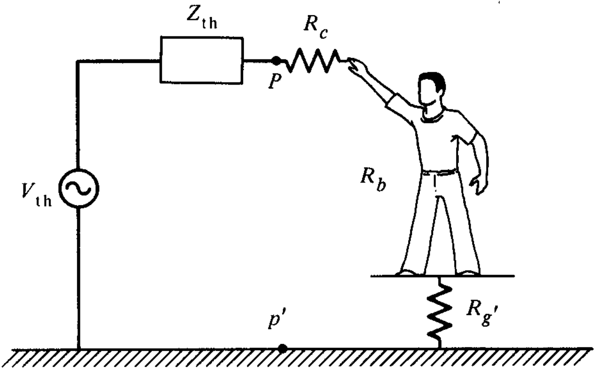

Figure 6.25 Equivalent circuit for estimating the current in a body contacting a grounded "incidental" line.

$$V_{th} = V_{oc} \text{ at } P_{p'},\ Z_{th} = R_g + Z_0 \tanh \gamma d,\ Z_m = R_{contact} + R_{body} + R_{ground'}$$

$$R_c = R_{contact} \approx 500\ \Omega,\ R_b = R_{body} \approx 1000\ \Omega,\ R_{g'} = R_{ground} \approx 600\ \Omega$$

6.7.3 Hazards to People

According to Dalziel's[4] electrocution formula, the 60-Hz body current leading to a possible fatality through ventricular fibrillation is

$$I = 0.165/\sqrt{t} \tag{6.9}$$

where t is in seconds in the range from about 8.3 ms to 5 s. The current that will flow in the body depends on the body resistance, the effective grounding resistance of the body, the open circuit voltage at the point on the conductor which contacts the individual, and the impedance seen looking back into the system at the point of contact. The latter two factors will be recognized as the essentials in the Thévenin equivalent circuit for the contact points (ground and the conductor). The current can then be calculated with the aid of this equivalent circuit,

$$I = \frac{V_{\mathrm{th}}}{Z_{\mathrm{th}} + Z_m}$$

where Z_m is the total impedance in the "human" part of the circuit as described above. The idea is indicated graphically in Fig. 6.25. Typical values of resistances are shown in the table, but these are subject to wide variations. A typical person to ground capacitance when on a 1-cm-thick insulator is about 200 pF.

TABLE 6.1 Typical Electric Shock Energy Effects

	Millijoules
1. Minimum primary shock energy	50,000
2. More conservative value (IEEE)	25,000
3. "Unpleasant" shock energy	250
4. Minimum secondary shock energy	0.5–1.5
5. Perception threshold	0.12

The current calculated in this way may be used in Eq. 6.9 to determine the time to a possible fatality, for if the current exceeds that given by Eq. (6.9) for a stated time, a fatality is likely. Moreover, currents in the range from 5 to 20 mA, if sustained, are likely to interfere with muscular control so that the individual cannot "let go" and free him or herself from the circuits. If currents in excess of the "let go" levels for an individual are continued, the result is likely to be a collapse, unconsciousness, asphyxia, and death. Tables 6.1 and 6.2 show effects of current on people.

TABLE 6.2 Typical Effects of Currents on People[a]

Effect	Current (mA)			
	Direct Current		60 Hz rms	
	Men	Women	Men	Women
1. No sensation on hand	1	0.6	0.4	0.3
2. Slight tingling. Perception threshold	5.2	3.5	1.1	0.7
3. Shock—not painful but muscular control not lost	9	6	1.8	1.2
4. Painful shock—painful but muscular control not lost	62	41	9	6
5. Painful shock—let go threshold	76	51	16.0	10.5
6. Painful and severe shock, muscular contractions, breathing difficult	90	60	23	15
7. Possible ventricular fibrillation from short shocks:				
(a) shock duration 0.03 s	1300	1300	1000	1000
(b) Shock duration 3.0 s	500	500	100	100
(c) Almost certain ventricular fibrillation (if shock duration is over one heart beat interval)	1375	1375	275	275

[a] Threshold for 50% of the males and females tested.

If the field strength levels are high enough, people in the region begin to sense the field in one way or another even though there are no significant currents to ground. The threshold of perception is in the range of 10–15 kV/m, where the sensations are body hair erection and tingling between body and clothes. Such activity is apparently not harmful, but at higher fields, perhaps as low as 15–20 kV/m, sparks may form and contacts to grass, weeds, or the handling of conducting objects are unpleasant, at least. At higher fields, corona may form at some places with small radii such as ears. This phenomenon is also disturbing to animals.

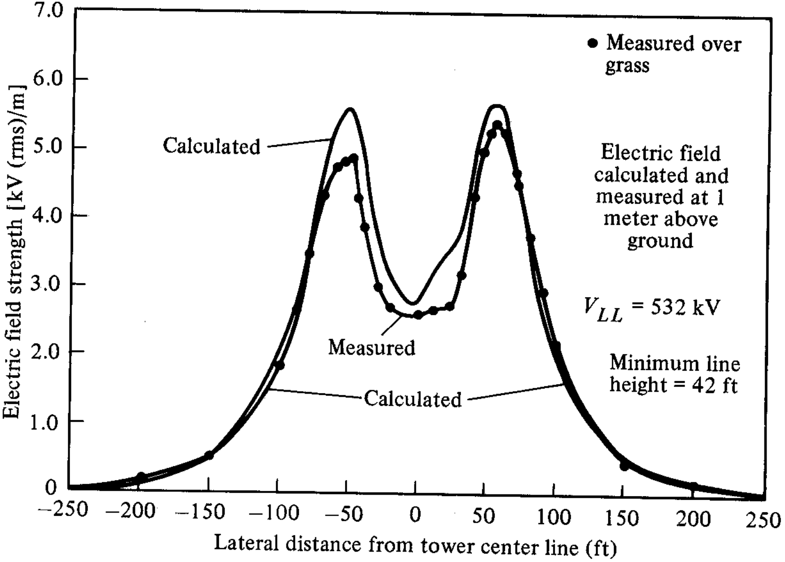

Figure 6.24 Comparison of measured and computer calculated electric field 500-kV line.