Fundamentals of Electric Power Systems

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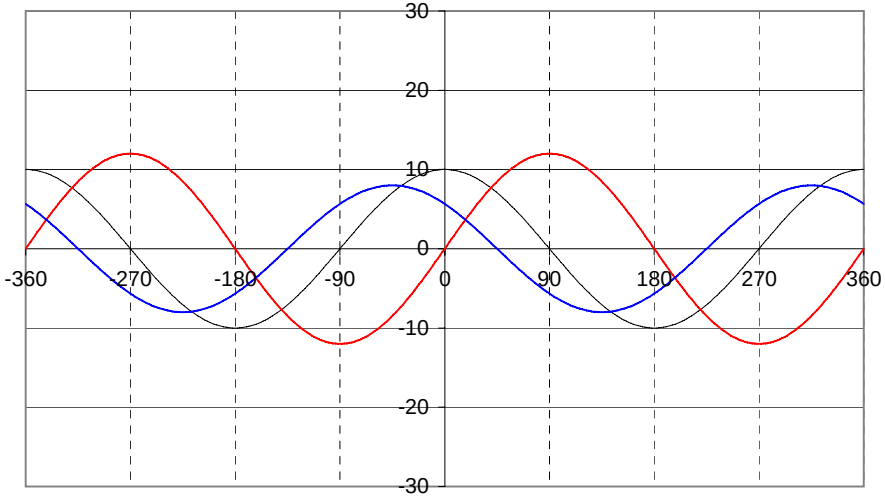
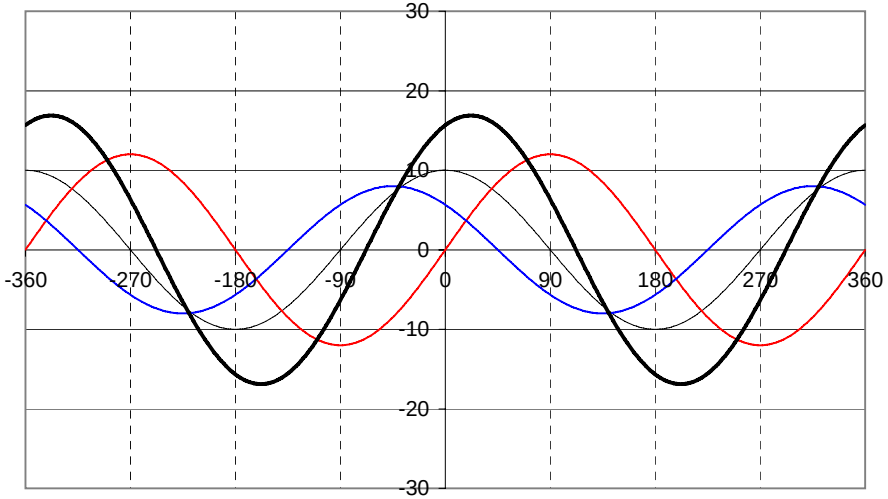
	Mag =	10	12	8
	Ang =	0	-90	45
-360	10	3.68E-15	5.656854	15.65685
-358	9.993908	0.418794	5.455987	15.86869
-356	9.975641	0.837078	5.248472	16.06119
-354	9.945219	1.254342	5.034563	16.23412
-352	9.902681	1.670077	4.81452	16.38728
-350	9.848078	2.083778	4.588611	16.52047
-348	9.781476	2.49494	4.357112	16.63353
-346	9.702957	2.903063	4.120305	16.72632
-344	9.612617	3.307648	3.878477	16.79874
-342	9.510565	3.708204	3.631924	16.85069
-340	9.396926	4.104242	3.380946	16.88211
-338	9.271839	4.495279	3.125849	16.89297
-336	9.135455	4.88084	2.866944	16.88324
-334	8.98794	5.260454	2.604545	16.85294
-332	8.829476	5.633659	2.338974	16.80211
-330	8.660254	6	2.070552	16.73081
-328	8.480481	6.359031	1.799608	16.63912
-326	8.290376	6.710315	1.526472	16.52716
-324	8.09017	7.053423	1.251476	16.39507
-322	7.880108	7.387938	0.974955	16.243
-320	7.660444	7.713451	0.697246	16.07114
-318	7.431448	8.029567	0.418688	15.8797
-316	7.193398	8.3359	0.139619	15.66892
-314	6.946584	8.632078	-0.13962	15.43904
-312	6.691306	8.917738	-0.41869	15.19036
-310	6.427876	9.192533	-0.69725	14.92316
-308	6.156615	9.456129	-0.97495	14.63779
-306	5.877853	9.708204	-1.25148	14.33458
-304	5.591929	9.948451	-1.52647	14.01391
-302	5.299193	10.17658	-1.79961	13.67616
-300	5	10.3923	-2.07055	13.32175
-298	4.694716	10.59537	-2.33897	12.95111
-296	4.383711	10.78553	-2.60455	12.56469
-294	4.067366	10.96255	-2.86694	12.16297
-292	3.746066	11.12621	-3.12585	11.74642
-290	3.420201	11.27631	-3.38095	11.31557
-288	3.09017	11.41268	-3.63192	10.87092
-286	2.756374	11.53514	-3.87848	10.41304
-284	2.419219	11.64355	-4.1203	9.942463
-282	2.079117	11.73777	-4.35711	9.459776
-280	1.736482	11.81769	-4.58861	8.965563
-278	1.391731	11.88322	-4.81452	8.460428
-276	1.045285	11.93426	-5.03456	7.944984
-274	0.697565	11.97077	-5.24847	7.419861



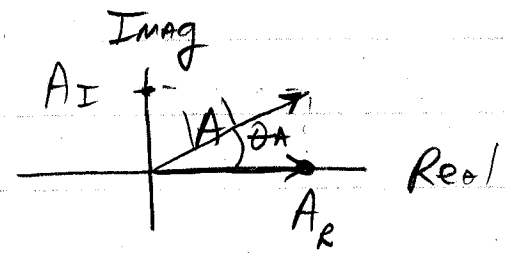


Complex Numbers

$j = \sqrt{-1}$

$$A = \overbrace{\underbrace{A_R}_{\text{Real}} + \underbrace{jA_I}_{\text{Imag}}}^{\text{Rectangular}} = \overbrace{|A| \angle \theta_A}^{\text{Polar}}$$

$$|A| = \sqrt{A_R^2 + A_I^2},$$



$$\tan\theta_A = \frac{A_I}{A_R}$$

$$A_R = \underbrace{|A| \cos\theta_A}_{\text{Projection on Real Axis}}$$

$$A_I = \underbrace{|A| \sin\theta_A}_{\text{Projection on Imag Axis}}$$

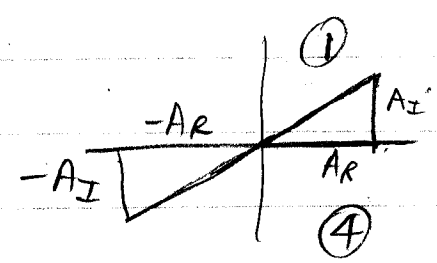
But $\tan\theta_A = \tan(\theta_A + 180°)$!

Be careful with $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ functions. Have two θ solutions.
Calculators pick θ in quadrants ① & ④ for $\sin^{-1}$, $\tan^{-1}$, but quadrants ① & ② for $\cos^{-1}$

②	①
③	④

$\pm 90°$

A_R	A_I	Calculator Correction for $\tan^{-1}$
+	+	None
+	−	None
−	+	Add 180°
−	−	Add 180°



- Addition & Subtraction, Use Rectangular Form

$$A = A_R + jA_I$$
$$+\ B = B_R + jB_I$$
$$(A_R + B_R) + j(A_I + B_I)$$

- Multiplication & Division, Use Polar Form

$$A \cdot B = (|A|\angle\theta_A)(|B|\angle\theta_B) = |A||B|\angle\theta_A + \theta_B$$

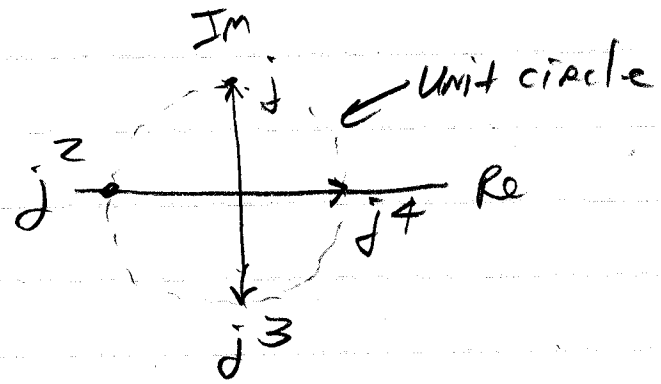
$$\frac{A}{B} = \frac{|A|}{|B|}\angle\theta_A - \theta_B$$

- Complex Conjg $A^* = A_R - jA_I$ Reverse sign of imag.

$$A + A^* = 2A_R \text{ (twice the Real part)}$$
$$A - A^* = j2A_I \text{ (twice the imag part)}$$

$$A \cdot A^* = (A_R + jA_I)(A_R - jA_I) = A_R^2 + A_I^2$$
$$= |A|^2$$

$j = \sqrt{-1}$
$j^2 = -1$
$j^3 = j\,j^2 = -j$
$j^4 = j^2\,j^2 = 1$
$j^5 = j\,j^4 = j$



Eulers Rule

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$|A|e^{j\theta} = \underbrace{|A|\cos\theta}_{A_R} + j\underbrace{|A|\sin\theta}_{A_I}$$

$$A_R + jA_I$$

Shorthand, $|A|e^{j\theta} \rightarrow |A|\angle\theta$

$$\left.\begin{aligned} e^{j0} &= 1 + j0 = 1 \\ e^{j90^\circ} &= e^{j\pi/2} = 0 + j1 = j \\ e^{j180^\circ} &= -1 + j0 = -1 \\ e^{j270} &= 0 - j1 = -j \end{aligned}\right] \text{The circle}$$

$$e^{ja} \cdot e^{jb} = e^{j(a+b)}$$

$$e^{ja}/e^{jb} = e^{j(a-b)}$$

$$(e^{ja})^c = e^{jac}$$

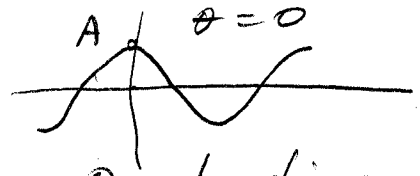
$$\text{so } \sqrt{j} = \left(e^{j90^\circ}\right)^{1/2} = e^{j45^\circ} = \frac{1+j1}{\sqrt{2}}$$

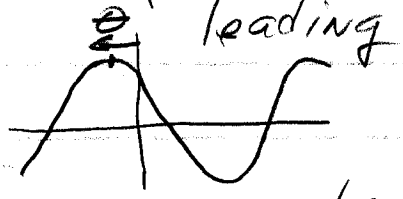
$$= \cos 45^\circ + j\sin 45^\circ$$

$$\frac{1}{j} = \frac{j}{j^2} = -j$$

Another useful fact

$A\cos(\omega t+\theta)$

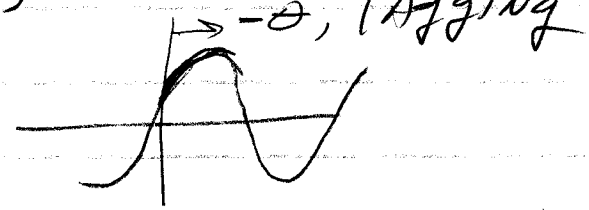




$\cos(\omega t) = \sin(\omega t+90)$

$\cos(\omega t-90) = \sin(\omega t)$

$\cos(\omega t+90) = -\sin(\omega t)$



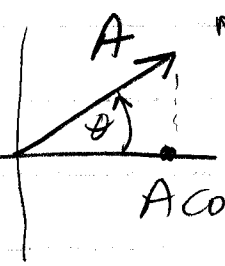
$$A\cos(\omega t+\theta) = A\left[\cos\omega t\cos\theta - \sin\omega t\sin\theta\right]$$

Euler's Rule $e^{j\theta} = \underbrace{\cos\theta}_{\text{Real}} + \underbrace{j\sin\theta}_{\text{Imag}}$

$$e^{j(\omega t+\theta)} = \underbrace{\cos(\omega t+\theta)}_{\text{Real}\{e^{j(\omega t+\theta)}\}} + j\sin(\omega t+\theta)$$

Phasor

Projection of Phasor on Real Axis is the instantaneous value

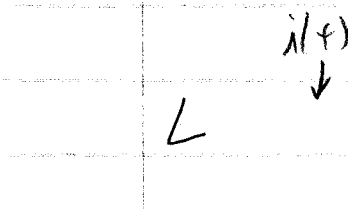


↑ω. Snapshot at $t=0$.

$A\cos(\omega t+\theta) = \mathrm{Re}\{Ae^{j(\omega t+\theta)}\}$

A Phasor is time varying

A Complex # is NOT



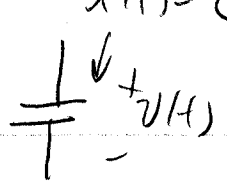
$v(t) = L\dfrac{di(t)}{dt}$

$\dfrac{di}{dt} = \dfrac{1}{L}v(t)$. If $v(t) = V\cos(\omega t+\theta)$

$i(t) = \dfrac{V}{\omega L}\sin(\omega t+\theta) + \text{const}$

$= \dfrac{V}{\omega L}\cos(\omega t+\theta-90^\circ)$ lagging by 90°

$$i(t) = C\frac{dv(t)}{dt}$$



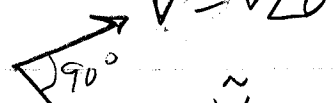
$$V\cos(\omega t+\theta) = v(t)$$

$$-C\omega V\sin(\omega t+\theta) = i(t)$$

$$= +C\omega V\cos(\omega t+\theta+90^\circ)$$ (leading by 90°

For L

$$\frac{\tilde{V}}{\tilde{I}}$$



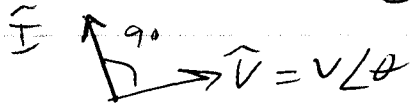
$$\tilde{V} = V\angle\theta$$

$$Z_L = j\omega L$$

$$\tilde{I} = \frac{\tilde{V}}{j\omega L} = \frac{V}{\omega L}\angle\theta - 90^\circ$$ lagging

FOR C

$$\tilde{I} = V(j\omega C) = V\omega C\angle\theta + 90^\circ$$



leading

$$Z_C = \frac{\tilde{V}}{\tilde{I}} = \frac{1}{j\omega C}$$

Adding or Subtracting Phasors

$$V_1 \cos(\omega t + \theta_1) + V_2 \cos(\omega t + \theta_2)$$

$$Re\left\{V_1 e^{j(\omega t+\theta_1)}\right\} + Re\left\{V_2 e^{j(\omega t+\theta_2)}\right\} = Re\left\{V_1 e^{j(\omega t+\theta_1)} + V_2 e^{j(\omega t+\theta_2)}\right\}$$

$$= Re\left\{e^{j\omega t}\underbrace{\left(V_1 e^{j\theta_1} + V_2 e^{j\theta_2}\right)}_{\tilde{V}_1 + \tilde{V}_2 = \tilde{V}_3}\right\}$$

$$V_1 \cos\theta_1 + jV_1 \sin\theta_1$$
$$V_2 \cos\theta_2 + jV_2 \sin\theta_2$$

$+$ Add Reals $=$ $V_3 \cos\theta_3$

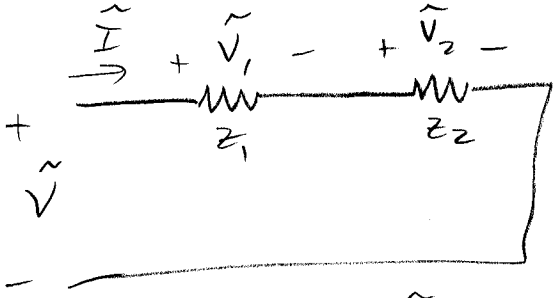
$+$ Add imaginaries $=$ $jV_3 \sin\theta_3$

$$= Re\left\{\tilde{V}_3 e^{j\omega t}\right\}$$

$$= Re\left\{V_3 e^{j\theta_3} e^{j\omega t}\right\} = Re\left\{V_3 e^{j(\omega t+\theta_3)}\right\} = V_3 \cos(\omega t + \theta_3)$$

Thus, KVL & KCL work in freq domain

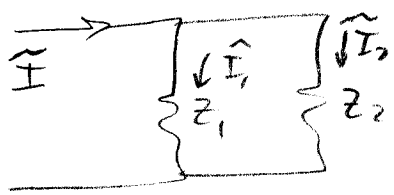
Series Impedances



$$-\tilde{V} + \tilde{V}_1 + \tilde{V}_2 = 0 \ , \quad -\tilde{V} + \hat{I} Z_1 + \hat{I} Z_2 = 0$$

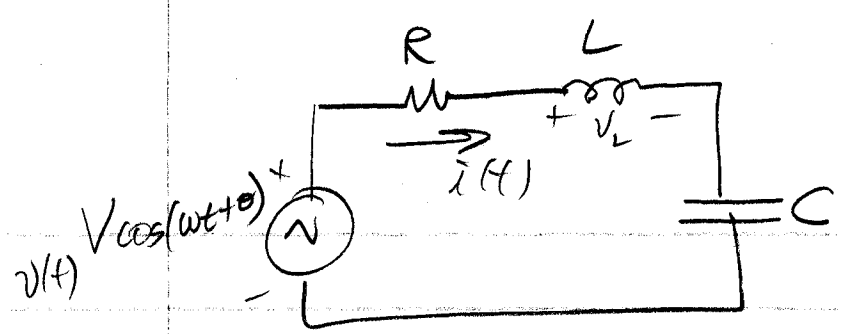
$$-V + \hat{I}\underbrace{(Z_1 + Z_2)}_{Z_{NET}} = 0$$

Parallel Impedances



$$\frac{\tilde{V}}{Z_1} + \frac{\hat{V}}{Z_2} = \frac{\hat{V}}{Z_{NET}}$$

$$\frac{1}{Z_{NET}} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

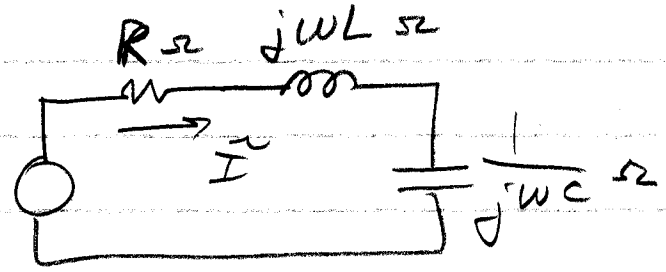


$$-v(t) + Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = 0$$

Find $i(t)$?

Phasors

$$\tilde{V} = V\angle\theta$$



$$-\tilde{V} + \left(R + j\omega L + \frac{1}{j\omega C}\right)\tilde{I} = 0$$

$$\tilde{I} = \frac{\tilde{V}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{V\angle\theta}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$V = 100\,V$, $f = 1\,kHz$, $\omega = 2\pi f = 6283\,rad/s$ (Fixed)

$\theta = 30°$

$R = 10\,\Omega$

$L = 1\,mH$

$C = 10\,\mu F$

$$\tilde{I} = \frac{100\angle 30°}{10 + j6283(0.001) + \frac{1}{j6283(10^{-5})}}$$

$$= \frac{100\angle 30}{10 + j6.283 - j15.916}$$

$$= \frac{100\angle 30}{10 - j9.633} = \frac{100\angle 30}{13.885\angle -43.9} = 7.202\angle +73.9\ A$$

$v(t) = 100\cos(6283t + 30^\circ)$ V

$i(t) = 7.202\cos(6283t + 73.9^\circ)$ A

$\tilde{V}_L = \tilde{I}(j\omega L) = (7.202\angle{+73.9})(j\,6.283)$

$= 45.25\angle{163.9}$

$v_L(t) = 45.25\cos(\omega t + 163.9^\circ)$

$i(t)$ lags $v_L(t)$ by 90°

$\tilde{V}_C = \tilde{I}\left(\frac{1}{j\omega C}\right) = (7.202\angle{+73.9})(-j15.917)$

$\tilde{V}_C = 114.6\angle{-16.1^\circ}$

$v_C(t) = 114.6\cos(\omega t - 16.1^\circ)$

$i(t)$ leads $v_C(t)$ by 90°

Note - can be bigger than source voltage!

Definitions

Definitions related to power. RMS values, active and reactive power, power factor, power flow in transmission lines. Review of per unit system and the advantages that it offers.

1. Single-Phase Definitions

The root-mean-squared (RMS) value of a periodic voltage (or current) waveform is

$$V_{RMS} = \sqrt{\frac{1}{T}\int_{t_o}^{t_o+T} v^2(t)dt}$$, where T is the period of $v(t)$.

If $v(t) = V\sin(\omega t + \delta)$, where V is the peak value, then using $\sin^2(A) = \frac{1-\cos(2A)}{2}$, V_{RMS} becomes

$$V_{RMS} = \frac{V}{\sqrt{2}} .$$

Instantaneous power flowing to a load, using the sign convention shown in Figure 1, is defined as

$$p(t) = v(t) \bullet i(t) .$$

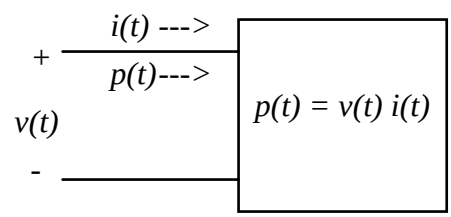


Figure 1. Instantaneous Power Flowing Into a Load

Average power flowing to a load is defined as

$$P = \frac{1}{T}\int_{t_o}^{t_o+T} p(t)dt$$, where T is the period of $p(t)$.

If $v(t) = V\sin(\omega t + \delta)$, $i(t) = I\sin(\omega t + \theta)$, then the instantaneous power becomes

$$p(t) = v(t) \bullet i(t) = \frac{VI}{2}\left[\cos(\delta - \theta) - \cos(2\omega t + \delta + \theta)\right] .$$

Note that $p(t)$ has double-frequency and time-invariant components. The average value of $p(t)$ is the time-invariant component, or

$$P = \frac{VI}{2}\cos(\delta - \theta) \ .$$

Since $V_{RMS} = \frac{V}{\sqrt{2}}$ and $I_{RMS} = \frac{I}{\sqrt{2}}$, then

$$P = V_{RMS} I_{RMS} \cos(\delta - \theta) \ ,$$

where $\cos(\delta - \theta)$ is known as the displacement power factor (*DPF*).

Reactive power Q is defined as

$$Q = V_{RMS} I_{RMS} \sin(\delta - \theta) \ .$$

Since $\sin^2(x) + \cos^2(x) = 1$, then

$$P^2 + Q^2 = V_{RMS}^2 \bullet I_{RMS}^2 \ .$$

Complex power S is defined as

$$S = P + jQ \ ,$$

so that

$$P = \mathrm{Real}\{S\} \ , \ Q = \mathrm{Imag}\{S\} \ .$$

The magnitude of S is

$$|S| = \sqrt{P^2 + Q^2} \ ,$$

which is identical to

$$|S| = V_{RMS} \bullet I_{RMS} \ .$$

Using voltage and current phasors $\tilde{V} = |V_{RMS}| \angle \delta$ and $\tilde{I} = |I_{RMS}| \angle \theta$, the product $\tilde{V}\tilde{I}^*$ is

$$|V_{RMS}| \bullet |I_{RMS}| \angle(\delta - \theta) = |V_{RMS}| \bullet |I_{RMS}| \cos(\delta - \theta) + j|V_{RMS}| \bullet |I_{RMS}| \sin(\delta - \theta) \ .$$

Therefore,

$$S = P + jQ = \tilde{V}\tilde{I}^* \ .$$

When $\tilde{I}$ lags $\tilde{V}$, Q is positive, and the power factor is lagging. When $\tilde{I}$ leads $\tilde{V}$, Q is negative, and the power factor is leading. Thus, an inductive load has a lagging power factor and absorbs Q, while a capacitive load has a leading power factor and produces Q.

The total power factor DPF is defined as

$$DPF = \frac{P}{S} \ .$$

For sinusoidal systems, total power factor is identical to displacement power factor defined previously as the cosine of the relative phase angle between voltage and current.

If P and DPF are given, Q can be calculated using

$$Q^2 = S^2 - P^2 = \frac{P^2}{DPF^2} - P^2 = P^2\left(\frac{1}{DPF^2} - 1\right) \text{, or}$$

$$Q = P\sqrt{\frac{1}{DPF^2} - 1} \ .$$

The relationships among P, Q, S, and pf are shown in the power factor triangle given in Figure 2.

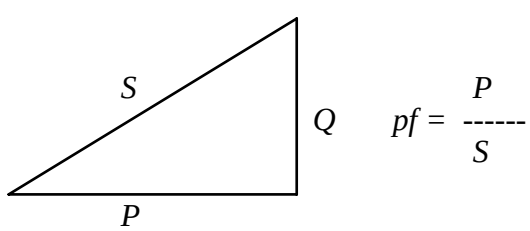


Figure 2. Power Factor Triangle

The impact of power factor on the Q/P ratio is given below in the Table 1.

DPF	Q / P
1.0	0.00
0.9	0.48
0.8	0.75
0.707	1.00
0.6	1.33

Table 1. Impact of Power Factor on Reactive Power *Q*

Now, consider the power flow through a purely inductive circuit element, such as a lossless transmission line or transformer shown in Figure 3.

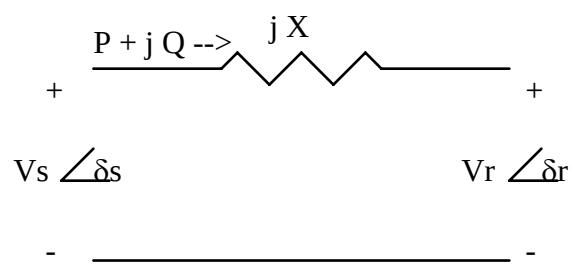


Figure 3. Power Flow Through a Purely Inductive Circuit Element

The active and reactive power flows, measured at the sending end S, can be shown to be

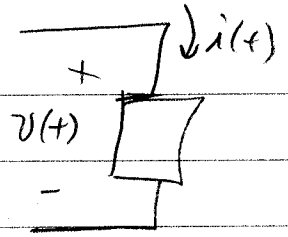
$$P = \frac{V_S V_R}{X}\sin(\theta_S - \theta_R)\,,\; Q = \frac{V_S}{X}\left[V_S - V_R\cos(\theta_S - \theta_R)\right].$$

Usually, $(\theta_S - \theta_R)$ is small, so that

$$P\,\alpha\,(\theta_S - \theta_R)\,,\; Q\,\alpha\,\frac{V_S}{X}\left[V_S - V_R\right].$$

Therefore, in inductive circuit elements, *P* tends to be proportional to voltage angle difference, and *Q* tends to be proportional to voltage magnitude difference.

Power - 1Φ



Instantaneous $p(t) = v(t)\,i(t)$

↑ Absorbed

Let $v(t) = V_p \cos(\omega t + \theta_V)$, $i(t) = I_p \cos(\omega t + \theta_I)$

$$p(t) = V_p I_p \cos(\omega t + \theta_V)\cos(\omega t + \theta_I)$$

$$= \frac{V_p I_p}{2}\left[\cos(\theta_V - \theta_I) + \cos(2\omega t + \theta_V + \theta_I)\right]$$

$$P_{AVG} = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)\,dt \qquad \underset{R/S\ \cdot s}{\omega T} = 2\pi$$

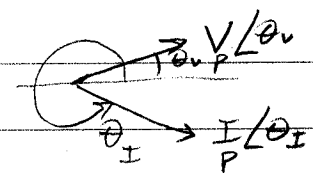
Avg value of first term $= \frac{V_p I_p}{2}\cos(\theta_V - \theta_I)$

Avg value of $2\omega t$ term $= 0$

$$P_{AVG} = \frac{V_p I_p}{2}\cos(\theta_V - \theta_I) = \left(\frac{V_p}{\sqrt{2}}\right)\cdot\left(\frac{I_p}{\sqrt{2}}\right)\cos(\theta_V - \theta_I)$$

Phasors $v(t) \Rightarrow \tilde{V} = V_p\angle\theta_V$, $i(t) \Rightarrow \tilde{I} = I_p\angle\theta_I$

$$v(t) = \mathrm{Re}\left\{\underbrace{V_p e^{j(\omega t + \theta_V)}}_{V_p\angle\theta_V}\right\} = \mathrm{Re}\left\{V_p\cos(\omega t + \theta_V) + j V_p \sin(\omega t + \theta_V)\right\}$$



$$V_p\angle\theta_V, \quad \left[I_p\angle\theta_I\right]^* = I_p\angle-\theta_I$$

$$(V_p\angle\theta_V)(I_p\angle-\theta_I) = V_p I_p\angle\theta_V - \theta_I$$

so $P_{AVG} = \mathrm{Re}\left\{\frac{V_p I_p}{2}\angle\theta_V - \theta_I\right\}$

↑ call this <u>Active</u> or <u>Real</u> power

What about $\mathrm{Im}\left\{\frac{V_p I_p}{2}\angle\theta_V - \theta_I\right\}$,

Call this <u>reactive power</u> or <u>imaginary power</u>

Now, the $\frac{V_p I_p}{2}$

$$V_{RMS}^2 = \frac{1}{T}\int_{t_0}^{t_0+T} v^2(t)\,dt. \quad \text{Use } v(t) = V_p\cos(\omega t + \theta_V)$$

~~Easier to use $V\cos\omega t$ (the θ_V can't make any difference here)~~

$$V_{RMS}^2 = \frac{1}{T}\int_0^T V^2\cos^2(\omega t+\theta_V)\,dt = \frac{V^2}{T}\int_0^T\left[\frac{1+\cos(2\omega t+2\theta_V)}{2}\right]dt$$

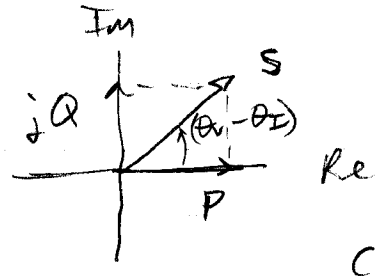
$$= \frac{V^2}{T}\left[\frac{t}{2} + \cancel{\frac{\sin(2\omega t+2\theta_V)}{4}}\right]_{t=0}^{T} = \frac{V^2 T}{T\cdot 2} = \frac{V^2}{2}$$

$$\boxed{V_{RMS} = \frac{V}{\sqrt{2}}}$$

So, $P = \mathrm{Re}\left\{V_{RMS}I_{RMS}\angle\theta_V - \theta_I\right\} = V_{RMS}I_{RMS}\cos(\theta_V - \theta_I)$ W

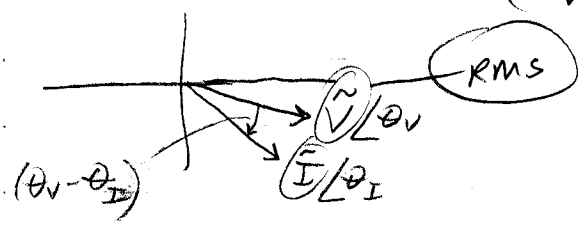
$Q = \mathrm{Im}\left\{V_{RMS}I_{RMS}\angle\theta_V - \theta_I\right\} = V_{RMS}I_{RMS}\sin(\theta_V - \theta_I)$ VAr

Define $S = (V_{RMS}\angle\theta_V)(I_{RMS}\angle\theta_I)^* = V_{RMS}I_{RMS}\angle\theta_V - \theta_I$ VA



Power Triangle

$\cos(\theta_V - \theta_I) \Rightarrow$ displacement power factor



power factor angle is the angle that $\tilde{I}$ lags $\tilde{V}$

R

$$\tilde{I} = \frac{\tilde{V}}{R} = \frac{V\angle\theta_v}{R}$$

In phase,

$$S = (V\angle\theta_v)\left(\frac{V\angle\theta_v}{R}\right)^*$$

$$= \frac{V^2\angle\theta_v-\theta_v}{R} = \frac{V^2}{R} \leftarrow P$$

$Q=0$

L

$$Z_L = j\omega L$$

$$\hat{I} = \frac{\tilde{V}}{j\omega L} = \frac{V\angle\theta_v}{\omega L\angle 90^\circ} = \frac{V}{\omega L}\angle\theta_v - 90^\circ$$

lags by 90

$$S = VI^* = (V\angle\theta_v)\left(\frac{V}{\omega L}\angle\theta_v-90^\circ\right)^*$$

$$= \frac{V^2}{\omega L}\angle\theta_v-\theta_v+90^\circ = \frac{jV^2}{\omega L} \Leftarrow P=0,\ Q=\frac{V^2}{\omega L}$$

(Absorbs Q)

$$Z_c = \frac{1}{j\omega C}$$

$$\hat{I} = \frac{\hat{V}}{\frac{1}{j\omega C}} = j\omega C\tilde{V} = V\omega C\angle\theta_v+90^\circ$$

$$S = VI^* = (V\angle\theta_v)(V\omega C\angle\theta_v+90^\circ)^* = V^2\omega C\angle -90^\circ$$

$$S = -jV^2\omega C \Leftarrow P=0,\ Q=-V^2\omega C$$

(produces Q)

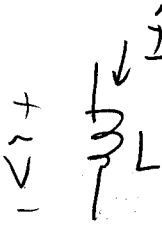
Static Load Model

$P = 10\,KW$, $V = 230V$
@ pf = 0.80, 60Hz
lagging

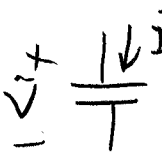
Find I, P, Q,
Equivalent series and parallel models

Power – 1d. Also, from a current perspective

$$S = V I^* = (\tilde{I} R)(\tilde{I})^* = (R I \angle\theta_I)(I \angle -\theta_I)$$
$$= I^2 R = P + jQ$$
$$P = I^2 R, \quad Q = 0$$



$$S = V I^* = (j\omega L I \angle\theta_I)(I \angle -\theta_I) = j\omega L I^2$$
$$P = 0, \quad Q = I^2 \omega L$$

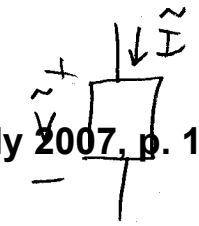


$$S = V I^* = \left(\frac{1}{j\omega C} I \angle\theta_I\right)(I \angle -\theta_I) = \frac{I^2}{j\omega C}$$
$$S = -j\frac{I^2}{\omega C}, \quad \text{so } P = 0, \; Q = \frac{-I^2}{\omega C}$$

Summary

	P (Absorbed)	Q (Absorbed)
R	$\frac{V^2}{R}$, $I^2 R$	0
L	0	$\frac{V^2}{\omega L}$, $I^2 \omega L$
C	0	$-V^2 \omega C$, $\frac{-I^2}{\omega C}$

V, I are RMS magnitudes of $\tilde{V}$, $\tilde{I}$



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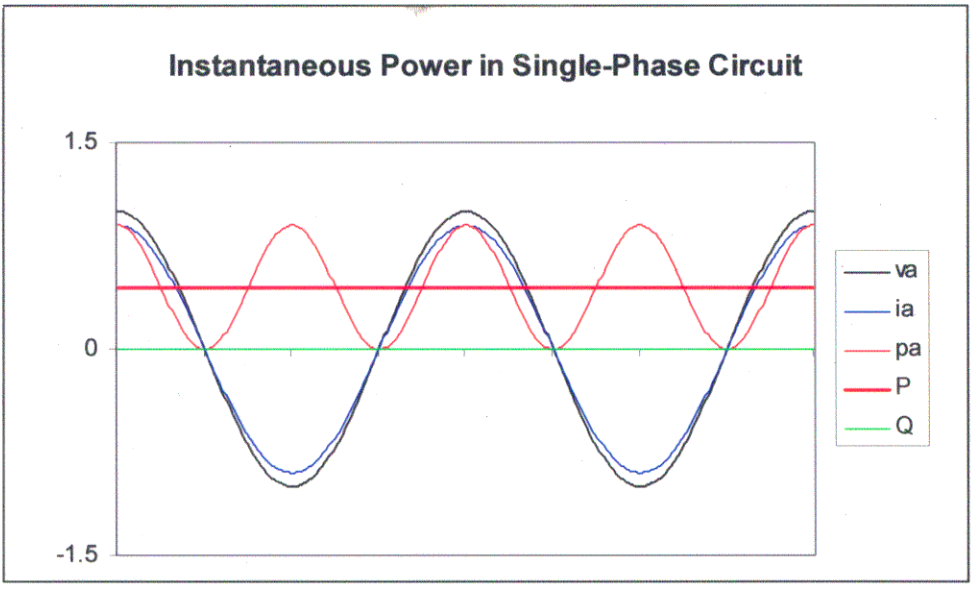

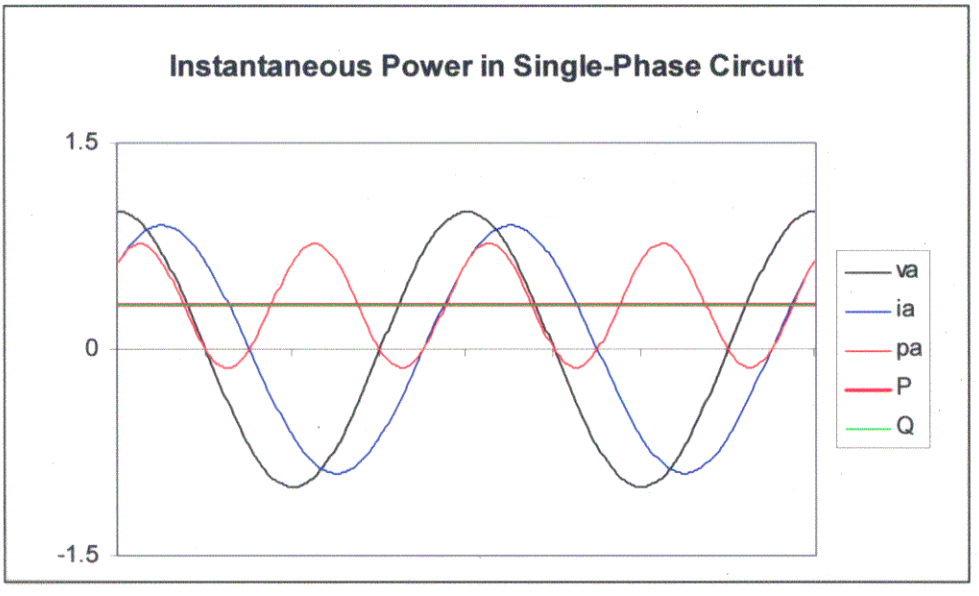

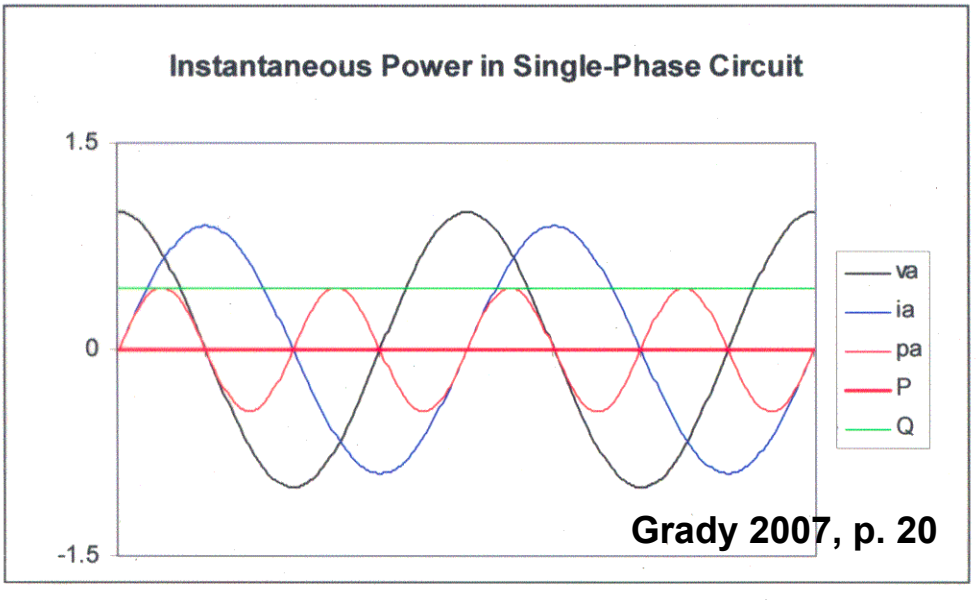

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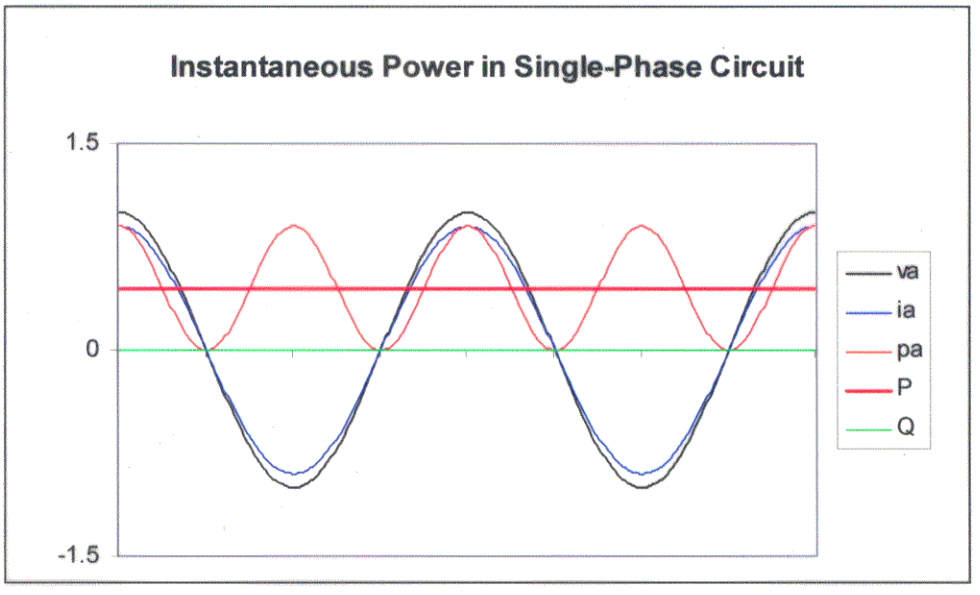

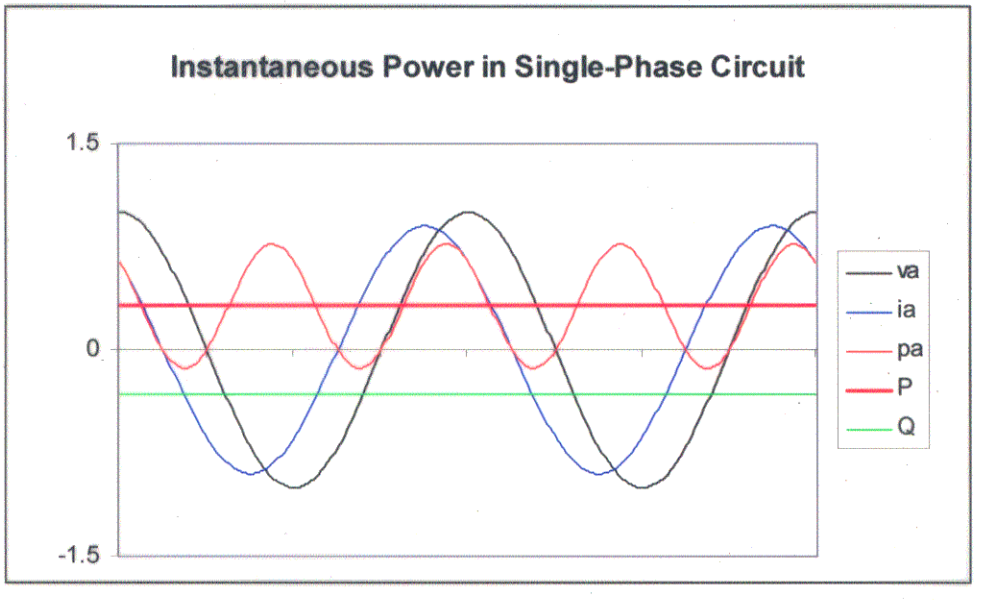

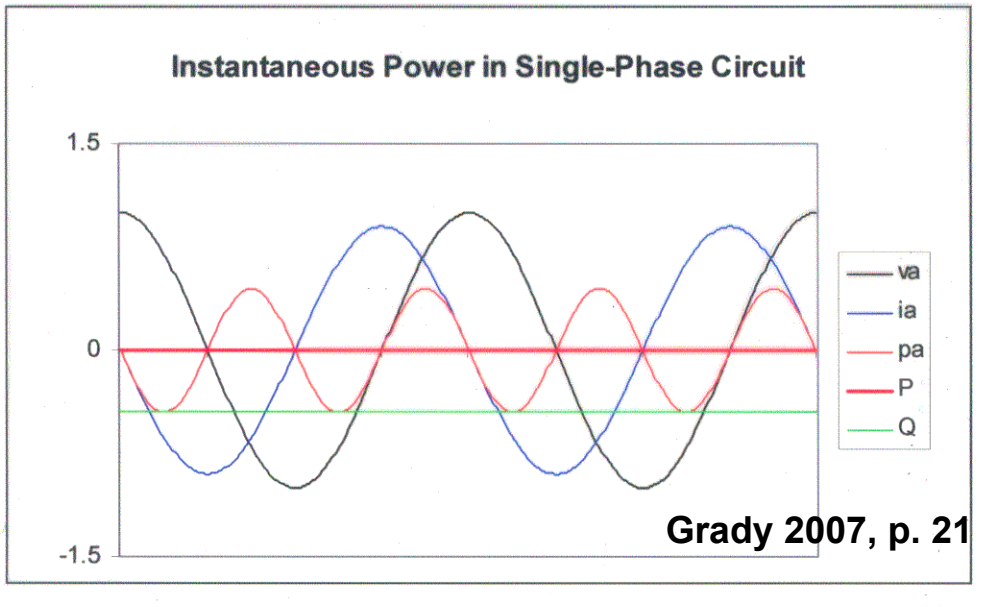

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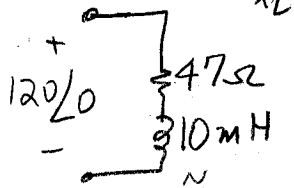
A load consists of a 47Ω resistor and 10mH inductor in series. The load is energized by a 120V, 60Hz voltage source. The phase angle of the voltage source is zero.

a. Determine the phasor current
b. Determine the load P, pf, Q, and S.
c. Find an expression for instantaneous p(t)

Problem 1. A load consists of a 47Ω resistor and 10mH inductor in series. The load is energized by a 120V, 60Hz voltage source. The phase angle of the voltage source is zero.

a. Determine the phasor current
b. Determine the load P, pf, Q, and S.
c. Find an expression for instantaneous p(t)

$$X_L = 2\pi f L = (2\pi)(60)(10\times10^{-3}) = 1200\pi\times10^{-3} = 3.77\,\Omega$$



$$Z_L = 47 + j3.77\,\Omega = 47.2\angle 4.59^\circ$$

a) $\tilde{I} = \dfrac{\tilde{V}}{Z_L} = \dfrac{120\angle 0}{47.2\angle 4.59}$

$$\boxed{\tilde{I} = 2.54\angle -4.59 \text{ A(RMS)}}$$

b) $S = VI^* = (120\angle 0)(2.54\angle -4.59)^* = \boxed{305\angle 4.59^\circ \text{ VA}}$

$P = Re\{S\} = 305\cos(4.59^\circ) = \boxed{304\text{W}}$

$Q = Im\{S\} = 305\sin(4.59^\circ) = \boxed{24.4\text{VAr}}$

$pf = \cos(4.59^\circ) = \boxed{0.997}$

Also, $P = |I|^2 R = (2.54)^2(47) = 303\text{W}$

$Q = |I|^2 X_L = (2.54)^2(3.77) = 24.3\text{VAr}$

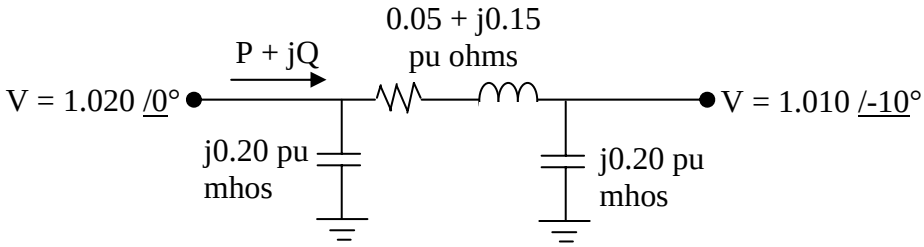
$|S| = 305$, $Q = 24.4$, $P = 304$

c) $p(t) = V_p\cos(\omega t + \theta_V) I_p \cos(\omega t + \theta_I) = \dfrac{V_p I_p}{2}\left[\cos(\theta_V - \theta_I) + \cos(2\omega t + \theta_V + \theta_I)\right]$

$p(t) = 305\left[0.997 + \cos(754t - 4.59^\circ)\right]$ (Note, $P = 305(0.997) = 304\text{W}$)

Problem 2. A three phase, 460V motor draws 5kW with a power factor of 0.80 lagging

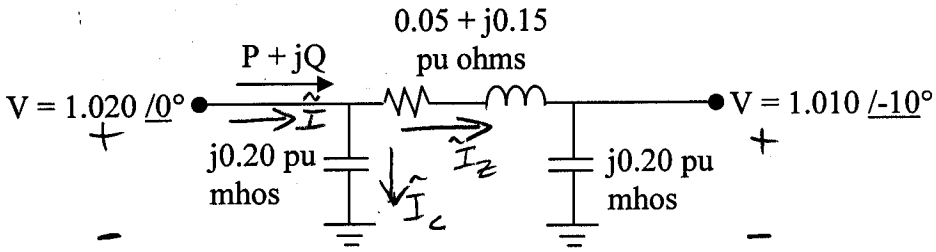
Calculate the P and Q flows (in per unit) for the loadflow situation shown below.



EE369, Final Exam, 2-5pm, Dec. 9, WRW102. Name: KEY

Six equally-weighted problems. Three sheets of notes permitted. Show all work.

Problem 5. Calculate the P and Q flows (in per unit) for the loadflow situation shown below.



$$P + jQ = V\tilde{I}^* = V(\hat{I}_c + \hat{I}_z)^*$$

$$\hat{I}_z = \frac{1.020\angle 0 - 1.010\angle -10^\circ}{0.05 + j0.15} = \frac{1.02 - (0.995 - j0.1754)}{0.1581\angle 71.6^\circ}$$

$$\hat{I}_z = \frac{0.0250 + j0.1754}{0.1581\angle 71.6} = \frac{0.1772\angle 81.9}{0.1581\angle 71.6} = 1.121\angle 10.3^\circ$$

$$\tilde{I}_z = 1.103 + j0.200$$

$$\hat{I}_c = \tilde{V}Y_c = (1.02\angle 0)(0.20\angle 90) = 0.204\angle 90^\circ = j0.204$$

$$\tilde{I} = \hat{I}_z + \hat{I}_c = 1.103 + j0.404 = 1.175\angle 20.1^\circ$$

$$P + jQ = (1.020\angle 0)(1.175\angle 20.1)^* = 1.199\angle -20.1$$

$$P + jQ = \underbrace{1.126}_{P} - j\underbrace{0.412}_{Q = -0.412}$$

Definitions

Node: A point or set of points at the same potential that have at least two branches connected to them.

Branch: A circuit element that connects nodes.

Major Node: A node with three or more branches connected to it.

Super Node: Two major nodes with an ideal voltage source between them.

Reference Node: The node to which all other node potentials are referenced. The relative voltage of the reference node is zero.

Solution Procedure

1. Draw a neat circuit diagram and try to eliminate as many branch crossings as possible.

2. Choose a reference node. All other node voltages will be referenced to it. Ideally, it should be the node with the most branches connected to it, so that the number of terms in the admittance matrix is minimal.

3. If the circuit contains voltage sources, do either of the following:

   - Convert them to current sources (if they have series impedances)
   - Create super nodes by encircling the corresponding end nodes of each voltage source.

4. Assign a number to every major node (except the reference node) that is not part of a super node (N1 of these).

5. Assign a number to either end (but not both ends) of every super node that does not touch the reference node (N2 of these)

6. Apply KCL to every numbered node from Step 4 (N1 equations)

7. Apply KCL to every numbered super node from Step 5 (N2 equations)

8. The dimension of the problem is now N1 + N2. Solve the set of linear equations for the node voltages. At this point, the circuit has been "solved."

9. Using your results, check KCL for at least one node to make sure that your currents sum to zero.

9. Use Ohm's Law, KCL, and the voltage divider principle to find other node voltages, branch currents, and powers as needed.

Notes on Mesh Analysis

Definitions

Branch: A circuit element that connects nodes

Planar Network: A network whose circuit diagram can be drawn on a plane in such a manner that no branches pass over or under other branches

Loop: A closed path

Mesh: A loop that is the only loop passing through at least one branch

Solution Procedure

1. Draw a neat circuit diagram and make sure that the circuit is planar (if not planar, then the circuit is not a candidate for mesh analysis)

2. If the circuit contains current sources, do either of the following:

   A. Convert them to voltage sources (if they have internal impedances), or
   B. Create super meshes by making sure in Step 3 that two (and not more than two) meshes pass through each current source. SM super meshes.

3. Draw clockwise mesh currents, where each one passes through at least one new branch. M meshes.

4. Apply KVL for every mesh that is not part of a super mesh (M – 2SM equations)

5. For meshes that form super meshes, apply KVL to the portion of the loop formed by the two meshes that does not pass through the current source (SM equations)

6. For each super mesh, write an equation that relates the corresponding mesh currents to the current source (SM equations)

7. The dimension of the problem is now M. Solve the set of M linear equations for the mesh currents. At this point, the network has been "solved."

8. Using your results, check KVL around at least one mesh to make sure that the net voltage drop is zero.

9. Use Ohm's law, loop currents, KVL, KCL and the voltage divider principle to find node voltages, branch currents, and powers as needed.

Notes on Thevenin Equivalents

The three cases to consider are

- **Case 1.** All sources are independent
- **Case 2.** The circuit has dependent sources and independent sources
- **Case 3.** The circuit has only dependent sources

Depending on the case, one or more of the following methods can be used to find the Thevenin equivalent:

Direct R_{th}. (Applies only to Case 1).

- Turn off all independent sources (i.e., set V = 0 for voltage sources, and I = 0 for current sources). Note - this is the same thing as replacing voltage sources with short circuits, and current sources with open circuits.
- Connect a fictitious ohmmeter across terminals a-b, and "measure" R_{th} directly.
- Find I_{sc} (or, alternatively, find $V_{oc} = V_{th}$).
- Compute $V_{th} = V_{oc} = I_{sc} \bullet R_{th}$ (or, alternatively, $I_{sc} = V_{oc} / R_{th}$).
- If time permits, find V_{th} (or, alternatively, I_{sc}) directly from the circuit, and then double-check with the above.

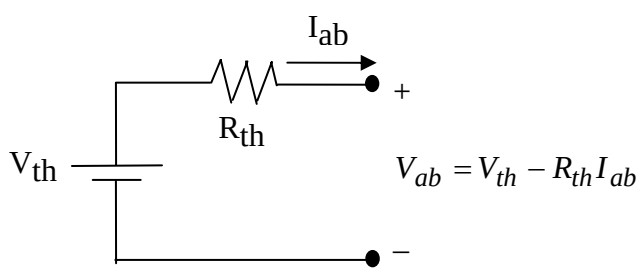
V_{oc} , I_{sc} (Applies to Cases 1 and 2).

- Find $V_{oc} = V_{th}$.
- Find I_{sc} .
- Compute $R_{th} = V_{th} / I_{sc}$.

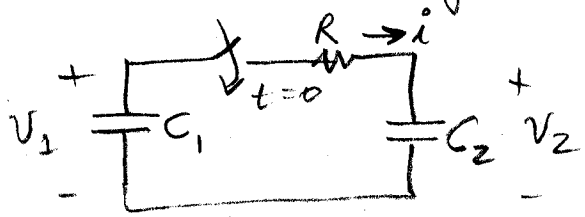
Fictitious Source (Applies to all Cases)

- Attach a fictitious source V_{ab} across terminals a-b. Find a linear equation with the following form: $V_{ab} = A - BI_{ab}$.
- By definition the linear equation must match Thevenin equation $V_{ab} = V_{th} - R_{th} I_{ab}$, term by term. Thus, matching the terms yields $V_{th} = A, R_{th} = B$.

Case	Direct R_{th}.	V_{oc} , I_{sc}	Fictitious Source
Case 1	**OK**	**OK**	**OK**
Case 2		**OK**	**OK**
Case 3			**OK**



Conservation of Charge Example



KVL: $-v_1 + Ri + v_2 = 0$

$$-\frac{1}{C_1}\int(-i)dt + Ri + \frac{1}{C_2}\int i\,dt = 0$$

$$\text{so } \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int i\,dt + Ri = 0$$

Let $\frac{1}{C_{TOT}} = \frac{1}{C_1} + \frac{1}{C_2}$, $\frac{1}{C_{TOT}}\int i\,dt + Ri = 0$,

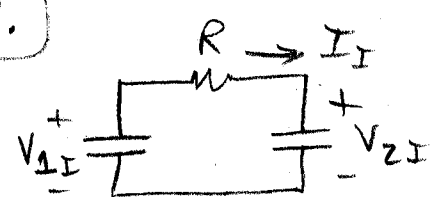
Now, $\frac{d}{dt} \Longrightarrow \frac{i}{C_{TOT}} + R\frac{di}{dt} = 0$, $\frac{di}{dt} = \frac{i}{RC_{TOT}}$,

Suggests $i = I_F + (I_I - I_F)e^{-t/\tau}$, $\tau = RC_{TOT}$

In this case, $I_F = 0$ because the stored energy is finite.

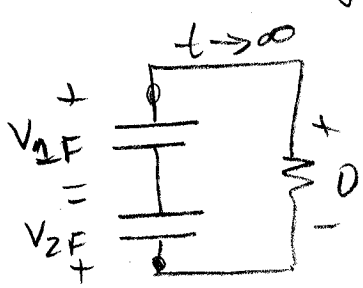
So, $\boxed{i = I_I e^{-t/\tau}, \quad t \geq 0.}$

The circuit at $t = 0^+$ is



so $\boxed{I_I = \frac{V_{1I} - V_{2I}}{R}}$, $V_{1I} = v_1(0^-)$, $V_{2I} = v_2(0^-)$

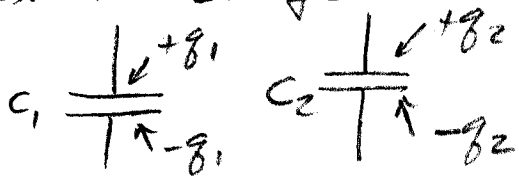
After a long time, when $i(t) \to 0$, then $\boxed{V_{1F} = V_{2F}}$



KVL $V_{2F} - V_{1F} = 0$, so $V_{1F} = V_{2F}$

But V_{1F} and V_{2F} are not zero in most cases!

Examine charge



$q_1 = C_1 v_1$

$q_2 = C_2 v_2$

$$q_1(t) = Q_{1I} - \int_{0^-}^{t} i\,dt$$

$$\oplus \quad q_2(t) = Q_{2I} + \int_{0^-}^{t} i\,dt$$

$$= q_1(t) + q_2(t) = Q_{1I} + Q_{2I} \quad \text{(Conservation of charge)}$$

So, after a long time, $\boxed{Q_{1F} + Q_{2F} = Q_{1I} + Q_{2I}}$

and $V_{1F} = V_{2F}$, so $C_1 V_{1F} + C_2 V_{2F} = Q_{1I} + Q_{2I}$

$$C_1 V_{1F} + C_2 V_{1F} = Q_{1I} + Q_{2I}$$

$$V_{1F} = \frac{Q_{1I} + Q_{2I}}{C_1 + C_2} = V_{2F}$$

Now, the stored energy diminishes because the resistor consumes power.

$P_R = i^2 R = I_I^2 e^{-2t/\tau} R$, so the energy absorbed as $t \to \infty$ is $W_R = \int_0^\infty p_R\,dt = I_I^2 R \int_0^\infty e^{-2t/\tau}\,dt$

$$W_R = I_I^2 R\left(\frac{-\tau}{2}\right) e^{-2t/\tau}\Big|_{t=0}^{\infty} = I_I^2 R\left(\frac{-\tau}{2}\right)(0-1) = I_I^2 R\frac{\tau}{2}$$

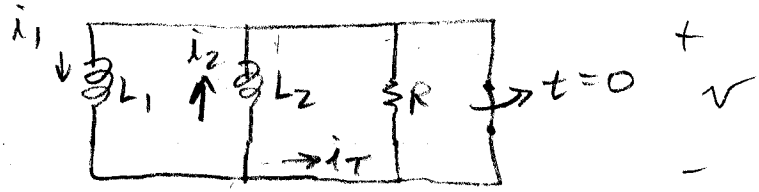
$$W_R = I_I^2 \frac{R^2 C_{TOT}}{2} = \left(\frac{V_{1I} - V_{2I}}{R}\right)^2 R^2 \frac{C_{TOT}}{2}$$

$$\boxed{W_R = \frac{(V_{1I} - V_{2I})^2}{2} C_{TOT}}.$$

Note - R is gone in the expression! So, it has nothing to do with R. As $R \to 0$, nothing changes.

Now, sanity check. If $V_{1I} = V_{2I}$, $W_R = 0$ because no current flows (ok). Also, if we took out C_2, then $W_R = \frac{1}{2} C_1 V_{1I}^2$ (ok). Not rigorous proof.

For Inductors – Conservation of flux linkages



KCL top, $\frac{1}{L_1}\int v\,dt + \frac{1}{L_2}\int v\,dt + \frac{v}{R} = 0, \quad t>0$

$$\frac{d}{dt} \longrightarrow \frac{v}{L_1} + \frac{v}{L_2} + \frac{1}{R}\frac{dv}{dt} = 0$$

$$\frac{dv}{dt} = -R\left(\frac{v}{L_1} + \frac{v}{L_2}\right) = -R\underbrace{\left(\frac{1}{L_1} + \frac{1}{L_2}\right)}_{\frac{1}{L_{TOT}} = \frac{1}{L_1} + \frac{1}{L_2}} v$$

$$\frac{dv}{dt} = -\frac{R}{L_{TOT}} v, \quad \text{so} \quad v = V_F + (V_I - V_F)e^{-t/\tau}$$

Also, $i_T = \cancel{I_{TF}}^{\,0} + (I_{TI} - \cancel{I_{TF}}^{\,0})e^{-t/\tau}, \quad \tau = \frac{L_{TOT}}{R_{eq}}$

$V_I = -R(I_{1I} - I_{2I}), \quad V_F = 0$ (finite stored energy)

So, $v = -R(I_{1I} - I_{2I})e^{-t/\tau}, \quad t \geq 0$

But I_{1F} & I_{2F} Not necessarily zero

$$P_R = \frac{v^2}{R} = \frac{R^2(I_{1I} - I_{2I})^2}{R} e^{-2t/\tau}$$

$$W_R = \int_0^\infty P_R\,dt = \frac{R^{\cancel{2}}(I_{1I} - I_{2I})^2}{\cancel{R}}\left(-\frac{\tau}{2}\right)e^{-2t/\tau}\Big|_{t=0}^{\infty}$$

long time

$$= R(I_{1I} - I_{2I})^2\left(\frac{-L_{TOT}}{2R}\right)\left[e^{-\infty} - 1\right]$$

$$\boxed{W_R = \frac{(I_{1I} - I_{2I})^2 (L_{TOT})}{2} \text{ lost in Resistor, independent of } R}$$

Check. Equal but opposite, $W_R = 0$, OK

If either coil missing, $\frac{1}{2}LI^2$, OK

$$i_1(t) = I_{1I} + \frac{1}{L_1}\int_{0^-}^{t} v\,dt \quad \leftarrow \text{mult. by } L_1$$

$$i_2(t) = I_{2I} - \frac{1}{L_2}\int_{0^-}^{t} v\,dt \quad \leftarrow \text{mult by } L_2$$

$$L_1 i_1(t) = L_1 I_{1I} + \int_{0^-}^{t} v\,dt$$

$$L_2 i_2(t) = L_2 I_{2I} - \int_{0^-}^{t} v\,dt$$

If $R \to \infty$, or if $t \gg \frac{L_{eq}}{R}$, $i_1(t) = i_2(t)$

define as $i(t)$

$$L_1 i(t) = L_1 I_{1I} + \int$$

$$\oplus \quad L_2 i(t) = L_2 I_{2I} - \int$$

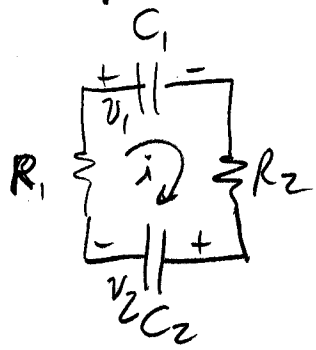
$$L_1 i(t) + L_2 i(t) = L_1 I_{1I} + L_2 I_{2I}$$

$$\boxed{L_1 I_{1F} + L_2 I_{2F} = L_1 I_{1I} + L_2 I_{2I}}$$

Conservation of flux linkages

$$\boxed{I_{1F} = \frac{L_1 I_{1I} + L_2 I_{2I}}{L_1 + L_2} = I_{2F}}$$

Multiple C's, L's



Series C's

KVL

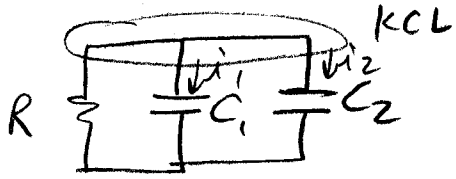
$$R_1 i + \frac{1}{C_1}\int i\,dt + R_2 i + \frac{1}{C_2}\int i\,dt = 0$$

$$(R_1+R_2)i + \left(\frac{1}{C_1}+\frac{1}{C_2}\right)\int i\,dt = 0$$

$$(R_1+R_2)\frac{di}{dt} + \left(\frac{1}{C_1}+\frac{1}{C_2}\right)i = 0$$

$$\frac{di}{dt} = \frac{1}{(R_1+R_2)}\cdot\left(\frac{1}{C_1}+\frac{1}{C_2}\right)i$$

$$\frac{di}{dt} = \frac{1}{R_{eq}C_{eq}}\,i \rightarrow R_{eq} = R_1+R_2, \quad \frac{1}{C_{eq}} = \frac{1}{C_1}+\frac{1}{C_2}$$



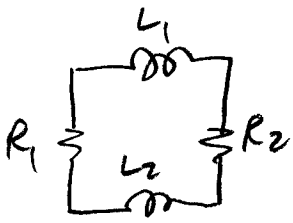
Parallel C's

KCL

$$\frac{v}{R} + C_1\frac{dv}{dt} + C_2\frac{dv}{dt} = 0$$

$$(C_1+C_2)\frac{dv}{dt} = -\frac{v}{R}$$

$$\frac{dv}{dt} = \frac{-1}{R(C_1+C_2)}\,v, \quad R_{eq} = R, \quad C_{eq} = C_1+C_2$$



Series L's

KVL

$$R_1 i + L_1\frac{di}{dt} + R_2 i + L_2\frac{di}{dt} = 0$$

$$(L_1+L_2)\frac{di}{dt} = -(R_1+R_2)i$$

$$\frac{di}{dt} = -\frac{(R_1+R_2)}{(L_1+L_2)}\,i, \quad R_{eq} = R_1+R_2, \quad L_{eq} = L_1+L_2$$

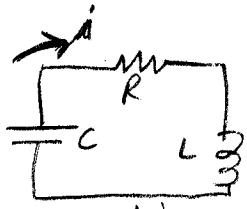
Parallel L's. $\frac{v}{R} + \frac{1}{L_1}\int v\,dt + \frac{1}{L_2}\int v\,dt = 0$

KCL $\frac{1}{R}\frac{dv}{dt} + \frac{v}{L_1} + \frac{v}{L_2} = 0, \quad \frac{dv}{dt} = -R\left(\frac{1}{L_1}+\frac{1}{L_2}\right)v$

$$\frac{1}{L_{eq}} = \frac{1}{L_1}+\frac{1}{L_2}$$

2nd Order Systems — Have two non-combinable storage elements. Classics — series R, L, C, and parallel R, L, C

Series (Source Free)



Start with known stored energy $\frac{1}{2}Cv_c^2$, $\frac{1}{2}Li^2$

$$-v_c + Ri + L\frac{di}{dt} = 0$$

$$-\frac{1}{C}\int(-i)dt + Ri + L\frac{di}{dt} = 0$$

$$\frac{1}{C}i + R\frac{di}{dt} + L\frac{d^2i}{dt^2} = 0, \quad \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

Try $i = Ae^{st}$, then $\frac{di}{dt} = Ase^{st}$, $\frac{d^2i}{dt^2} = As^2e^{st}$

so, $As^2e^{st} + \frac{R}{L}Ase^{st} + \frac{A}{LC}e^{st} = 0$

$$Ae^{st}\underbrace{\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right]}_{} = 0$$ the "characteristic equation"

In general, this must be 0 because $A \neq 0$ except in trivial case (like no initial stored energy)

Solving,

$$s = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Define $\alpha = \frac{R}{2L}$ (damping coef.), $\omega_0 = \frac{1}{\sqrt{LC}}$ (undamped resonant freq (r/s)), Re-write

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Case 1, $\alpha^2 > \omega_0^2$, so Negative Real Roots (Overdamped)

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$v(t)$, $i(t)$ above have the form (but A_1 & A_2 are not same for v & i)

$$A_1e^{s_1t} + A_2e^{s_2t}$$

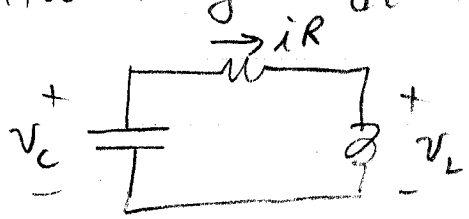
Negative, so don't $\rightarrow \infty$!

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$i(0^+) = A_1 + A_2$$

$$\boxed{\left.\frac{di}{dt}\right|_{0^+}} = \left.\left(s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}\right)\right|_{0^+} = \boxed{s_1 A_1 + s_2 A_2}$$

How to get $\frac{di}{dt}$? Use $v_L = L\frac{di}{dt}$



$$-v_c + iR + v_L = 0,$$

so $v_L = v_c - iR$

$$v_L\Big|_{t=0} = L\left.\frac{di}{dt}\right|_{t=0} = \left.(v_c - iR)\right|_{t=0}$$

Case 2. $\alpha^2 < \omega_0^2$, so $\sqrt{-\text{value}}$

$$s = -\alpha \pm \sqrt{-1(\omega_0^2 - \alpha^2)} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

Define $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ (damped resonant freq, i.e., $< \omega_0$)

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$$

$$= e^{-\alpha t}\left[A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right] \qquad \text{Eulers } e^{j\theta} = \cos\theta + j\sin\theta$$

$$= e^{-\alpha t}\left[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos(-\omega_d t) + j\sin(-\omega_d t))\right]$$

$$= e^{-\alpha t}\left[A_1\cos\omega_d t + jA_1\sin\omega_d t + A_2\cos\omega_d t - jA_2\sin\omega_d t\right]$$

$$i(t) = e^{-\alpha t}\left[\underbrace{(A_1 + A_2)}_{B_1}\overbrace{\cos\omega_d t}^{\text{vary with time}} + \underbrace{j(A_1 - A_2)}_{B_2}\overbrace{\sin\omega_d t}^{\text{vary with time}}\right]$$

Now, since $i(t)$ is real, then $\overset{B_1}{(A_1+A_2)}$ must be real, and $\underset{B_2}{j(A_1-A_2)}$ must be real

$$\left.\begin{aligned} A_1 &= A_{1R} + jA_{1I} \\ A_2 &= A_{2R} + jA_{2I} \end{aligned}\right] \quad \overset{B_1}{A_1 + A_2} = (A_{1R}+A_{2R}) + j(A_{1I}+A_{2I})$$

$$\underset{B_2}{j(A_1 - A_2)} = j(A_{1R}-A_{2R}) - (A_{1I}-A_{2I})$$

The only possibility is for $A_{1R} = A_{2R}$, $A_{1I} = -A_{2I}$

OR, $A_1 = A_2^*$, so $B_1 = A_{1R} + A_{2R}$, ← Both

$B_2 = -A_{I1} + A_{2I}$ ← Real

So, $i(t) = e^{-\alpha t}\left[B_1 \cos\omega_d t + B_2 \sin\omega_d t\right]$ Real

$\boxed{i(0) = B_1}$, $\dfrac{di}{dt} = -\alpha e^{-\alpha t}\left[B_1 \cos\omega_d t + B_2 \sin\omega_d t\right]$

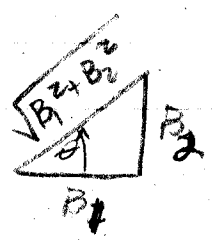
$+ e^{-\alpha t}\left[-\omega_d B_1 \sin\omega_d t + \omega_d B_2 \cos\omega_d t\right]$

$\boxed{\left.\dfrac{di}{dt}\right|_0 = -\alpha B_1 + \omega_d B_2}$ Get B_1, B_2

Handy Relation

$e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$

$$e^{-\alpha t}\sqrt{B_1^2+B_2^2}\left[\underbrace{\frac{B_1}{\sqrt{B_1^2+B_2^2}}}_{\cos\theta}\cos\omega_d t + \underbrace{\frac{B_2}{\sqrt{B_1^2+B_2^2}}}_{\sin\theta}\sin\omega_d t\right]$$



$\sqrt{B_1^2+B_2^2}\, e^{-\alpha t}\left[\cos(\omega_d t - \theta)\right]$, Use $A\cos()$ to get θ on calc. Check quadrant using signs of B_1, B_2. If calc. value wrong use $-\theta$ instead

$B_1 -$, $B_2 +$	$B_1 +$, $B_2 +$
$B_1 -$, $B_2 -$	$B_1 +$, $B_2 -$

quadrant

Case 3, $\alpha^2 = \omega_0^2$, critically damped

$$s = -\alpha \pm \sqrt{0}$$

so $i(t) = Ae^{st} = Ae^{-\alpha t}$, $\alpha = \frac{R}{2L}$

But this doesn't permit $i(0)$ to be zero unless $A = 0$! Not good enough

Try $i(t) = A_1 t e^{-\alpha t} + A_2 e^{-\alpha t}$ (old trick)

$$\frac{di}{dt} = A_1 e^{-\alpha t} - A_1 \alpha t e^{-\alpha t} - A_2 \alpha e^{-\alpha t} = e^{-\alpha t}\left[A_1 - A_1 \alpha t - A_2 \alpha\right]$$

$$\frac{d^2 i}{dt^2} = -\alpha A_1 e^{-\alpha t} - A_1 \alpha\left[e^{-\alpha t} - \alpha t e^{-\alpha t}\right] + A_2 \alpha^2 e^{-\alpha t}$$

$$= e^{-\alpha t}\left[-\alpha A_1 - A_1 \alpha + A_1 \alpha^2 t + A_2 \alpha^2\right]$$

Sub in

$$e^{-\alpha t}\left[-\alpha A_1 - A_1 \alpha + A_1 \alpha^2 t + A_2 \alpha^2\right]$$

$$+ e^{-\alpha t}\left(\frac{R}{L}\right)\left[A_1 - A_1 \alpha t - A_2 \alpha\right]$$

$$+ e^{-\alpha t}\left(\frac{1}{LC}\right)\left[A_1 t + A_2\right] = 0 \ ?$$

Gather t terms, and non-t terms, set $\stackrel{?}{=} 0$

$$t\left(A_1 \alpha^2 - \frac{A_1 R}{L}\alpha + \frac{A_1}{LC}\right) + \left(-2\alpha A_1 + A_2 \alpha^2 + A_1 \frac{R}{L} - A_2 \alpha \frac{R}{L} + \frac{A_2}{LC}\right) \stackrel{?}{=} 0$$

$$tA_1\left(\alpha^2 - \frac{R}{L}\alpha + \frac{1}{LC}\right) +$$

$$tA_1\left(\frac{R^2}{4L^2} - \frac{R^2}{2L^2} + \frac{1}{LC}\right)$$

$$tA_1\underbrace{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}_{\text{yes}} = 0 \ ?$$

$$\left(-2\left(\frac{R}{2L}\right)A_1 + A_2\frac{R^2}{4L^2} + A_1\frac{R}{L} - A_2\left(\frac{R^2}{2L^2}\right) + \frac{A_2}{LC}\right) = 0 \ ?$$

$$A_2\left(\frac{R^2}{4L^2} - \frac{R^2}{2L^2} + \frac{1}{LC}\right) = 0 \text{ (yes)}$$

So, $i(t) = (A_1 t + A_2)e^{-\alpha t}$ OK for critically damped

$i(0) = A_2$

$$\left.\frac{di}{dt}\right|_0 = e^{-\alpha t}\left[A_1 - A_1\alpha t - A_2\alpha\right]\Big|_0 = A_1 - A_2\alpha$$

What happens to the $te^{-\alpha t}$ as $t \to \infty$?

Use series expansion $e^{\alpha t} = 1 + \frac{(\alpha t)}{1!} + \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} \cdots$

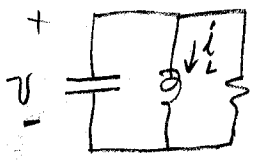
$$te^{-\alpha t} = \frac{t}{e^{\alpha t}} = \frac{t}{1 + \alpha t + \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} + \cdots}$$

$$\frac{0}{0}\ t \quad \frac{1}{\cancel{\frac{1}{t}}^{\,0} + \alpha + \frac{\alpha^2 t}{2!} + \frac{\alpha^3 t^2}{3!}}$$

function goes to zero as $t \to \infty$

$t \to \infty$

Parallel RLC
(Source Free)



KCL top $C\frac{dv}{dt} + \frac{1}{L}\int v\,dt + \frac{v}{R} = 0$

$$C\frac{d^2v}{dt^2} + \frac{v}{L} + \frac{1}{R}\frac{dv}{dt} = 0$$

$$\frac{d^2v}{dt} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

Try $v = A_1 e^{st}$, so $Ae^{st}\left[s^2 + \frac{s}{RC} + \frac{1}{LC}\right] = 0$

$$s = \frac{-\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}{2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$, like before except $\boxed{\alpha = \frac{1}{2RC}}$

All other developments follow

Case 1, overdamped, $\alpha^2 > \omega_0^2$

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}, \quad \boxed{v(0) = A_1 + A_2}$$

$$\frac{dv}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}, \quad \boxed{\left.\frac{dv}{dt}\right|_{0^+} = A_1 s_1 + A_2 s_2}$$

but $i_C(0^+) = C\left.\frac{dv}{dt}\right|_{0^+}$, so $\boxed{\left.\frac{dv}{dt}\right|_{0^+} = \frac{i_C(0^+)}{C}}$

AND SO ON.

Now, for series & parallel RLC with DC source, add the final solution to the (above) natural solution (which decays to zero)

$$v_{TOT} = V_F + \underbrace{v_{NAT}(t)}_{\text{solved above}}, \text{ AND SO ON.}$$

Three Properties of Balanced Three-Phase Systems

First, some trigonometry identities:

cos(A – B) = cos(A)cos(B) + sin(A)sin(B)

cos(A + B) = cos(A)cos(B) – sin(A)sin(B)

sin(A – B) = sin(A)cos(B) – cos(A)sin(B)

sin(A + B) = sin(A)cos(B) + cos(A)sin(B)

1. **Because they form a balanced set, the a-b-c currents sum to zero. Thus, there is no return current through the neutral or ground, which reduces wiring losses.**

$$i_a(t) = I\cos(\omega t + \theta),\ i_b(t) = I\cos(\omega t + \theta - 120°),\ i_c(t) = I\cos(\omega t + \theta + 120°),$$

$$i_N(t) = i_a(t) + i_b(t) + i_c(t) = I\left[\cos(\omega t + \theta) + \cos(\omega t + \theta - 120°) + \cos(\omega t + \theta + 120°)\right],$$

$$\begin{aligned} i_N(t) = \frac{I}{2}\big[&\cos(\omega t)\cos(\theta) - \sin(\omega t)\sin(\theta) \\ &+ \cos(\omega t)\cos(\theta - 120°) - \sin(\omega t)\sin(\theta - 120°) \\ &+ \cos(\omega t)\cos(\theta + 120°) - \sin(\omega t)\sin(\theta + 120°)\big], \end{aligned}$$

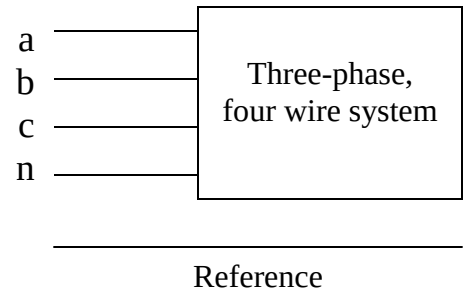
$$\begin{aligned} i_N(t) = \frac{I}{2}\big[&\cos(\omega t)\cos(\theta) - \sin(\omega t)\sin(\theta) \\ &+ \cos(\omega t)\left[\cos(\theta)\cos(120°) + \sin(\theta)\sin(120°)\right] - \sin(\omega t)\left[\sin(\theta)\cos(120°) - \cos(\theta)\sin(120°)\right] \\ &+ \cos(\omega t)\left[\cos(\theta)\cos(120°) - \sin(\theta)\sin(120°)\right] - \sin(\omega t)\left[\sin(\theta)\cos(120°) + \cos(\theta)\sin(120°)\right]\big], \end{aligned}$$

$$\begin{aligned} i_N(t) = &\frac{I}{2}\cos(\omega t)\left\{\cos(\theta)\left[1 + \cos(120°) + \cos(120°)\right] + \sin(\theta)\left[\sin(120°) - \sin(120°)\right]\right\} \\ &- \frac{I}{2}\sin(\omega t)\left\{\sin(\theta)\left[1 + \cos(120°) + \cos(120°)\right] - \cos(\theta)\left[\sin(120°) - \sin(120°)\right]\right\}, \end{aligned}$$

$$\begin{aligned} i_N(t) = &\frac{I}{2}\cos(\omega t)\left\{\cos(\theta)\left[1 - \frac{1}{2} - \frac{1}{2}\right] + \sin(\theta)\left[\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right]\right\} \\ &- \frac{I}{2}\sin(\omega t)\left\{\sin(\theta)\left[1 - \frac{1}{2} - \frac{1}{2}\right] - \cos(\theta)\left[\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right]\right\}. \end{aligned}$$

Thus $i_N(t) = 0$.

2. A N-wire system needs (N – 1) meters. A three-phase, four-wire system needs three meters. A three-phase, three-wire system needs only two meters.



$$p_{tot}(t) = p_a(t) + p_b(t) + p_c(t),$$

$$= v_{a,ref}(t)i_a(t) + v_{b,ref}(t)i_b(t) + v_{c,ref}(t)i_c(t) + v_{n,ref}(t)i_n(t),$$

$$i_n(t) = -i_a(t) - i_b(t) - i_c(t),$$

$$p_{tot}(t) = v_{a,ref}(t)i_a(t) + v_{b,ref}(t)i_b(t) + v_{c,ref}(t)i_c(t)$$

$$+ v_{n,ref}(t)\big(-i_a(t) - i_b(t) - i_c(t)\big),$$

$$p_{tot}(t) = \big(v_{a,ref}(t) - v_{n,ref}(t)\big)i_a(t) + \big(v_{b,ref}(t) - v_{n,ref}(t)\big)i_b(t) + \big(v_{c,ref}(t) - v_{n,ref}(t)\big)i_c(t).$$

Thus, for three-phase, four-wire, three wattmeters are needed to compute

$$p_{tot}(t) = v_{an}(t)i_a(t) + v_{bn}(t)i_b(t) + v_{cn}(t)i_c(t).$$

For three-phase, three-wire, the neutral wire is not present. Thus,

$$i_c(t) = -i_a(t) - i_b(t),$$

and letting phase c become the reference, then

$$p_{tot}(t) = v_{an}(t)i_a(t) + v_{bn}(t)i_b(t) + v_{cn}(t)\big(-i_a(t) - i_b(t)\big),$$

$$p_{tot}(t) = \big(v_{an}(t) - v_{cn}(t)\big)i_a(t) + \big(v_{bn}(t) - v_{cn}(t)\big)i_b(t).$$

Thus, for three-phase, three-wire, two wattmeters are needed to compute

$$p_{tot}(t) = v_{ac}(t)i_a(t) + v_{bc}(t)i_b(t).$$

More trigonometry identities:

$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

3. The instantaneous power is constant

$v_a(t) = V\cos(\omega t + \delta)$, $v_b(t) = V\cos(\omega t + \delta - 120°)$, $v_c(t) = V\cos(\omega t + \delta + 120°)$,

$$p_{tot}(t) = p_a(t) + p_b(t) + p_c(t),$$

$$= VI\left[\cos(\omega t + \delta)\cos(\omega t + \theta) + \cos(\omega t + \delta - 120°)\cos(\omega t + \theta - 120°)\right.$$

$$\left. + \cos(\omega t + \delta + 120°)\cos(\omega t + \theta + 120°)\right],$$

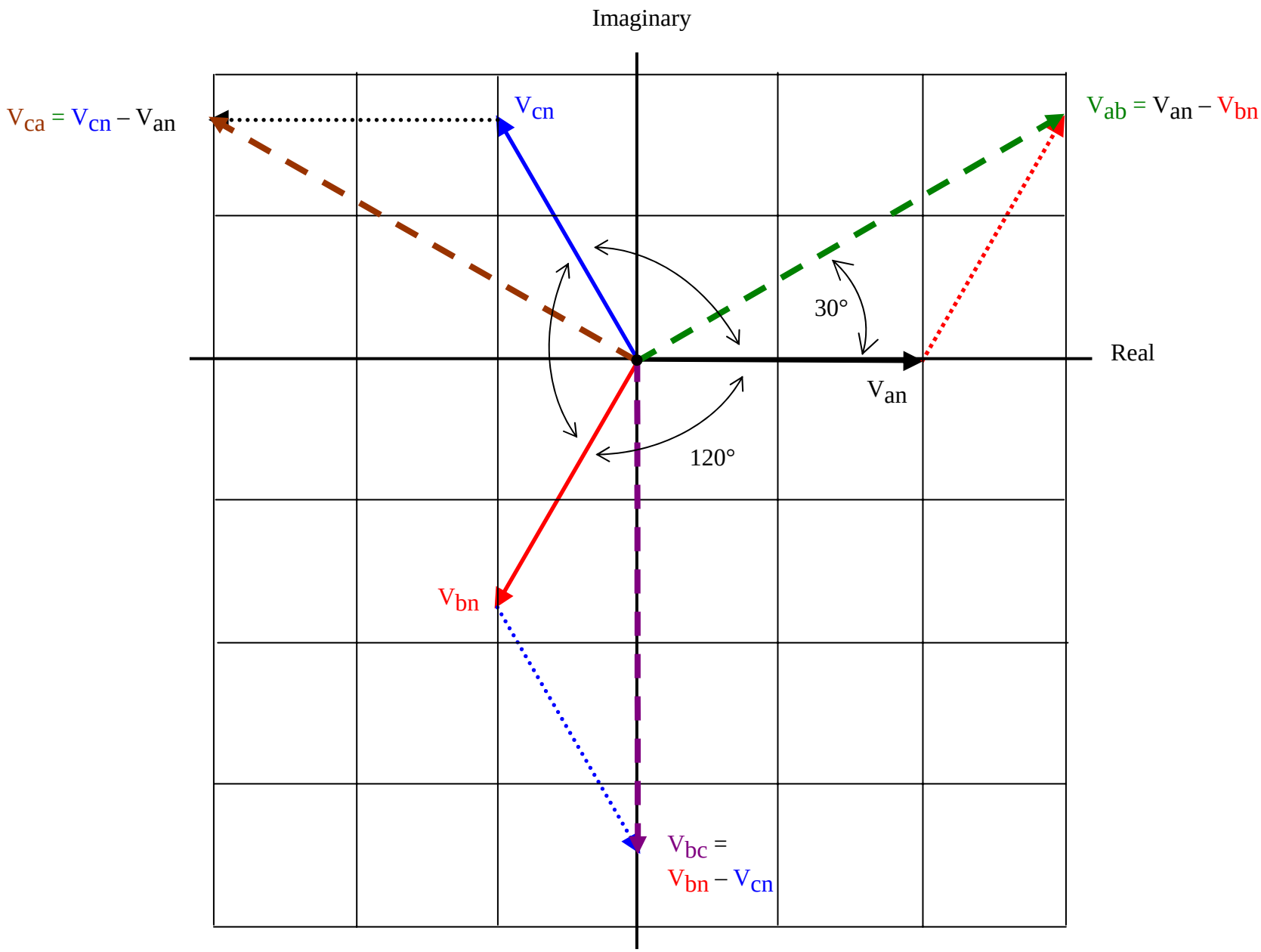
$$= \frac{VI}{2}\left[\cos(\delta - \theta) + \cos(2\omega t + \delta + \theta) + \cos(\delta - \theta) + \cos(2\omega t + \delta + \theta - 240°)\right.$$

$$\left. + \cos(\delta - \theta) + \cos(2\omega t + \delta + \theta + 240°)\right],$$

$$= \frac{3VI}{2}\cos(\delta - \theta) + \frac{VI}{2}\left[\cos(2\omega t + \delta + \theta) + \cos(2\omega t + \delta + \theta + 120°) + \cos(2\omega t + \delta + \theta - 120°)\right].$$

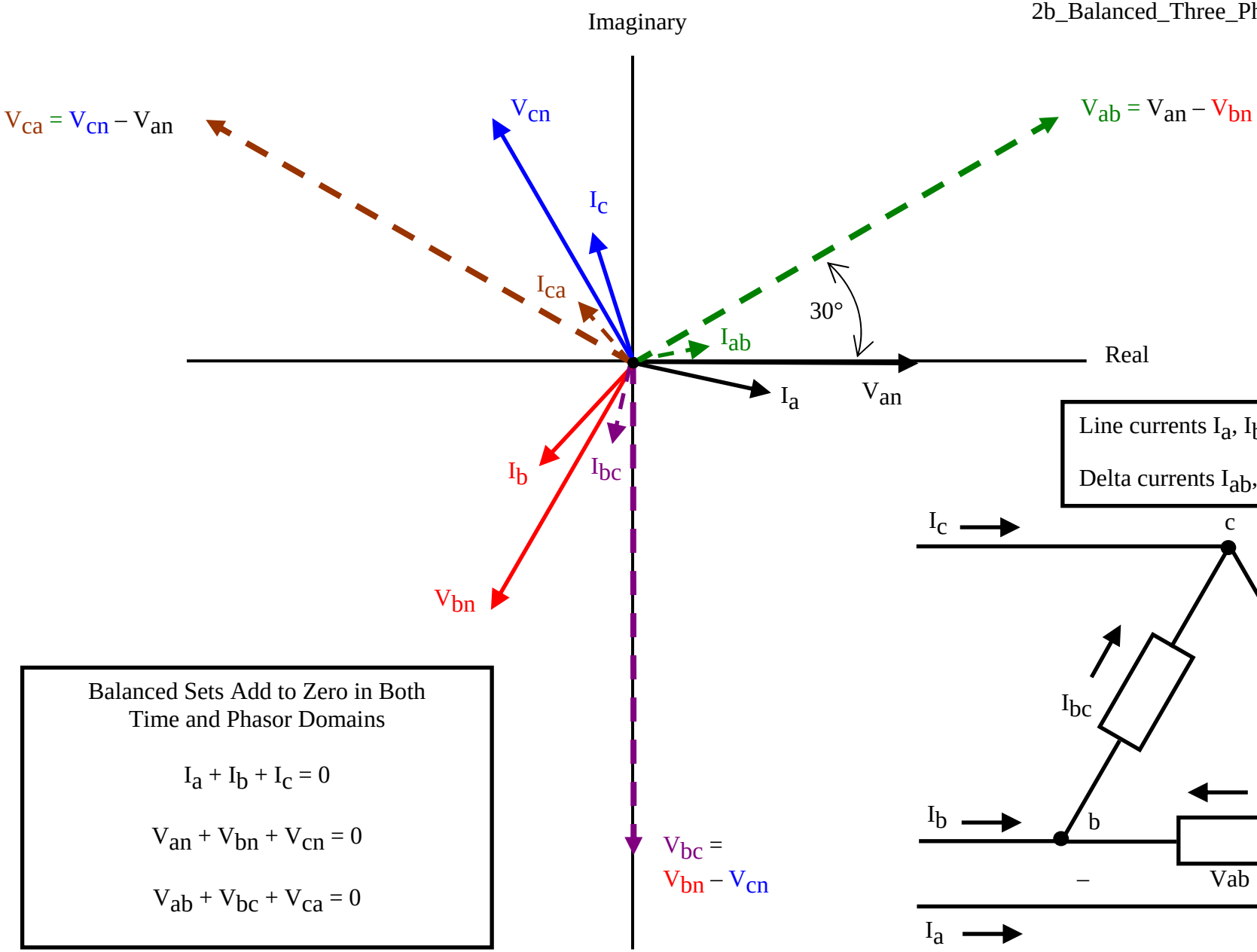
The terms in side the bracket form a balanced set, and thus equal zero. Thus, $p_{tot}(t)$ is a constant and equals its average value, which is

$$p_{tot}(t) = p_a(t) + p_b(t) + p_c(t) = \frac{3VI}{2}\cos(\delta - \theta) = 3V_{rms}I_{rms}\cos(\delta - \theta).$$



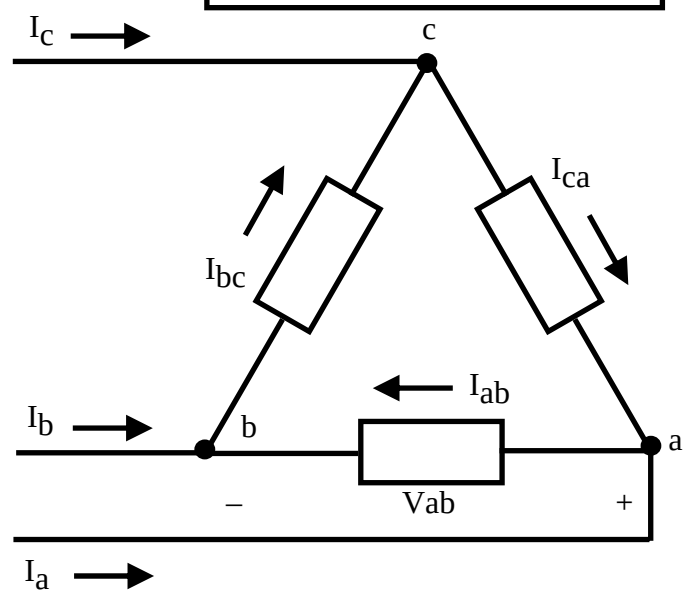
The phasors are rotating counter-clockwise.

The magnitude of line-to-line voltage phasors is $\sqrt{3}$ times the magnitude of line-to-neutral voltage phasors.



Line currents I_a, I_b, and I_c

Delta currents I_{ab}, I_{bc}, and I_{ca}



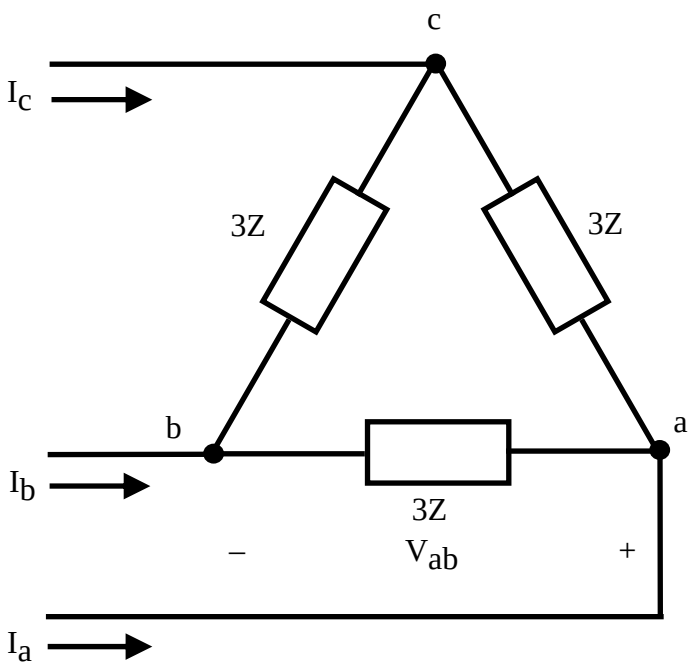
Balanced Sets Add to Zero in Both Time and Phasor Domains

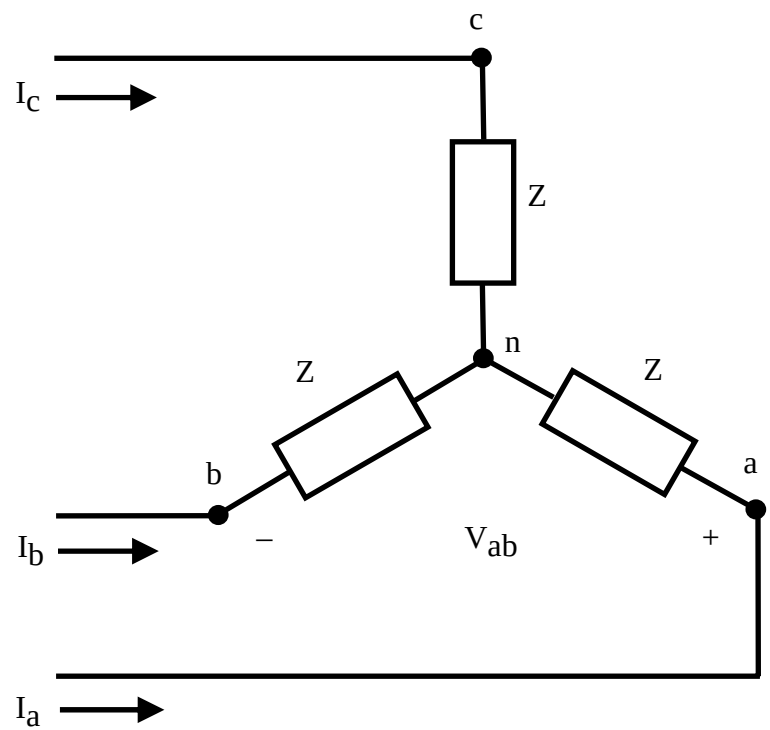
$$I_a + I_b + I_c = 0$$

$$V_{an} + V_{bn} + V_{cn} = 0$$

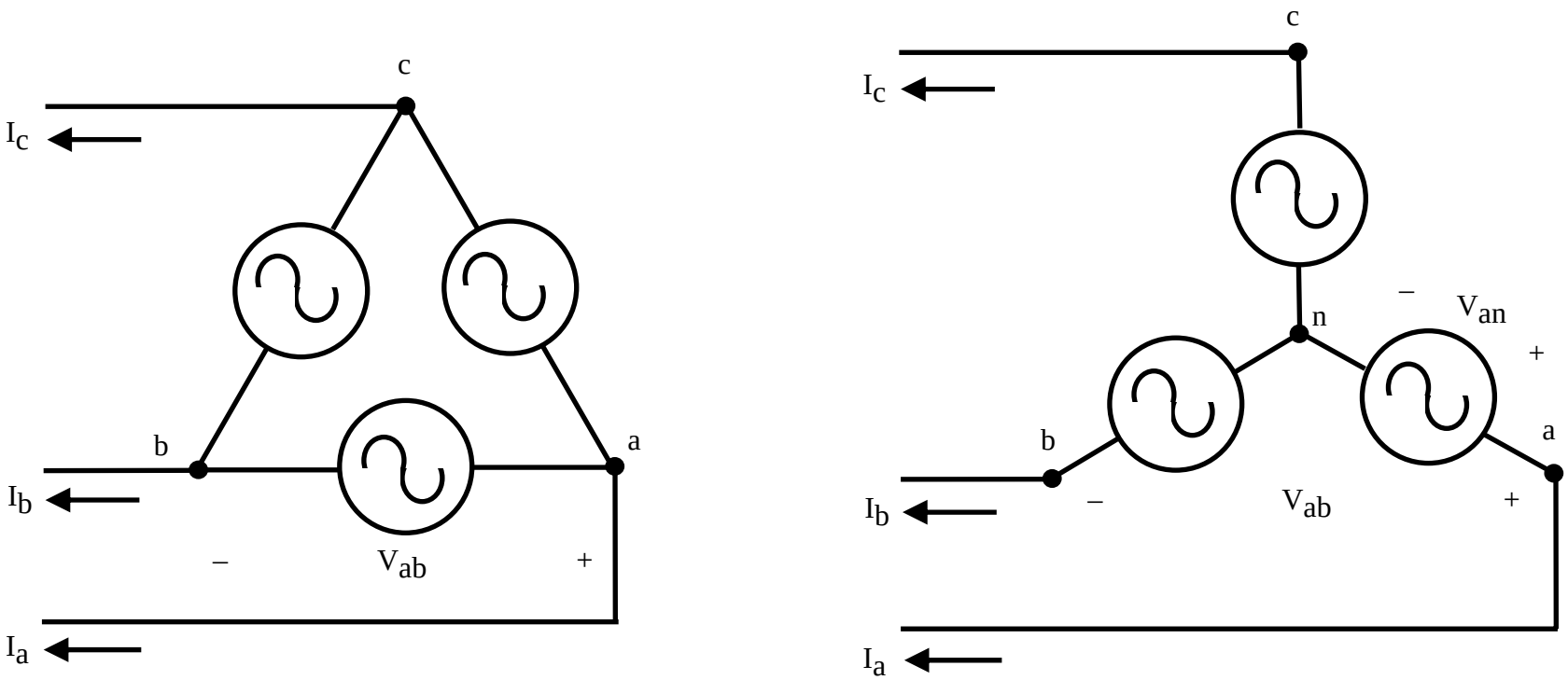
$$V_{ab} + V_{bc} + V_{ca} = 0$$

Conservation of power requires that the magnitudes of delta currents I_{ab}, I_{ca}, and I_{bc} are $\frac{1}{\sqrt{3}}$ times the magnitude of line currents I_a, I_b, I_c.

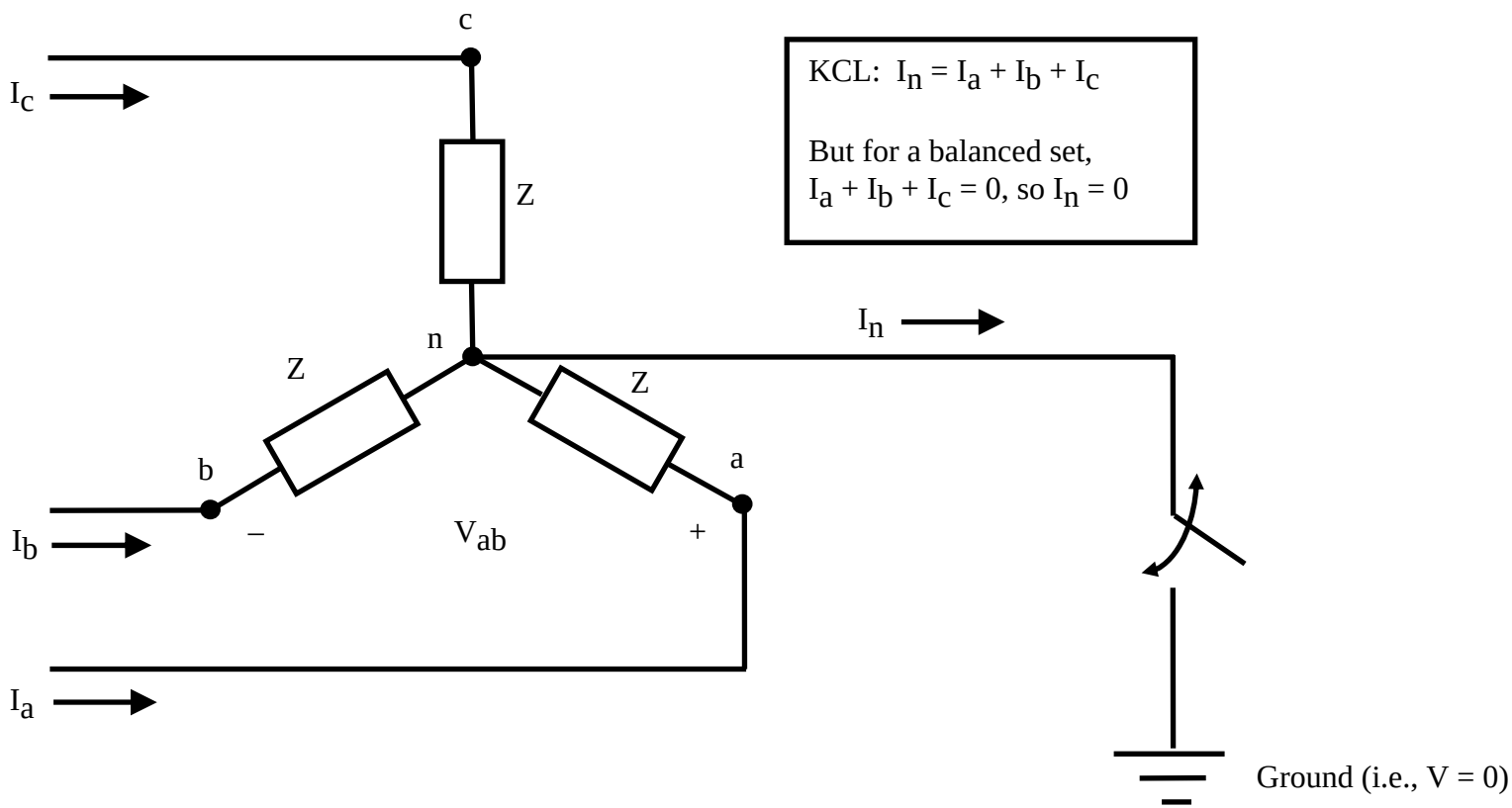




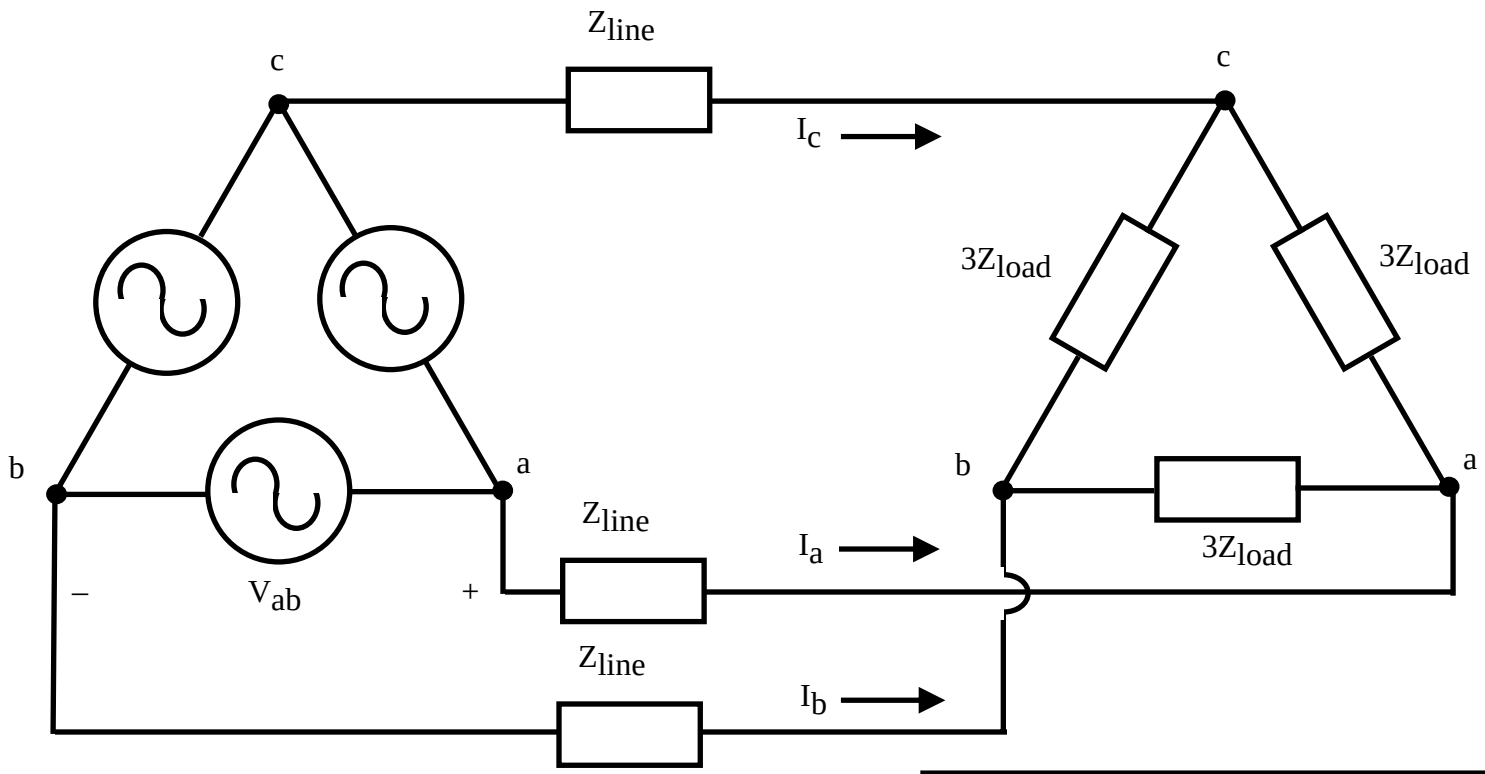
The Two Above Loads are Equivalent in Balanced Systems
(i.e., same line currents I_a, I_b, I_c and phase-to-phase voltages V_{ab}, V_{bc}, V_{ca} in both cases)

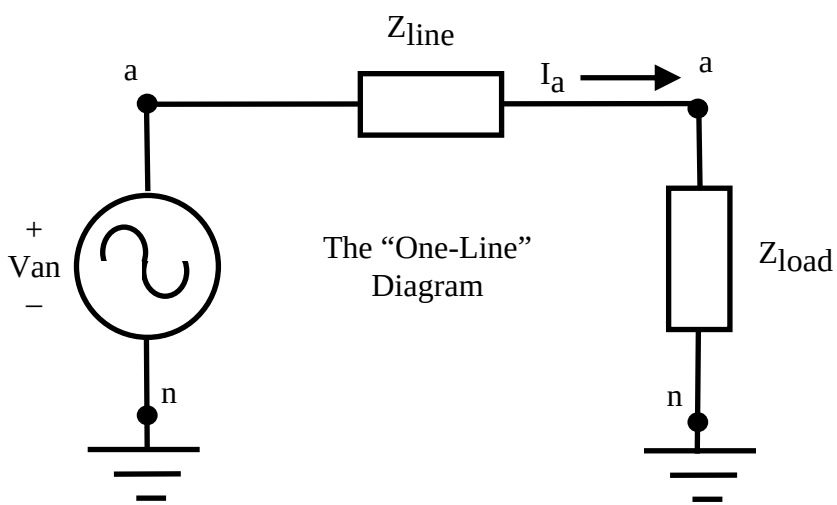


The Two Above Sources are Equivalent in Balanced Systems
(i.e., same line currents I_a, I_b, I_c and phase-to-phase voltages V_{ab}, V_{bc}, V_{ca} in both cases)



The Experiment: Opening and closing the switch has no effect because I_n is already zero for a three-phase balanced set. Since no current flows, even if there is a resistance in the grounding path, we must conclude that $V_n = 0$ at the neutral point (or equivalent neutral point) of any balanced three phase load or source in a balanced system. This allows us to draw a "one-line" diagram (typically for phase a) and solve a single-phase problem. Solutions for phases b and c follow from the phase shifts that must exist.

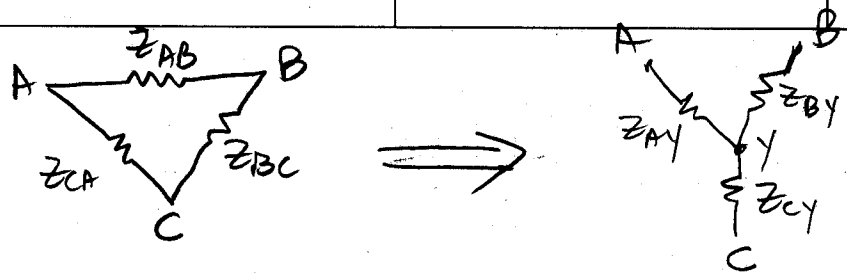




Balanced three-phase systems, no matter if they are delta connected, wye connected, or a mix, are easy to solve if you follow these steps:

1. Convert the entire circuit to an equivalent wye with a grounded neutral.
2. Draw the one-line diagram for phase a, recognizing that phase a has one third of the P and Q.
3. Solve the one-line diagram for line-to-neutral voltages and line currents.
4. If needed, compute line-to-neutral voltages and line currents for phases b and c using the ±120° relationships.
5. If needed, compute line-to-line voltages and delta currents using the $\sqrt{3}$ and ±30° relationships.

DELTA-to-WYE



Measure $Z_{A\,to\,B}$, $Z_{AB} \| (Z_{CA} + Z_{BC}) = Z_{AY} + Z_{BY}$

$$\frac{Z_{AB} Z_{CA} + Z_{AB} Z_{BC}}{Z_{AB} + Z_{BC} + Z_{CA}} = Z_{AY} + Z_{BY} \quad (1)$$

Measure $Z_{B\,to\,C}$, $Z_{BC} \| (Z_{AB} + Z_{CA}) = Z_{BY} + Z_{CY}$

$$\frac{Z_{AB} Z_{BC} + Z_{BC} Z_{CA}}{Z_{AB} + Z_{BC} + Z_{CA}} = Z_{BY} + Z_{CY} \quad (2)$$

Measure $Z_{C\,to\,A}$, $Z_{CA} \| (Z_{AB} + Z_{BC}) = Z_{AY} + Z_{CY}$

$$\frac{Z_{AB} Z_{CA} + Z_{BC} Z_{CA}}{Z_{AB} + Z_{BC} + Z_{CA}} = Z_{AY} + Z_{CY} \quad (3)$$

$$(1) - (2) + (3) = Z_{AY} + \cancel{Z_{BY}} - \cancel{Z_{BY}} - \cancel{Z_{CY}} + Z_{AY} + \cancel{Z_{CY}} = 2 Z_{AY}$$

$$= \frac{Z_{AB} Z_{CA} + \cancel{Z_{AB} Z_{BC}} - \cancel{Z_{AB} Z_{BC}} - \cancel{Z_{BC} Z_{CA}} + Z_{AB} Z_{CA} + \cancel{Z_{BC} Z_{CA}}}{Z_{AB} + Z_{BC} + Z_{CA}}$$

$$\therefore \; Z_{AY} = \frac{Z_{AB} Z_{CA}}{Z_{AB} + Z_{BC} + Z_{CA}}$$

product of adjacents

sum

From symmetry

$$Z_{BY} = \frac{Z_{AB} Z_{BC}}{Z_{AB} + Z_{BC} + Z_{CA}}$$

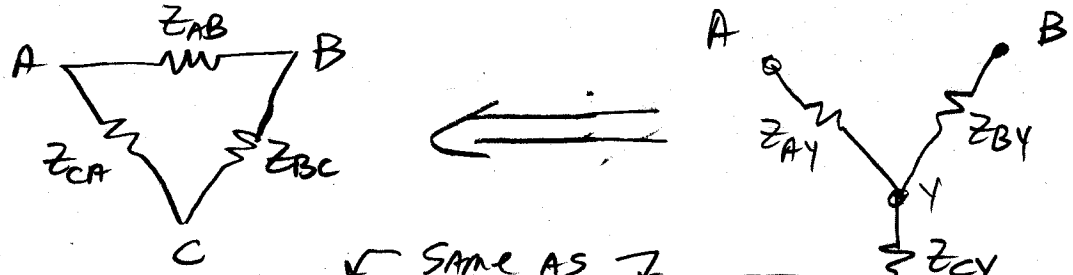
product of adjacents

$$Z_{CY} = \frac{Z_{BC} Z_{CA}}{Z_{AB} + Z_{BC} + Z_{CA}}$$

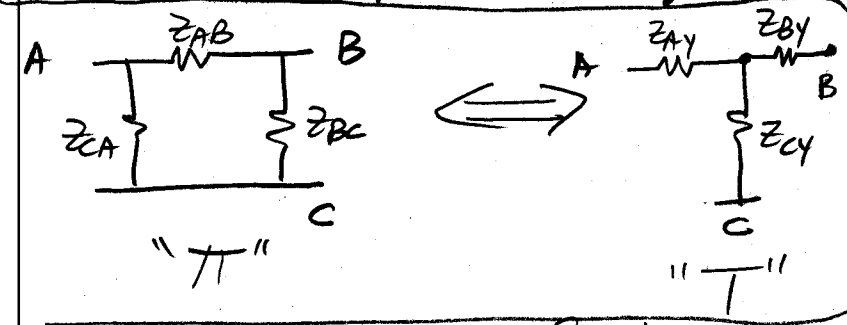
product of adjacents

No. 5505 Engineer's Computation Pad ALVIN

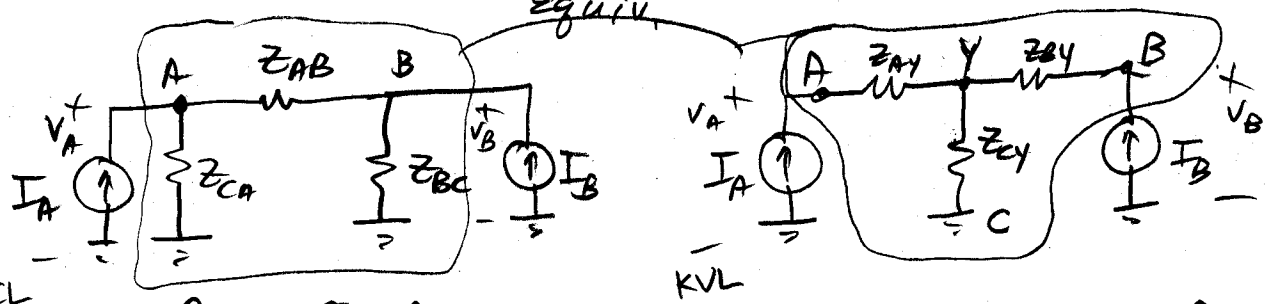
WYE-DELTA Equivalents, Going WYE-to-Delta



SAME AS



Equiv.



KCL

$$-\tilde{I}_A + \frac{\hat{V}_A}{Z_{CA}} + \frac{\tilde{V}_A - \hat{V}_B}{Z_{AB}} = 0$$

$$-\hat{I}_B + \frac{\tilde{V}_B}{Z_{BC}} + \frac{\tilde{V}_B - \tilde{V}_A}{Z_{AB}} = 0$$

KVL

$$-\tilde{V}_A + \tilde{I}_A(Z_{AY} + Z_{CY}) + \tilde{I}_B Z_{CY} = 0$$

$$-\hat{V}_B + \tilde{I}_B(Z_{BY} + Z_{CY}) + \hat{I}_A Z_{CY} = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\underbrace{\begin{bmatrix} \frac{1}{Z_{CA}} + \frac{1}{Z_{AB}} & -\frac{1}{Z_{AB}} \\ -\frac{1}{Z_{AB}} & \frac{1}{Z_{BC}} + \frac{1}{Z_{AB}} \end{bmatrix}}_{Y} \underbrace{\begin{bmatrix} \hat{V}_A \\ \tilde{V}_B \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} \hat{I}_A \\ \tilde{I}_B \end{bmatrix}}_{I}, \qquad \underbrace{\begin{bmatrix} V_A \\ V_B \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} Z_{AY} + Z_{CY} & Z_{CY} \\ Z_{CY} & Z_{BY} + Z_{CY} \end{bmatrix}}_{Z} \underbrace{\begin{bmatrix} \hat{I}_A \\ \tilde{I}_B \end{bmatrix}}_{I}$$

$$Z = Y^{-1}, \quad Y = Z^{-1}, \quad Z^{-1} = \frac{\begin{bmatrix} Z_{BY} + Z_{CY} & -Z_{CY} \\ -Z_{CY} & Z_{AY} + Z_{CY} \end{bmatrix}}{(Z_{AY} + Z_{CY})(Z_{BY} + Z_{CY}) - Z_{CY}^2}$$

$$Z_{AY}Z_{BY} + Z_{BY}Z_{CY} + Z_{CY}Z_{AY}$$

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Going from WYE-to-Delta, cont.

Term by term $-\frac{1}{Z_{AB}} = \frac{-Z_{CY}}{Z_{AY}Z_{BY} + Z_{BY}Z_{CY} + Z_{CY}Z_{AY}}$, so

$$Z_{AB} = \frac{\Sigma}{Z_{CY}}$$

$\equiv \Sigma$ of products

$$\frac{1}{Z_{CA}} + \frac{1}{Z_{AB}} = \frac{Z_{BY} + Z_{CY}}{\Sigma} = \frac{Z_{BY}}{\Sigma} + \frac{Z_{CY}}{\Sigma}$$

← From above, this is $\frac{1}{Z_{AB}}$

so $$Z_{CA} = \frac{\Sigma}{Z_{BY}}$$

Likewise, $$\frac{1}{Z_{BC}} + \frac{1}{Z_{AB}} = \frac{Z_{AY} + Z_{CY}}{\Sigma},$$

← This is $\frac{1}{Z_{AB}}$

so $$Z_{BC} = \frac{\Sigma}{Z_{AY}}$$

No. 5505 Engineer's Computation Pad ALVIN

A three-phase, 460V motor draws 5kW with a power factor of 0.80 lagging. Assuming that $\tilde{V}_{an}$ has phase angle zero,

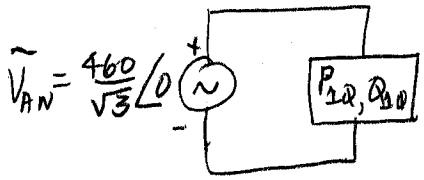
a. Find phasor currents $\tilde{I}_a$ and $\tilde{I}_{ab}$ (note – $\tilde{I}_{ab}$ is inside the motor delta windings)
b. Find the three phase motor Q and S
c. How much capacitive kVAr (three-phase) should be connected in parallel with the motor to improve the net power factor to 0.95?
d. Assuming no change in supply voltage, what will be the new $\tilde{I}_a$ after the kVArs are added?

Problem 2. A three-phase, 460V motor draws 5kW with a power factor of 0.80 lagging.

Assuming that $\tilde{V}_{an}$ has phase angle zero,

a. Find phasor currents $\tilde{I}_a$ and $\tilde{I}_{ab}$ (note – $\tilde{I}_{ab}$ is inside the motor delta windings)
b. Find the three phase motor Q and S
c. How much capacitive kVAr (three-phase) should be connected in parallel with the motor to improve the net power factor to 0.95?

a) DRAW A ONE-LINE DIAGRAM



$$P_{1\phi} = \frac{P_{3\phi}}{3} = \frac{5000}{3} = 1667\text{W}$$

$$Q_{1\phi} = P_{1\phi}\sqrt{\left(\frac{1}{pf}\right)^2 - 1} = 1250\text{VAr}$$

$$S_{1\phi} = \frac{P_{1\phi}}{pf}\angle\cos^{-1}(pf) = \frac{1667}{0.80}\angle 36.9^\circ = 2084\angle 36.9 \text{ VA}$$

a) $S_{1\phi} = V_{AN}(I_A)^* \Rightarrow \boxed{I_A = \frac{S_{1\phi}^*}{V_{AN}^*} = \frac{2084\angle -36.9}{\frac{460}{\sqrt{3}}\angle 0} = 7.85\angle -36.9 \text{ A}}$

$$\tilde{I}_{ab} = \frac{\tilde{I}_a}{\sqrt{3}}\angle 30^\circ = \frac{7.85\angle -36.9}{\sqrt{3}}\angle 30 = \boxed{4.53\angle -6.9^\circ \text{ A} = \tilde{I}_{ab}}$$

b) $\boxed{Q_{3\phi} = 3Q_{1\phi} = 3750\text{VAr}}$

$\left(Q_{3\phi} = P_{3\phi}\sqrt{\left(\frac{1}{pf}\right)^2 - 1} = 3750\right)$

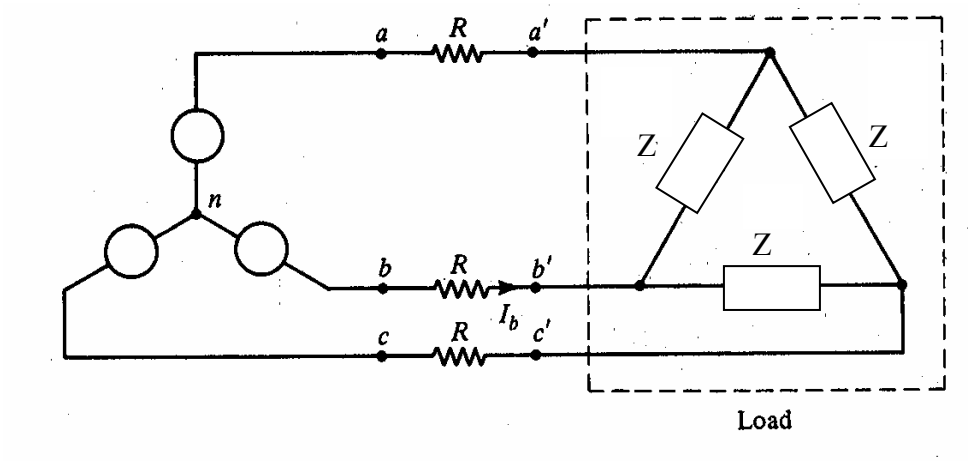
$\boxed{S_{3\phi} = 3S_{1\phi} = 6252\angle 36.9 \text{ VA}}$

c) $Q_{NET\,3\phi} = P_{NET\,3\phi}\sqrt{\left(\frac{1}{pf_{NET}}\right)^2 - 1} = 5000\sqrt{\left(\frac{1}{0.95}\right)^2 - 1} = 1643\text{VAr}$

$Q_{NET\,3\phi} = Q_{MOT\,3\phi} + Q_{CAP\,3\phi}$, $\boxed{Q_{CAP\,3\phi} = 1643 - 3750 = -2107\text{VAr}}$

Grady 2007, p. 53

The 60Hz system shown below is balanced. The line-to-line voltage of the source is 460V. Resistors R are each 5Ω.



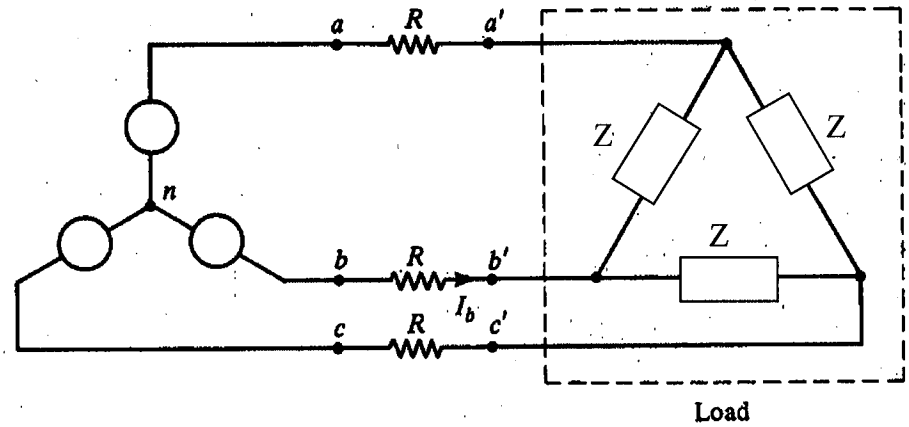
Part a. If each Z is (90 + j45)Ω, determine the three-phase complex power delivered by the source, and the three-phase complex power absorbed by the delta-connected Z loads.

Part b. If $\widetilde{V}_{an}$ at the source has phase angle zero, find $\widetilde{V}_{a'b'}$ at the load.

EE369/394J, Test #2, Sept. 10, 2004. Name KEY 1/2

One sheet of notes permitted. Show all steps.

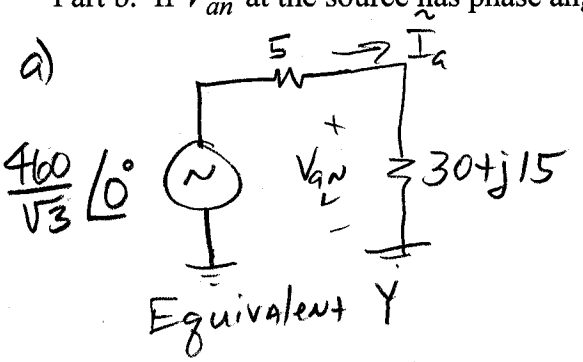
The 60Hz system shown below is balanced. The line-to-line voltage of the source is 460V. Resistors R are each 5Ω.



Part a. If each Z is (90 + j45)Ω, determine the three-phase complex power delivered by the source, and the three-phase complex power absorbed by the delta-connected Z loads.

Part b. If $\tilde{V}_{an}$ at the source has phase angle zero, find $\tilde{V}_{a'b'}$ at the load.

a)



$$\tilde{I}_a = \frac{\frac{460}{\sqrt{3}}\angle 0}{35+j15} = \frac{266\angle 0}{38.1\angle 23.2°} = 6.98\angle -23.2$$

$$P_{L1\phi} = |I_A|^2 R_{LY} = (6.98)^2(30) = 1462\,W$$

$$Q_{L1\phi} = |I_A|^2 X_{LY} = 731\ VAr$$

$$S_{L1\phi} = P_{L1\phi} + jQ_{L1\phi} = 1635\angle 26.6°\ VA$$

$$S_{L3\phi} = 3S_{1\phi} = 4905\angle 26.6\ VA$$

4385 + j2196

Source

$3V_{aN}I_a^* = 3\left(\frac{460}{\sqrt{3}}\angle 0\right)(6.98\angle 23.2)$

$= 5561\angle 23.2\ VA$

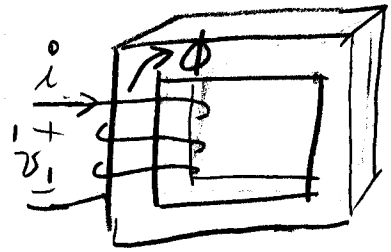
5111 + j2191

b) $\tilde{V}_{aN} = \tilde{I}_a Z_Y = (6.98\angle -23.2)\underbrace{(30+j15)}_{33.5\angle 26.6} = 234\angle 3.4°$

$\tilde{V}_{ab} = \sqrt{3}\,\tilde{V}_{aN} \cdot 1\angle 30$

$$\tilde{V}_{ab} = (\sqrt{3})(234\angle 3.4)(1\angle 30) = 405\angle 33.4°\ V$$

Inductance

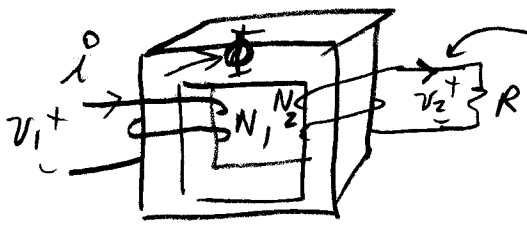


Right-hand rule, $L \equiv N\Phi / I$

Faraday's Law, $v = N \frac{d\Phi}{dt}$

$v_1 = L \frac{di_1}{dt}$

Lenz's Law – induced voltage in direction so that if a current is permitted to flow, it would try to cancel the flux.



I out to cancel or partially cancel.

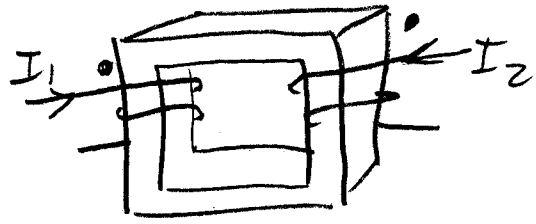
From $v = \frac{N\Phi}{dt}$, if all the flux links both coils,

then $\frac{v_1}{N_1} = \frac{d\Phi}{dt} = \frac{v_2}{N_2}$, so $v_2 = v_1 \left(\frac{N_2}{N_1}\right)$ Transformer

Conservation of power. For lossless transformer,

$v_1 i_1 = v_2 i_2$, so $i_2 = i_1 \left(\frac{N_1}{N_2}\right)$

Mutual Inductance & Dot Convention,

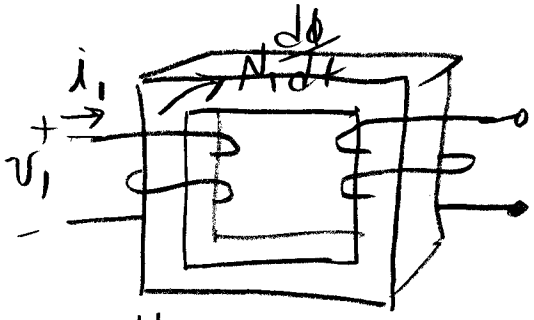


Current into dots produce Additive flux, so if you excite side 1, Lenz's law says the other current tries to flow out of dot 2.

For coupled coils, we use a matrix

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{bmatrix} L_{11} & \pm M_{12} & \pm M_{13} \\ \pm M_{21} & L_{22} & \pm M_{23} \\ \pm M_{31} & \pm M_{32} & L_{33} \end{bmatrix} \begin{bmatrix} di_1/dt \\ di_2/dt \\ di_3/dt \end{bmatrix}.$$

M's are positive, and $M_{JK} = M_{KJ}$ (show later)

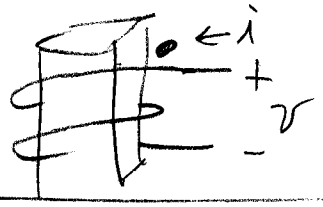


+ v_2 (open ckt) -

Lenz's Law say $v_2 = +N_2 \frac{d\phi}{dt}$

Here, $v_2 = M_{21}\frac{di_1}{dt} + \cancel{L_{22}\frac{di_2}{dt}}^{\,0}$

So, for all ⊕ terms, label all polarities so



Energy – consider a 2-coil system.

1. With $i_2 = 0$, bring i_1 up from 0 to I_1
2. With $i_1 = I_1 =$ const, bring i_2 up from 0 to I_2

For 1, $\Delta W_1 = \int_t v_1 i_1\, dt = \int_t L_{11}\frac{di_1}{dt} i_1\, dt = \frac{1}{2}L_{11}I_1^2$

For 2, $\Delta W_2 = \int v_1 I_1\, dt + \int v_2 i_2\, dt$

$$= \int_t \left[\cancel{L_{11}\frac{di_1}{dt}}^{\,0} + M_{12}\frac{di_2}{dt}\right] I_1\, dt + \int_t \left[\cancel{M_{21}\frac{di_1}{dt}}^{\,0} + L_{22}\frac{di_2}{dt}\right] i_2\, dt$$

$$= M_{12}I_1\int_{i_2} di_2 + L_{22}\int_{i_2} i_2\, di_2 = M_{12}I_1I_2 + \frac{1}{2}L_{22}I_2^2$$

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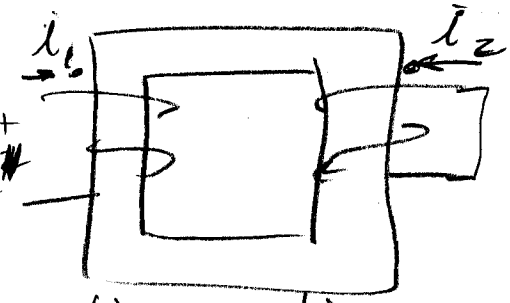
So the total energy stored for both steps

$$W = \Delta W_1 + \Delta W_2 = \tfrac{1}{2}L_{11}I_1^2 + \tfrac{1}{2}L_{22}I_2^2 + M_{12}I_1I_2$$

Now, for a linear system, the stored energy doesn't depend on whether we energize coil 1 first, or coil 2 first. So, $\boxed{M_{12} = M_{21}}$ by symmetry.

Consider



shorted, $+\ V_2 = 0\ -$

$$v_1 = L_{11}\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$v_2 = M\frac{di_1}{dt} + L_{22}\frac{di_2}{dt} \leftarrow 0, \text{ so } \frac{di_2}{dt} = \frac{-M}{L_{22}}\frac{di_1}{dt}$$

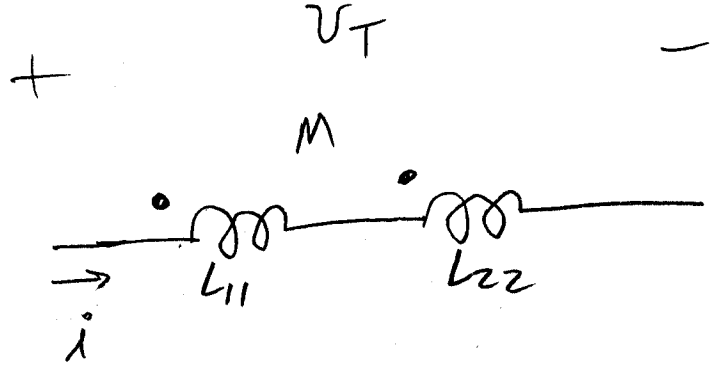
sub into v_1, $\quad v_1 = \left(L_{11} + M\frac{-M}{L_{22}}\right)\frac{di_2}{dt}$

$$v_1 = \left(L_{11} - \frac{M^2}{L_{22}}\right)\frac{di_2}{dt}$$

For perfect coupling, $v_1 = 0$, so $L_{11} = \frac{M^2}{L_{22}}$, $M = \sqrt{L_{11}L_{22}}$

For less than perfect coupling, let $M = K\sqrt{L_{11}L_{22}}$, $0 \le K \le 1$

$M = 0$ for NO coupling, so $M = 0\sqrt{L_{11}L_{22}} = 0$



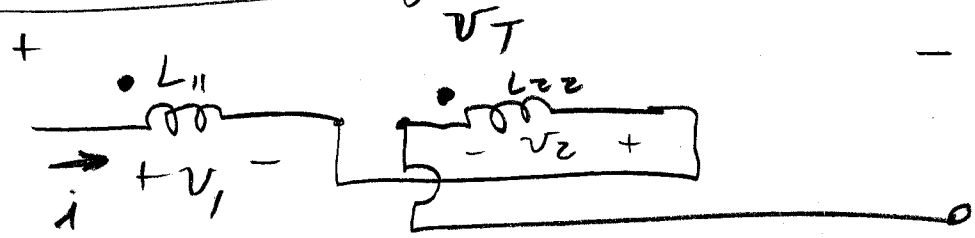
Same current enters both dots

$$v_2 = L_{22}\frac{di}{dt} + M\frac{di}{dt}$$

$$v_1 = L_{11}\frac{di}{dt} + M\frac{di}{dt}$$

$$v_T = v_1 + v_2 = (L_{11} + M + L_{22} + M)\frac{di}{dt}$$

$$v_T = \underbrace{(L_{11} + L_{22} + 2M)}_{\text{Equiv. } L}\frac{di}{dt}$$



$$v_T = v_1 + v_2$$

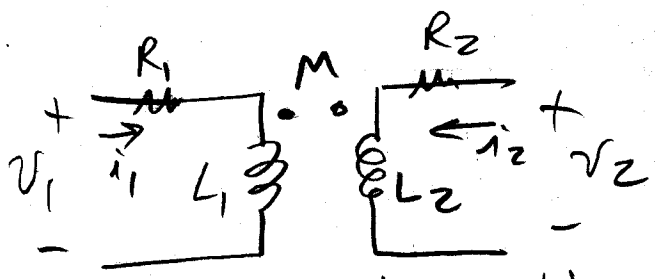
$$v_1 = L_{11}\frac{di}{dt} - M\frac{di}{dt} \qquad v_2 = L_{22}\frac{di}{dt} - M\frac{di}{dt}$$

$$v_T = v_1 + v_2 = \underbrace{(L_{11} + L_{22} - 2M)}_{\text{Equiv. } L}\frac{di}{dt}$$

Note $(L_{11} + L_{22} + 2M) - (L_{11} + L_{22} - 2M) = 4M$

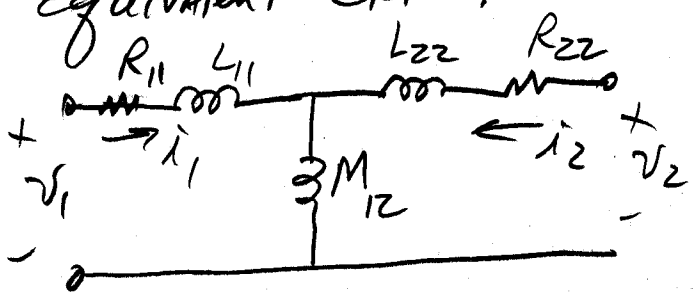
$\therefore$ Difference in measurements gives $4M$

Coupled Coils and Transformers



$$v_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$
$$v_2 = R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$
$$\Rightarrow \begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} R_1 + j\omega L_1 & j\omega M \\ j\omega M & R_2 + j\omega L_2 \end{bmatrix} \begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix}$$

Equivalent Ckt ?



Compare [Z]

$$\begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} R_{11} + j\omega L_{11} + j\omega M_{12} & j\omega M_{12} \\ j\omega M_{12} & R_{22} + j\omega L_{22} + j\omega M_{12} \end{bmatrix} \begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix}, \text{ so}$$

$M_{12} = M$,

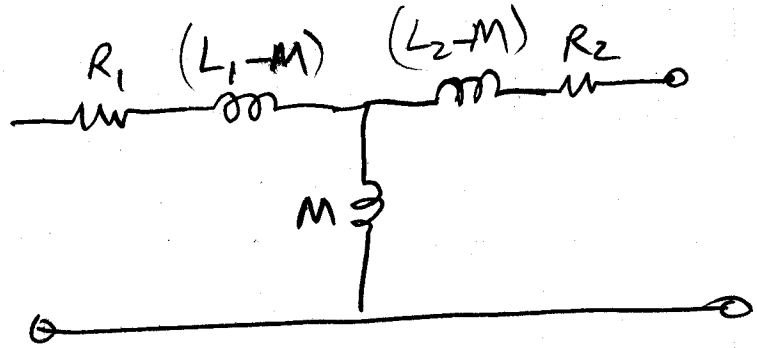
R_1 & R_2 same

$L_1 = L_{11} + M_{12}$, so

$L_{11} = L_1 - M$

$L_{22} + M_{12} = L_2$,

$L_{22} = L_2 - M$.



Problem, $M = k\sqrt{L_1 L_2}$, $0 \le k \le 1$, so M can easily be greater than either L_1 or L_2, leaving $(L_1 - M)$ or $(L_2 - M) < 0$, <u>Not physically realizable</u>

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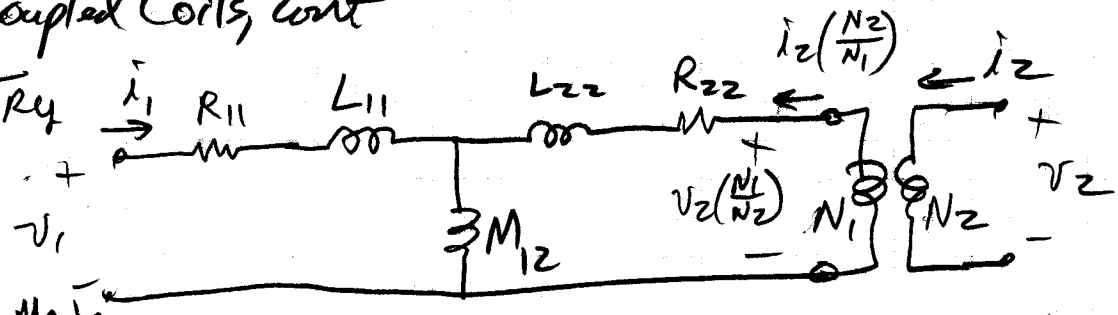
ALVIN

Coupled Coils, cont

Try



Two Meshes

$$\tilde{V}_1 = (R_{11} + j\omega L_{11} + j\omega M_{12})\tilde{I}_1 + j\omega M_{12}\tilde{I}_2\left(\frac{N_2}{N_1}\right)$$

$$\tilde{V}_2\left(\frac{N_1}{N_2}\right) = (R_{22} + j\omega L_{22} + j\omega M_{12})\tilde{I}_2\left(\frac{N_2}{N_1}\right) + j\omega M_{12}\tilde{I}_1$$

$$\tilde{V}_2 = (R_{22} + j\omega L_{22} + j\omega M_{12})\left(\frac{N_2}{N_1}\right)\tilde{I}_2\left(\frac{N_2}{N_1}\right) + j\omega M_{12}\left(\frac{N_2}{N_1}\right)\tilde{I}_1$$

$$\begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} R_{11} + j\omega L_{11} + j\omega M_{12} & j\omega M_{12}\left(\frac{N_2}{N_1}\right) \\ j\omega M_{12}\left(\frac{N_2}{N_1}\right) & (R_{22} + j\omega L_{22} + j\omega M_{12})\left(\frac{N_2}{N_1}\right)^2 \end{bmatrix} \begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix}$$

Matching terms in $[Z]$

$$R_1 + j\omega L_1 = R_{11} + j\omega L_{11} + j\omega M_{12} \Rightarrow R_{11} = R_1 \quad (1)$$

$$L_{11} + M_{12} = L_1$$

$$j\omega M = j\omega M_{12}\left(\frac{N_2}{N_1}\right) \Rightarrow M_{12} = M\left(\frac{N_1}{N_2}\right)$$

$$L_{11} = L_1 - M\left(\frac{N_1}{N_2}\right) \quad (2)$$

$$R_2 + j\omega L_2 = (R_{22} + j\omega L_{22} + j\omega M_{12})\left(\frac{N_2}{N_1}\right)^2$$

$$\Rightarrow R_{22} = R_2\left(\frac{N_1}{N_2}\right)^2 \quad (3)$$

$$L_{22} + M_{12} = \left(\frac{N_1}{N_2}\right)^2 L_2$$

$$L_{22} = \left(\frac{N_1}{N_2}\right)^2 L_2 - M\left(\frac{N_1}{N_2}\right) = \left(\frac{N_1}{N_2}\right)\left[L_2\left(\frac{N_1}{N_2}\right) - M\right] \quad (4)$$

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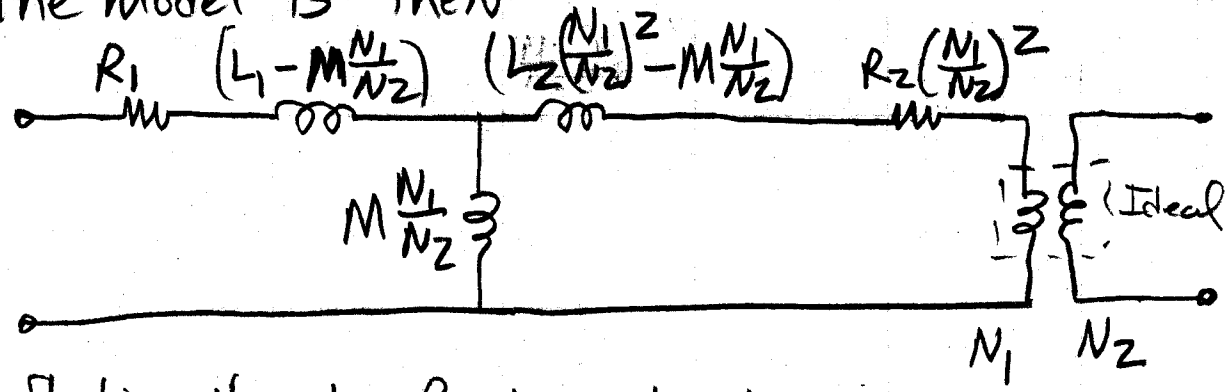
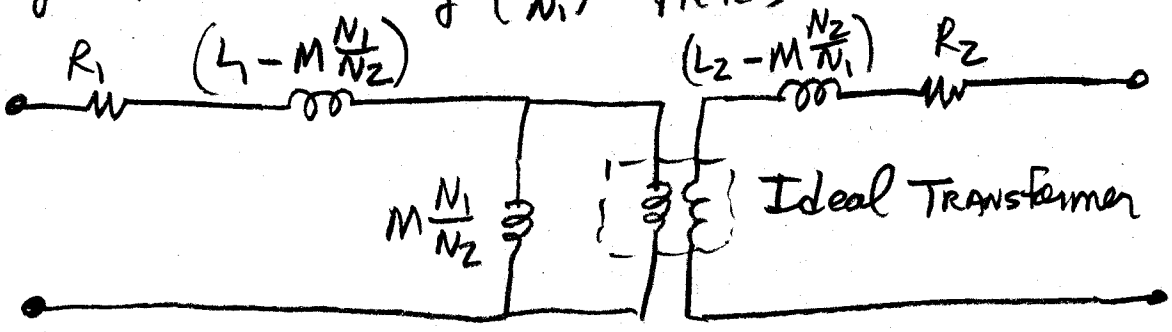
Coupled Coils, cont

The model is then



Reflecting the L_2, R_2 terms to the Right-hand side by $(\frac{N_2}{N_1})^2$ yields



Note – $M = K\sqrt{L_1 L_2}$, so $L_1 - M\frac{N_1}{N_2} = L_1 - K\sqrt{L_1 L_2}\frac{N_1}{N_2}$.

Since $L = \frac{\mu N^2 A}{\ell}$, then $L_1 \simeq L_2(\frac{N_1}{N_2})^2$, $L_2 = L_1(\frac{N_2}{N_1})^2$

Thus $\underbrace{L_1 - K\sqrt{L_1}\sqrt{L_1}(\frac{N_2}{N_1})(\frac{N_1}{N_2})} = L_1 - KL_1 = L_1(1-K)$

$$(L_1 - M\frac{N_1}{N_2}) \simeq L_1(1-K),\ K \le 1 \text{ (Physically realizable}$$

Likewise

$$L_2 - M\frac{N_2}{N_1} = L_2 - K\sqrt{L_1 L_2}\frac{N_2}{N_1} = L_2 - K\sqrt{L_2}(\frac{N_1}{N_2})\sqrt{L_2}\frac{N_2}{N_1}$$

$$= L_2(1-K)$$

(Physically realizable)

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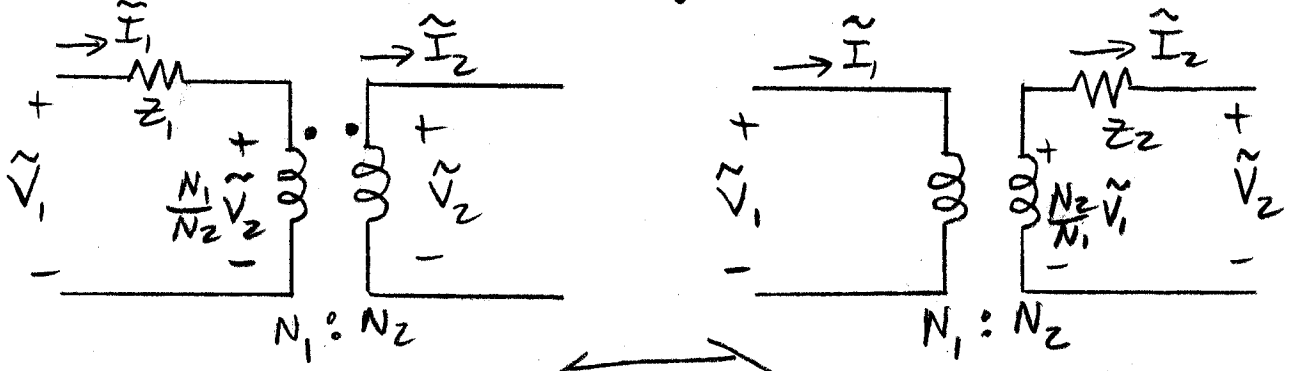
Two-Port Analysis of Single-Phase Transformer



How to make them equivalent?

$$-\tilde{V}_1 + Z_1\tilde{I}_1 + \frac{N_1}{N_2}\tilde{V}_2 = 0$$

$$\tilde{I}_1 = \frac{N_2}{N_1}\tilde{I}_2$$

$$-\frac{N_2}{N_1}\tilde{V}_1 + Z_2\tilde{I}_2 + \tilde{V}_2 = 0$$

$$\tilde{I}_1 = \frac{N_2}{N_1}\tilde{I}_2$$

Put in Two-Port Form

$$\tilde{V}_2 = \frac{N_2}{N_1}\tilde{V}_1 - \frac{N_2}{N_1}Z_1\tilde{I}_1$$

$$\tilde{I}_2 = \frac{N_1}{N_2}\tilde{I}_1$$

$$\tilde{V}_2 = \frac{N_2}{N_1}\tilde{V}_1 - Z_2\frac{N_1}{N_2}\tilde{I}_1$$

$$\tilde{I}_2 = \frac{N_1}{N_2}\tilde{I}_1$$

$$\begin{bmatrix}\tilde{V}_2\\ \tilde{I}_2\end{bmatrix} = \begin{bmatrix}\frac{N_2}{N_1} & -\frac{N_2}{N_1}Z_1\\ 0 & \frac{N_1}{N_2}\end{bmatrix}\begin{bmatrix}\tilde{V}_1\\ \tilde{I}_1\end{bmatrix}, \quad \begin{bmatrix}\tilde{V}_2\\ \tilde{I}_2\end{bmatrix} = \begin{bmatrix}\frac{N_2}{N_1} & -\frac{N_1}{N_2}Z_2\\ 0 & \frac{N_1}{N_2}\end{bmatrix}\begin{bmatrix}\tilde{V}_1\\ \tilde{I}_1\end{bmatrix}$$

Same equation if $\frac{N_2}{N_1}Z_1 = \frac{N_1}{N_2}Z_2$,

OR $\boxed{Z_2 = Z_1\left(\frac{N_2}{N_1}\right)^2}$

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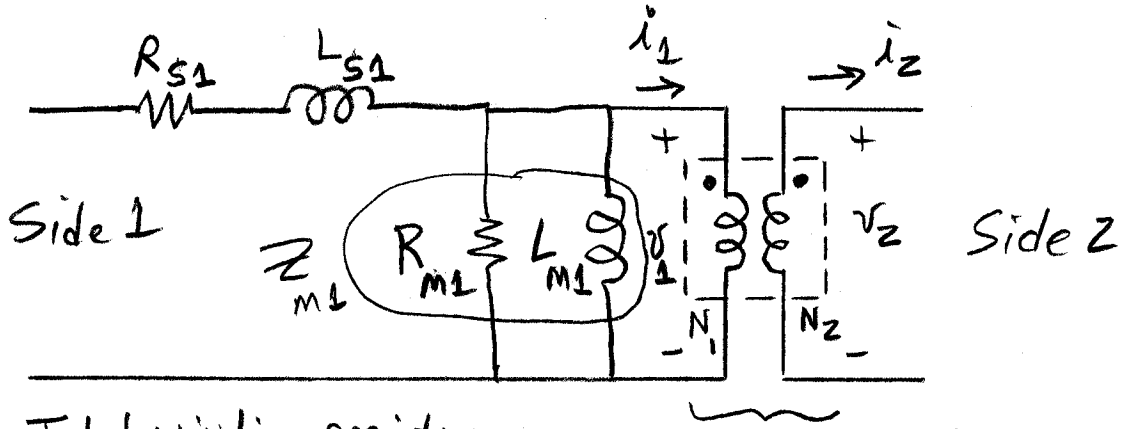
Transformer Model



R_{S1} : Total winding resistance Reflected to side 1

L_{S1} : Total leakage inductance Reflected to side 1

R_{M1} : Core loss term reflected to side 1

L_{M1} : Core magnetizing inductance reflected to side 1

$$v_2 = v_1\left(\frac{N_2}{N_1}\right)$$
$$i_2 = i_1\left(\frac{N_1}{N_2}\right)$$
$$v_1 i_1 = v_2 i_2$$

Facts: $L_{M1} >> L_{S1}$, $R_{M1} >> R_{S2}$

Reflection: Can reflect R, L terms to side 2 using $\left(\frac{N_2}{N_1}\right)^2$, e.g. $R_{S2} = R_{S1}\left(\frac{N_2}{N_1}\right)^2$, $L_{M2} = L_{M1}\left(\frac{N_2}{N_1}\right)^2$.

Open Circuit Test

- Open circuit the HV side
- Energize the LV side with rated voltage.
- Measure $\tilde{V}_{LV}$, $\tilde{I}_{LV}$. $\frac{\tilde{V}_{LV}}{\tilde{I}_{LV}} \simeq \boxed{R_m \| j\omega L_m}$ Z_M

because $Z_M >> (R_s + j\omega L_s)$

As seen on low voltage side

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Also, can use $\frac{V_{LV}^2}{R_m} = P_{oc}$, $\frac{V_{LV}^2}{\omega L_m} = Q_{oc}$ to get R_m, ωL_m

Short Circuit Test

- Short circuit the LV side
- Energize the HV side with rated current
- Measure $\tilde{V}_{HV}$, $\tilde{I}_{HV}$. $\dfrac{\tilde{V}_{HV}}{\tilde{I}_{HV}} \simeq \underbrace{R_s + j\omega L_s}_{\text{As seen on high voltage side.}}$,

because the voltage across R_m and L_m terms is small (few % of nominal, with most of $\tilde{I}_{HV}$ going to the ideal transformer which is short circuited

Also can use $I_{HV}^2 R_s = P_{sc}$ to get R_s, ωL_s

$$I_{HV}^2 \omega L_s = Q_{sc}$$

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
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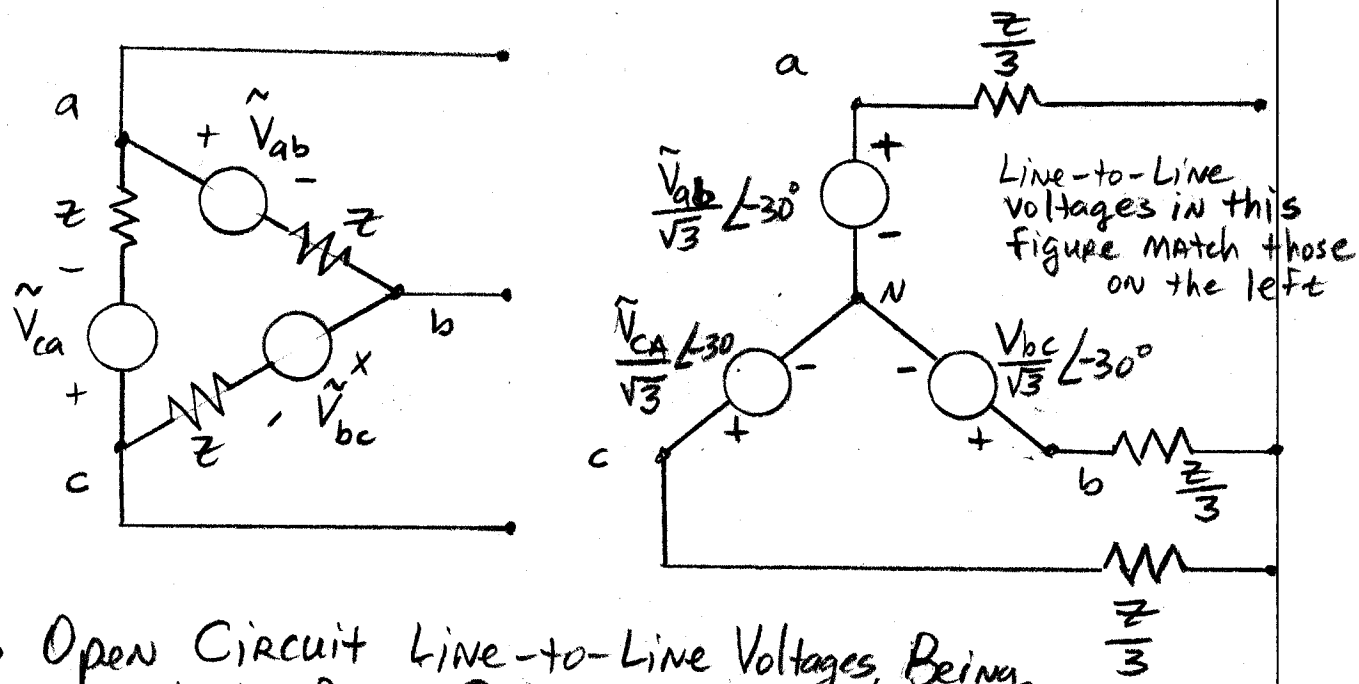
Wye Equivalent of Delta Transformer



- Open Circuit Line-to-Line Voltages Being Equal in Both Cases Dictate the $\sqrt{3}$ and $\angle -30^\circ$ Angles in Wye Equivalent

- By turning off the energizing windings (not shown), and then measuring the impedance, we see that the Z's in delta become $\frac{Z}{3}$ in wye

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WYE EQUIVALENT CIRCUIT FOR THREE PHASE TRANSFORMERS (ONLY ONE PHASE IS SHOWN)

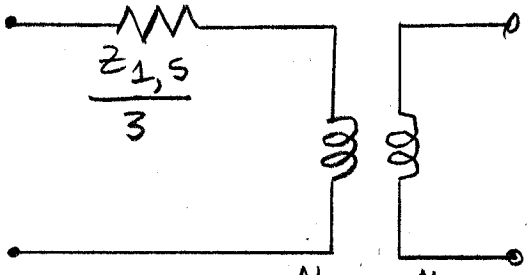
N_1 N_2

Y Y — Handle the same as single phase

N_1 N_2

Δ Δ — Convert each side to Y, then handle like single phase

Each transformer has $Z_{1,s}$ and turns ratio $N_1 : N_2$



OR, Bring the impedance to side 2 by square of turns ratio

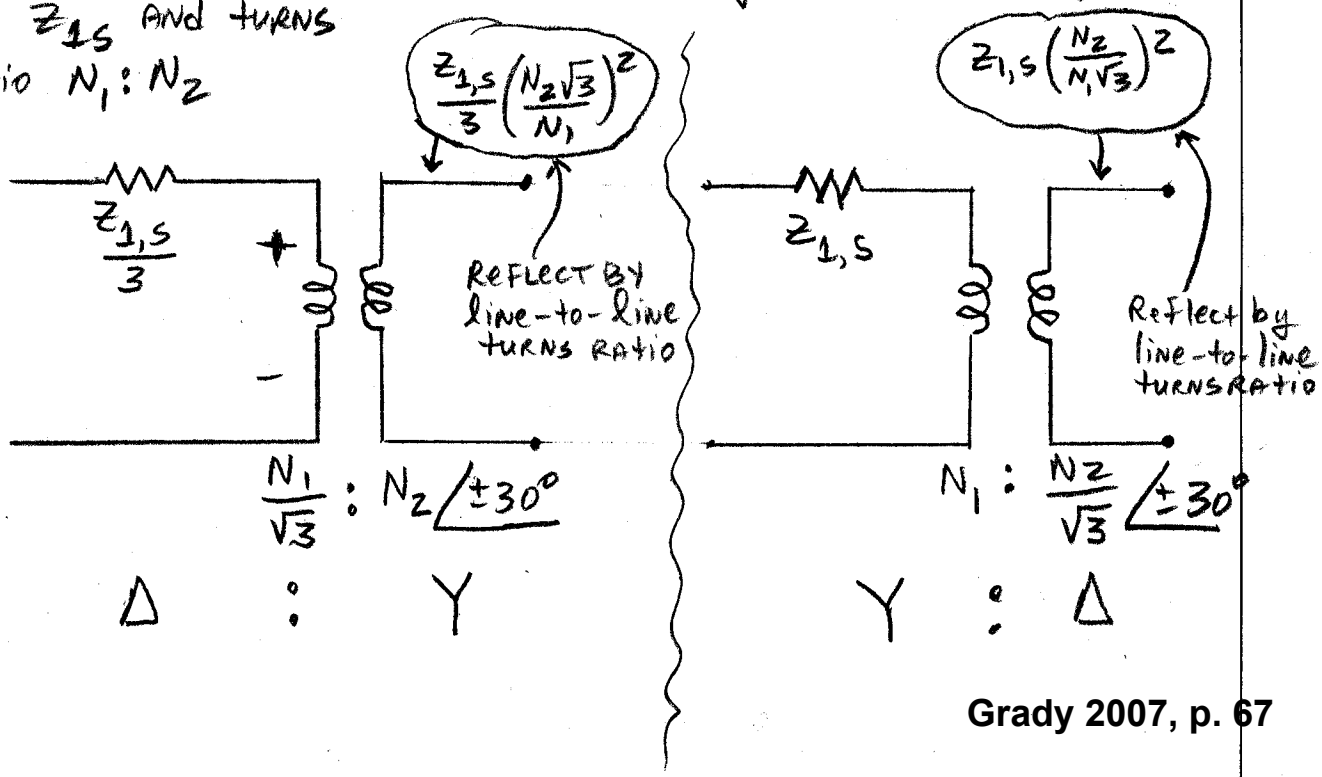
$\frac{N_1}{\sqrt{3}} : \frac{N_2}{\sqrt{3}}$ ← Line-to-Neutral turns ratio same as line-to-line

---

N_1 N_2 N_1 N_2

Δ Y OR Y Δ — Convert Δ side to Y, then handle like single phase. There will be a 30° phase shift with the sign dictated by ANSI.

Each transformer has Z_{1s} and turns ratio $N_1 : N_2$



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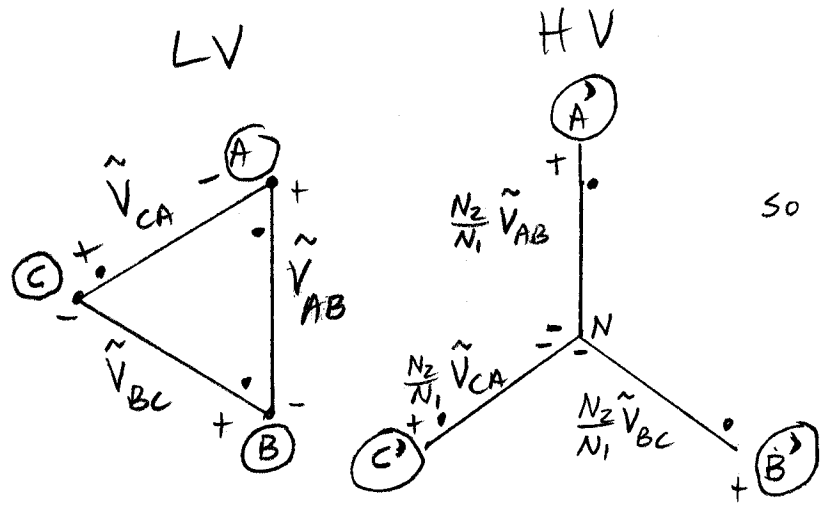
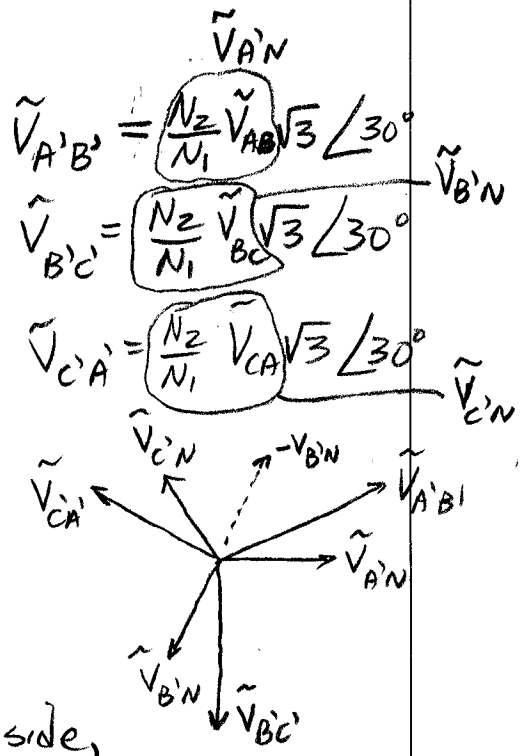
PHASE SHIFT AND ANSI STANDARD IN Y Δ



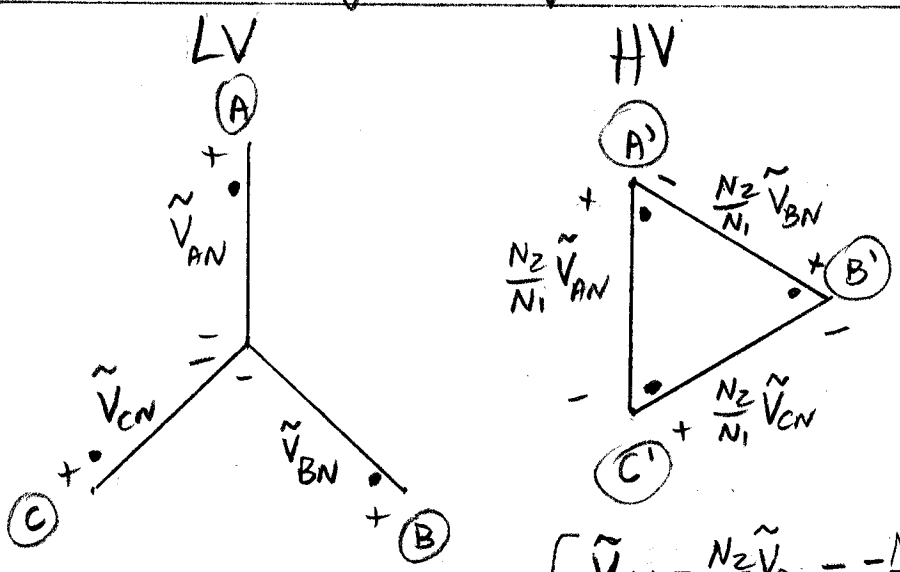
so $\tilde{V}_{A'B'} = \frac{N_2}{N_1}\tilde{V}_{AB}\sqrt{3}\angle 30^\circ$

$\tilde{V}_{B'C'} = \frac{N_2}{N_1}\tilde{V}_{BC}\sqrt{3}\angle 30^\circ$

$\tilde{V}_{C'A'} = \frac{N_2}{N_1}\tilde{V}_{CA}\sqrt{3}\angle 30^\circ$



LL voltages on left side lag LL voltages on Right side by 30°. ANSI std. says then left side has to be low voltage side, Right side is High voltage



see a-b-c diagram

$$\tilde{V}_{B'A'} = \frac{N_2}{N_1}\tilde{V}_{BN} = -\frac{N_2}{N_1}\left(-\tilde{V}_{BN}\right) = -\frac{N_2}{N_1}\left(+\frac{V_{AB}}{\sqrt{3}}\angle 30^\circ\right)$$

$$\tilde{V}_{C'B'} = \frac{N_2}{N_1}\tilde{V}_{CN} = -\frac{N_2}{N_1}\left(-\tilde{V}_{CN}\right) = -\frac{N_2}{N_1}\left(+\frac{V_{BC}}{\sqrt{3}}\angle 30^\circ\right)$$

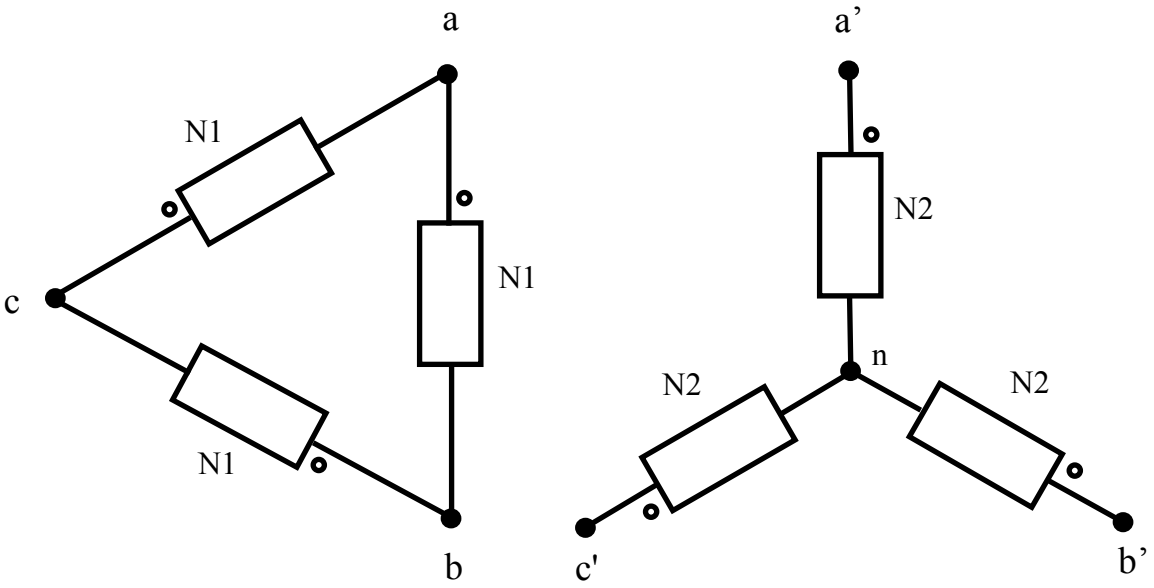
$$\tilde{V}_{A'C'} = \frac{N_2}{N_1}\tilde{V}_{AN} = -\frac{N_2}{N_1}\left(-\tilde{V}_{AN}\right) = -\frac{N_2}{N_1}\left(+\frac{V_{CA}}{\sqrt{3}}\angle 30^\circ\right)$$

Then

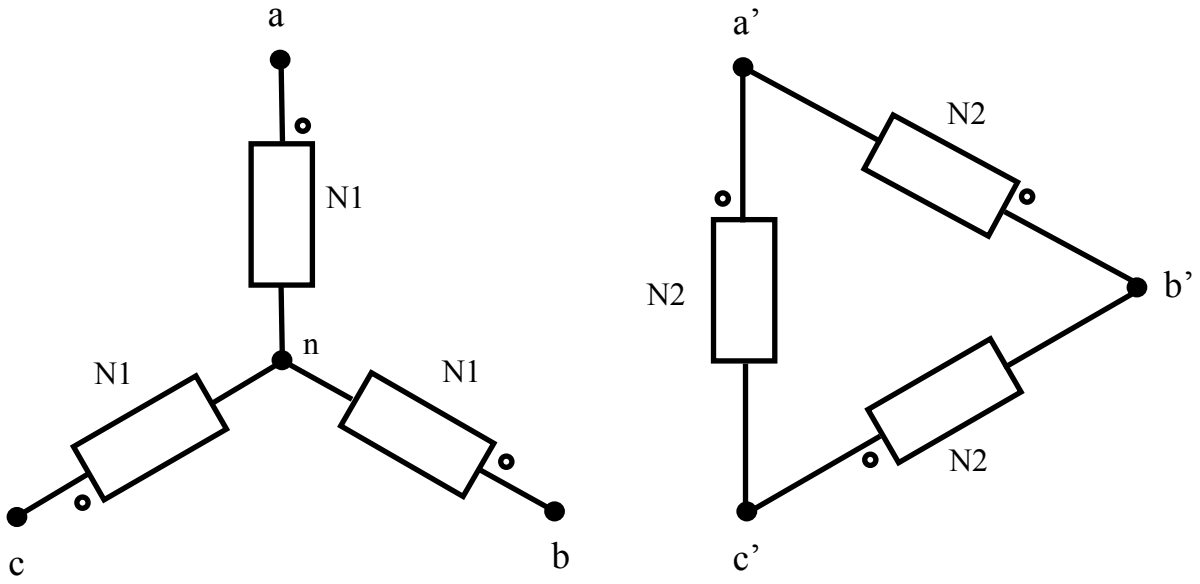
$$\tilde{V}_{A'B'} = -\tilde{V}_{B'A'} = \frac{N_2}{N_1\sqrt{3}}V_{AB}\angle 30^\circ,$$

$$\tilde{V}_{B'C'} = -\tilde{V}_{C'B'} = \frac{N_2}{N_1\sqrt{3}}V_{BC}\angle 30^\circ, \quad \tilde{V}_{C'A'} = -\tilde{V}_{A'C'} = \frac{N_2}{N_1\sqrt{3}}\tilde{V}_{CA}\angle 30^\circ$$

so, the A' B' C' side is the HV side



LV **HV**



LV **HV**

EE362L, Trans. Test
1/23/03

PHASE A VOLTAGE SPECTRUM

Fundamental volts: 114.9 Vrms

Fundamental freq: 60.0 Hz V_1

HARM	PCT	SINE PHASE	HARM	PCT	SINE PHASE
FUND	100.0%	0°	2nd		
3rd	0.7%	69°	4th		
5th	1.5%	-133°	6th		
7th	1.6%	15°	8th		
9th	0.1%	164°	10th		
11th	0.7%	-154°	12th		
13th	0.2%	74°	14th		
15th			16th		
17th	0.1%	-147°	18th		
19th			20th		
21st			22nd		
23rd			24th		
25th			26th		
27th			28th		
29th			30th		
31st			32nd		
33rd			34th		
35th			36th		
37th			38th		
39th			40th		
41st			42nd		
43rd			44th		
45th			46th		
47th			48th		
49th			50th		
ODD	2.4%		EVEN	0.1%	

THD: 2.4%

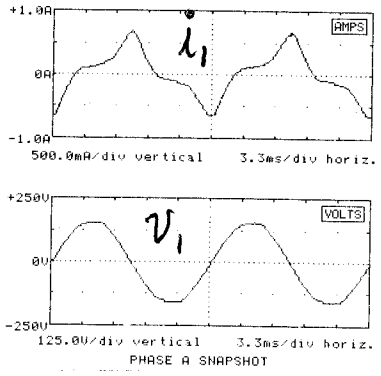
PHASE A CURRENT SPECTRUM

Fundamental amps: 320 mA rms

Fundamental freq: 60.0 Hz I_1

HARM	PCT	SINE PHASE	HARM	PCT	SINE PHASE
FUND	100.0%	-71°	2nd	0.3%	21°
3rd	38.7%	-79°	4th		
5th	9.6%	-82°	6th	0.2%	88
7th	2.2%	-65°	8th		
9th	1.8%	-85°	10th		
11th	1.1%	-107°	12th		
13th	0.2%	-166°	14th		
15th			16th		
17th	0.2%	-99°	18th	0.1%	167°
19th			20th		
21st			22nd		
23rd	0.1%	-141°	24th		
25th	0.1%	-65°	26th		
27th	0.1%	137°	28th		
29th	0.1%	-66°	30th		
31st	0.1%	33°	32nd		
33rd			34th		
35th			36th		
37th			38th	0.2%	-140°
39th			40th		
41st			42nd	0.1%	-110°
43rd			44th	0.1%	-67°
45th			46th		
47th	0.1%	-174°	48th	0.1%	102°
49th	0.1%	28°	50th	0.1%	-27°
ODD	40.0%		EVEN	0.5%	

THD: 40.0%



Open Circuit Test.
Energize 115V side,
Open Circuit 25V side.
Yields R_M, $j\omega L_M$.

Turns ratio measured to be 115.3/26.5
primary secondary

PHASE A VOLTAGE SPECTRUM

Fundamental volts: 8.2 Vrms

Fundamental freq: 60.0 Hz V_1

HARM	PCT	SINE PHASE	HARM	PCT	SINE PHASE
FUND	100.0%	0°	2nd	0.2%	38°
3rd	0.5%	69°	4th	0.1%	-145°
5th	1.4%	-148°	6th	0.1%	58°
7th	1.7%	-6°	8th		
9th	0.2%	105°	10th		
11th	0.7%	174°	12th	0.2%	21°
13th	0.2%	39°	14th		
15th			16th		
17th	0.1%	135°	18th		
19th	0.2%	12°	20th		
21st	0.1%	56°	22nd		
23rd			24th		
25th	0.2%	-8°	26th		
27th			28th		
29th	0.1%	52°	30th	0.2%	167°
31st			32nd		
33rd			34th		
35th			36th		
37th			38th	0.1%	168°
39th			40th	0.2%	117°
41st	0.2%	107°	42nd	0.1%	-70°
43rd	0.2%	176°	44th	0.1%	-74°
45th	0.1%	-119°	46th	0.3%	133°
47th	0.1%	-35°	48th		
49th	0.1%	55°	50th		
ODD	2.4%		EVEN	0.6%	

THD: 2.5%

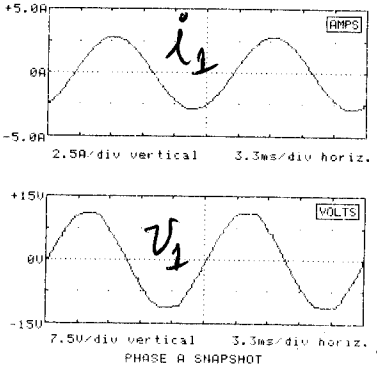
PHASE A CURRENT SPECTRUM

Fundamental amps: 2.049 A rms

Fundamental freq: 60.0 Hz I_1

HARM	PCT	SINE PHASE	HARM	PCT	SINE PHASE
FUND	100.0%	-56°	2nd		
3rd	0.2%	-12°	4th		
5th	0.4%	129°	6th		
7th	0.3%	-89°	8th		
9th			10th		
11th			12th		
13th			14th		
15th			16th		
17th			18th		
19th			20th		
21st			22nd		
23rd			24th		
25th			26th		
27th			28th		
29th			30th		
31st			32nd		
33rd			34th		
35th			36th		
37th			38th		
39th			40th		
41st			42nd		
43rd			44th		
45th			46th		
47th			48th		
49th			50th		
ODD	0.5%		EVEN	0.1%	

THD: 0.5%



Short Circuit Test
Energize 115V side,
Short Circuit 25V side.
Yields R_1, $j\omega L_1$

Question. If you energize the primary at 115V, and have a 3 ohm resistor connected to the secondary, what voltage is across the 3Ω?

Grady 2007 p.70

The following data were taken for a 25kVA, 7200/240V transformer:

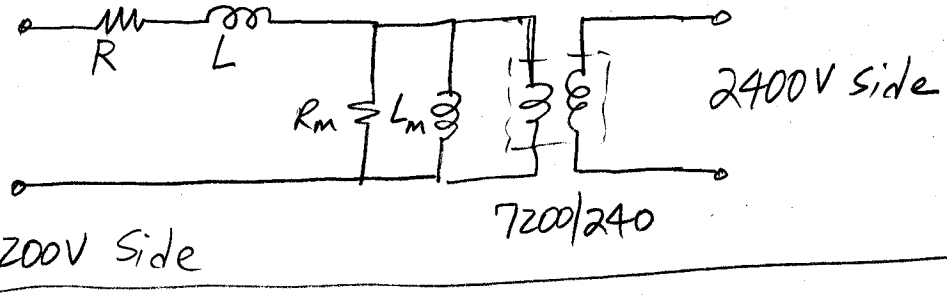
- Short Circuit Test: 240V side short-circuited, 7200V side energized at reduced voltage. The measurements (on the 7200V side) are Irms = 3.47A (i.e., rated current), Vrms = 216V, P = 335W, Q = 669VAr.

- Open Circuit Test: 7200V side open-circuited, 240V side energized. The measurements (on the 240V side) are Vrms = 240V (i.e., rated voltage), Irms = 2.08A, P = 171W, Q = 469VAr.

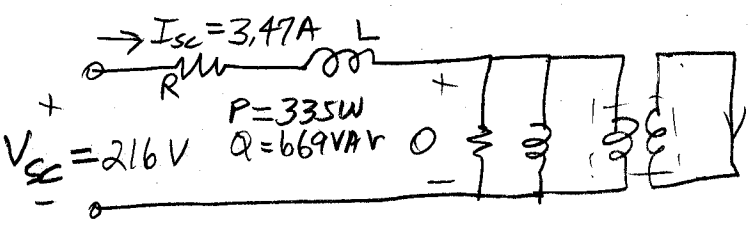
Draw the transformer equivalent circuit with all four circuit parameters (i.e., R, L_l, L_m, R_m) shown on the 7200V side.

HW 4

25KVA TRANSFORMER



Short Ckt Test



$I_{sc}^2 R = P$

$$R = \frac{P}{I_{sc}^2} = \frac{335}{(3.47)^2} = 27.8\,\Omega$$

$I_{sc}^2 \omega L = Q$

$$L = \frac{Q}{I_{sc}^2 \omega} = \frac{669}{(\boxed{3.47})^2(120\pi)}$$

$$L = 0.1474H$$

Check, $Z = R + j\omega L = 27.8 + j55.6\,\Omega$

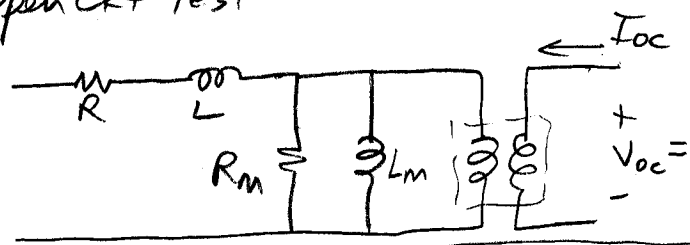
$$\tilde{I}_{sc} = \frac{\tilde{V}_{sc}}{Z} = \frac{216\angle 0}{62.2\angle 63.4^\circ}$$

$$\tilde{I}_{sc} = \underbrace{3.47}_{OK}\angle -63.4\ A$$

$$S = V_{sc} I_{sc}^* = (216\angle 0)(3.47\angle -63.4)^* = 750\angle 63.4\ VA = \underset{W}{336} + j\underset{VAr}{671}$$

OK

Open Ckt Test



$V_{oc} = 240V$, $I_{oc} = 2.08A$, $P = 171W$, $Q = 469VAr$

$$P = \frac{\left[V_{oc}\left(\frac{7200}{240}\right)\right]^2}{R_m}, \quad R_m = \frac{\left[240\left(\frac{7200}{240}\right)\right]^2}{171} = 303K\Omega$$

$$Q = \frac{\left[V_{oc}\left(\frac{7200}{240}\right)\right]^2}{\omega L_m}, \quad L_m = \frac{\left[240\left(\frac{7200}{240}\right)\right]^2}{(469)(120\pi)} = 29H$$

Check $Z_m = R_m \| j\omega L_m$

$$Y_m = \frac{1}{R_m} + \frac{1}{j\omega L_m} = 3.30\times10^{-6} - j9.05\times10^{-6}, \quad Z_m = Y_m^{-1} = 103.8K\Omega\angle 70.0^\circ$$

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STAEDTLER® No. 937 811E Engineer's Computation Pad

HW4, cont

Reflect Z_M to 240V side

$$Z_{M_{240}} = (103.8\,k\Omega \angle 70.0^\circ)\left(\frac{240}{7200}\right)^2 = 115.3\angle 70.0^\circ\ \Omega$$

$$\tilde{I}_{oc} = \frac{\tilde{V}_{oc}}{Z_M} = \frac{240\angle 0}{115.3\angle 70.0} = 2.09\angle -70.0^\circ\ A \quad \text{OK}$$

$$S = V_{oc} I_{oc}^* = (240\angle 0)(2.09\angle -70.0)^* = 502\angle 70.0^\circ$$

$$= \underset{W}{171.7} + j\,\underset{VAr}{471.7} \quad \text{OK}$$

Comments – For the short ckt test, it took $\frac{216V}{7200V} = 0.0300$ pu voltage to get rated current (1 pu)

This means that the series impedance of the transformer is 0.0300 pu on the transf. base. The $\frac{X}{R}$ ratio is $\frac{0.1474(120\pi)}{27.8} = 2.00$

For the open circuit test, the current drawn was 2.08A. The rated current of the transf. on the 240V side is $\frac{25000}{240} = 104.2A$.

Thus, the pu magnetizing current is

$$\frac{2.08A}{104.2A} = 0.020\ pu.$$

The full-load P losses of the transf. would be

(335W as measured in S.C. test)

$$\left(\frac{25000}{7200}\right)^2 (27.8) + \underbrace{\frac{(7200)^2}{303k\Omega}}_{171W,\ \text{as meas. in O.C. test}} = 335 + 171 = 506\ W$$

Rated current on 7200V side

(About 2% of 25000)

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STAEDTLER® No. 937 811E Engineer's Computation Pad

EE369/394J, Test #4, Sept. 24, 2004. Name _____

One sheet of notes permitted. Show all steps.

The following data were taken for a 75kVA, 7200/480V transformer:

- Short Circuit Test: 480V side short-circuited, 7200V side energized at reduced voltage. The measurements (on the 7200V side) are Irms = 10.0A, Vrms = 144V, P = 750W.
- Open Circuit Test: 7200V side open-circuited, 480V side energized. The measurements (on the 480V side) are Vrms = 480V, Irms = 1.50A, P = 500W.

Draw the transformer equivalent circuit with all four circuit parameters (i.e., R, L_l, L_m, R_m) shown on the 7200V side. Hint – remember that $S = V_{rms} I_{rms}$, $S^2 = P^2 + Q^2$.

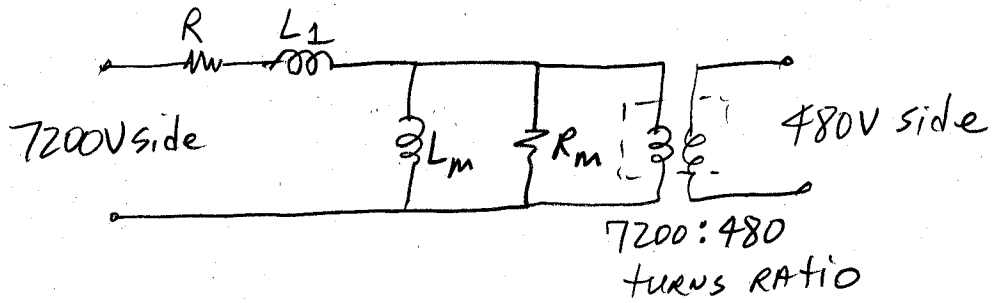
EE369/394J, Test #4, Sept. 24, 2004. Name KEY

One sheet of notes permitted. Show all steps.

The following data were taken for a 75kVA, 7200/480V transformer:

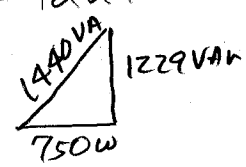
- Short Circuit Test: 480V side short-circuited, 7200V side energized at reduced voltage. The measurements (on the 7200V side) are Irms = 10.0A, Vrms = 144V, P = 750W.
- Open Circuit Test: 7200V side open-circuited, 480V side energized. The measurements (on the 480V side) are Vrms = 480V, Irms = 1.50A, P = 500W.

Draw the transformer equivalent circuit with all four circuit parameters (i.e., R, L_1, L_m, R_m) shown on the 7200V side. Hint – remember that $S = V_{rms} I_{rms}$, $S^2 = P^2 + Q^2$.



• Short Ckt Test (Measure R & L_1. Ignore L_m, R_m)

$S = V_{RMS} I_{RMS} = 1440$ VA. $P = 750$ W, $Q = \sqrt{S^2 - P^2} = 1229$ VAr



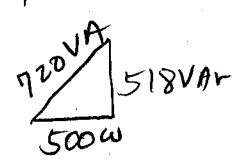
$I_{RMS}^2 R = P$, $I_{RMS}^2 X_1 = Q$

$R = \frac{750}{(10)^2} = 7.5\Omega$, $X_1 = \frac{1229}{(10)^2} = 12.29\Omega$,

$L_1 = \frac{X_1}{\omega} = 32.6$ mH

• Open Ckt Test (Measure L_m, R_m. Ignore R, L_1)

$S = V_{RMS} I_{RMS} = (480)(1.5) = 720$ VA. $Q = \sqrt{S^2 - P^2} = 518$ VAr



$\frac{(480)^2}{R_m} = P$. $\frac{(480)^2}{X_m} = Q$

R_m on 480 side, X_m on 480 side

$R_m = \frac{(480)^2}{500} = 461\Omega$, $X_m = \frac{(480)^2}{518} = 445\Omega$

Reflecting to 7200 V side

$R_m = (461)\left(\frac{7200}{480}\right)^2 = 103.7$ kΩ, $X_m = (445)\left(\frac{7200}{480}\right)^2 = 100.1$ kΩ

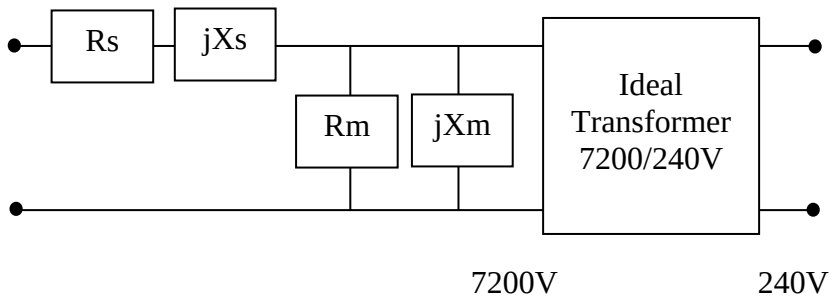
$L_m = \frac{X_m}{\omega} = 26$ H

Open circuit and short circuit tests are performed on a single-phase, 7200/240V, 25kVA, 60Hz distribution transformer. The results are:

- Short circuit test (short circuit the low-voltage side, energize the high-voltage side so that rated current flows, and measure P_{sc} and Q_{sc}). Measured P_{sc} = 400W, Q_{sc} = 200VAr.

- Open circuit test (open circuit the high-voltage side, apply rated voltage to the low-voltage side, and measure P_{oc} and Q_{oc}). Measured P_{oc} = 100W, Q_{oc} = 250VAr.

Determine the four impedance values (in ohms) for the transformer model shown.

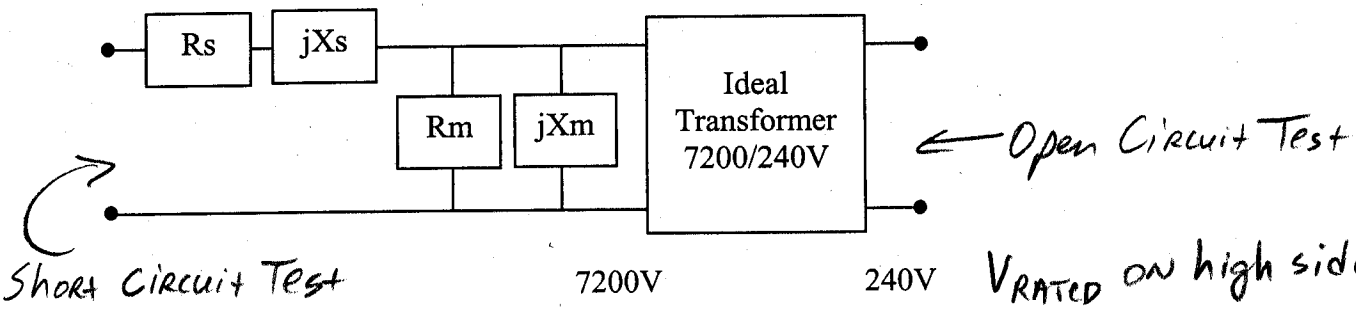


EE368, Test 1. March 3, 2006. One page of notes permitted. Name KEY
Show all work on these pages. Clearly show all steps needed to fully support your answers.
Four equally-weighted problems. a.

Problem 1. Open circuit and short circuit tests are performed on a single-phase, 7200/240V, 25kVA, 60Hz distribution transformer. The results are:

- Short circuit test (short circuit the low-voltage side, energize the high-voltage side so that rated current flows, and measure P_{sc} and Q_{sc}). Measured P_{sc} = 400W, Q_{sc} = 200VAr.
- Open circuit test (open circuit the high-voltage side, apply rated voltage to the low-voltage side, and measure P_{oc} and Q_{oc}). Measured P_{oc} = 100W, Q_{oc} = 250VAr.

Determine the four impedance values (in ohms) for the transformer model shown.



V_{RATED} on high side $= 7200V$

$$I_{RATED} = \frac{25000VA}{7200V} = 3.472A$$

$$I_{RATED}^2 R_s = P_{sc}, \quad R_s = \frac{P_{sc}}{I_{RATED}^2},$$

$$\boxed{R_s = \frac{400W}{(3.472A)^2} = 33.2\Omega \qquad X_s = \frac{Q_{sc}}{I_{RATED}^2} = 16.6\Omega}$$

$$P_{oc} = \frac{V_{RATED}^2}{R_m}$$

$$R_m = \frac{V_{RATED}^2}{P_{oc}}$$

$$\boxed{R_m = \frac{(7200)^2}{100W} = 518K\Omega}$$

$$X_m = \frac{V_{RATED}^2}{Q_{oc}}$$

$$\boxed{X_m = \frac{(7200)^2}{250} = 207K\Omega}$$

Other versions

(b) $R_s = \frac{350}{(3.472)^2} = 29.0\Omega$ $\quad R_m = \frac{(7200)^2}{80} = 648K\Omega$

$X_s = \frac{250}{(3.472)^2} = 20.7\Omega$ $\quad X_m = \frac{(7200)^2}{270} = 192K\Omega$

(c) $R_s = \frac{450}{(3.472)^2} = 37.3\Omega$ $\quad R_m = \frac{(7200)^2}{120} = 432K\Omega$

$X_s = \frac{150}{(3.472)^2} = 12.4\Omega$ $\quad X_m = \frac{(7200)^2}{230} = 225K\Omega$

Transformers

Transformers. Transformer phase shift. Wye-delta connections and impact on zero sequence. Inductance and capacitance calculations for transmission lines. GMR, GMD, L, and C matrices, effect of ground conductivity. Underground cables.

Equivalent Circuits

The standard transformer equivalent circuit used in power system simulation is shown below, where the R and X terms represent the series resistance and leakage reactance, and N1 and N2 represent the transformer turns. Note that the shunt terms are usually ignored in the model..

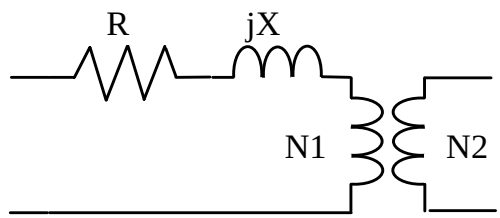


Figure 1. Power System Model for Transformer

Three-phase transformers can consist of either three separate single-phase transformers, or three windings on a three-legged, four-legged, or five-legged core. The high-voltage and low-voltage sides can be connected independently in either wye or delta.

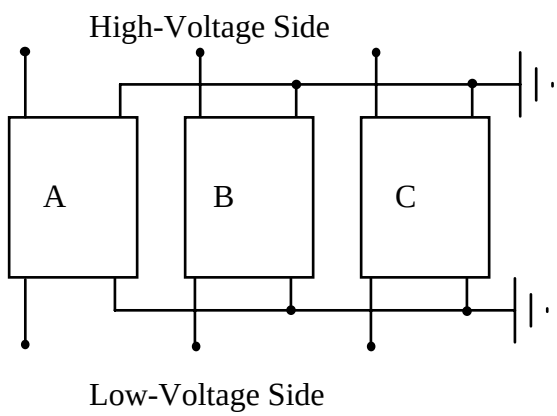


Figure 2. A Three-Phase Ground-Wye Grounded-Wye Transformer

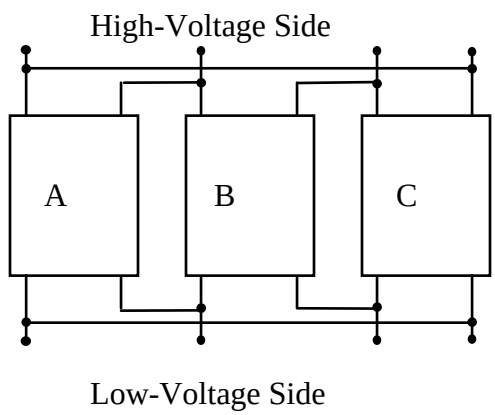


Figure 3. A Three-Phase Delta-Delta Transformer

The transformer impedances consist of winding resistances and leakage reactances. There are no mutual resistances, and the mutual leakage reactances between the separate phase a-b-c coils are negligible. Hence, in symmetrical components, $S = R + jX$, and $M = 0$, so that $S + 2M = S - M = R + jX$, so therefore the positive and negative sequence impedances of a transformer are

$$Z_1 = Z_2 = R + jX \quad .$$

One must remember that no zero sequence currents can flow into a three-wire connection. Therefore, the zero sequence impedance of a transformer depends on the winding connections. In the case where one side of a transformer is connected grounded-wye, and the other side is delta, circulating zero sequence currents can be induced in the delta winding. In that case, the zero sequence impedance "looking into" the transformer is different on the two sides.

The zero sequence equivalent circuits for three-phase transformers is given in Figure 4.

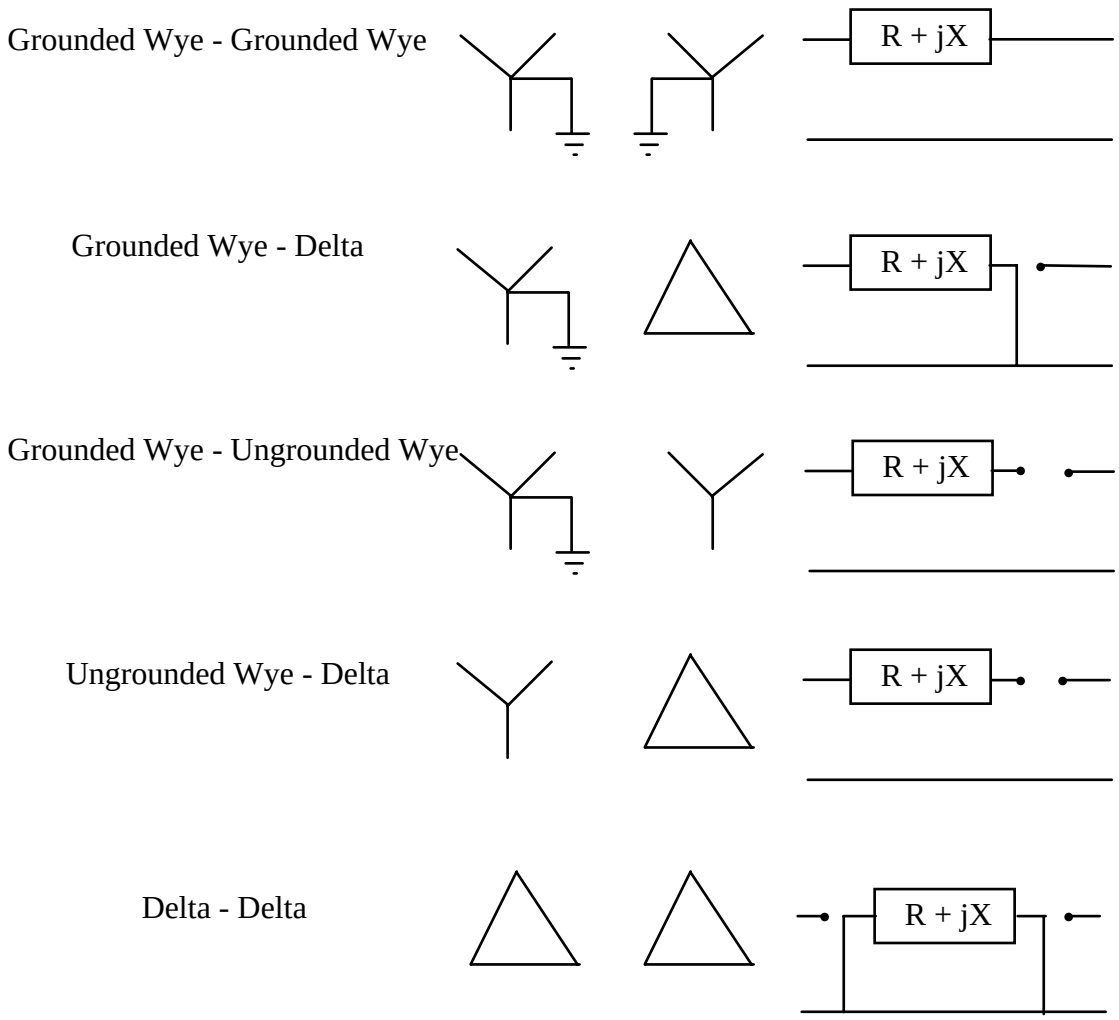


Figure 4. Zero Sequence Impedance Equivalent Circuits for Three-Phase Transformers

A wye-delta transformer connection introduces a 30° phase shift in positive/negative sequence voltages and currents because of the relative shift between line-to-neutral and line-to-ground voltages. Transformers are labeled so that

1. High side positive sequence voltages and currents lead those on the low side by 30°.

2. High side negative sequence voltages and currents lag those on the low side by 30°.

3. There is no phase shift for zero sequence.

Transformer tap magnitudes can be adjusted to control voltage, and transformer phase shifts can be adjusted to control active power flow. The effect of these "off-nominal" adjustments can be incorporated into a pi-equivalent circuit model for a transformer.

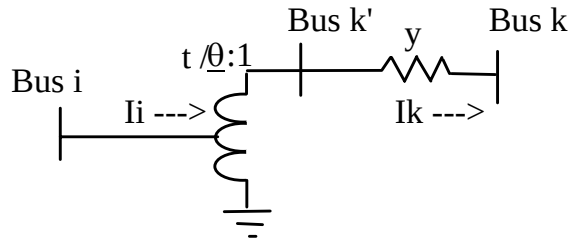


Figure 5. Off-Nominal Transformer Circuit Model

Assume that the transformer in Figure 5 has complex "off-nominal" tap $t\angle\theta_t$ and series admittance y. The relationship between the voltage on opposite sides of the transformer tap is $\tilde{V}_{k'} = \dfrac{\tilde{V}_i}{t\angle\theta_t}$, and since the power on both sides of the ideal transformer must be the same, then $\tilde{V}_i\tilde{I}_i^* = \tilde{V}_{k'}\tilde{I}_{k'}^*$, implying that $\tilde{I}_{k'} = \tilde{I}_i t\angle-\theta_t$. Now, suppose that the transformer can be modeled by the following pi-equivalent circuit of Figure 6:

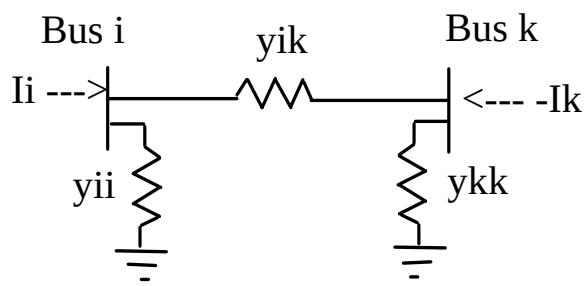


Figure 6. Pi-Equivalent Model of Transformer

Admittances y_{ii}, y_{ik}, and y_{kk} can be found so that the above circuit is equivalent to Figure 5. This can be accomplished by forcing the terminal behavior to be the same. For the above circuit, the appropriate equations are

$$\tilde{I}_i = \left(\tilde{V}_i - \tilde{V}_k\right)y_{ik} + \tilde{V}_i y_{ii} \text{ , and } -\tilde{I}_k = \left(\tilde{V}_k - \tilde{V}_i\right)y_{ik} + \tilde{V}_k y_{kk} \text{ ,}$$

or in matrix form

$$\begin{bmatrix} \tilde{I}_i \\ -\tilde{I}_k \end{bmatrix} = \begin{bmatrix} y_{ii} + y_{ik} & -y_{ik} \\ -y_{ik} & y_{kk} + y_{ik} \end{bmatrix} \begin{bmatrix} \tilde{V}_i \\ \tilde{V}_k \end{bmatrix} .$$

For Figure 5, the terminal equations are

$$\tilde{I}_k = \left(\tilde{V}_{k'} - \tilde{V}_k\right)y = \left(\frac{\tilde{V}_i}{t\angle\theta_t} - \tilde{V}_k\right)y \text{ ,}$$

and since

$$\tilde{I}_i = \frac{\tilde{I}_k}{t\angle -\theta_t} \ ,$$

then

$$\tilde{I}_i = \left(\frac{\tilde{V}_i}{t\angle\theta_t \bullet t\angle -\theta_t} - \frac{\tilde{V}_k}{t\angle -\theta_t} \right) y \ .$$

In matrix form,

$$\begin{bmatrix} \tilde{I}_i \\ -\tilde{I}_k \end{bmatrix} = \begin{bmatrix} \dfrac{y}{t\angle\theta_t \bullet t\angle -\theta_t} & \dfrac{-y}{t\angle -\theta_t} \\ \dfrac{-y}{t\angle\theta_t} & y \end{bmatrix} \begin{bmatrix} \tilde{V}_i \\ \tilde{V}_k \end{bmatrix} \ .$$

Comparing the two sets of terminal equations shows that equality can be reached if the shunt branch in the equivalent circuit, y_{ik}, can have two values:

$y_{ik} = \dfrac{y}{t\angle -\theta_t}$ from the perspective of Kirchhoff's current law at bus i, $y_{ik} = \dfrac{y}{t\angle\theta_t}$ from the perspective of Kirchhoff's current law at bus k.

Note that if the tap does not include an off-nominal phase shift, then $y_{ik} = \dfrac{y}{t}$ from either direction.

Next, solving for y_{ii} and y_{kk} yields

$$y_{ii} = \frac{y}{t\angle\theta_t \bullet t\angle -\theta_t} - \frac{y}{t\angle -\theta_t} = \frac{y}{t\angle -\theta_t}\left(\frac{1}{t\angle\theta_t} - 1 \right) ,$$

$$y_{kk} = y - \frac{y}{t\angle\theta_t} = y\left(1 - \frac{1}{t\angle\theta_t} \right) .$$

Neutral Grounding Impedance

If the wye-side of a transformer or wye-connected load is grounded through a grounding impedance Z_g, the grounding impedance is "invisible" to the positive and negative sequence currents since their corresponding voltages at the wye-point is always zero due to symmetry. However, since the neutral current is three-times the zero sequence current, the voltage drop on the grounding impedance is $3I_{ao}$. For that reason, the zero sequence equivalent circuit for a grounding impedance must contain $3Z_g$.

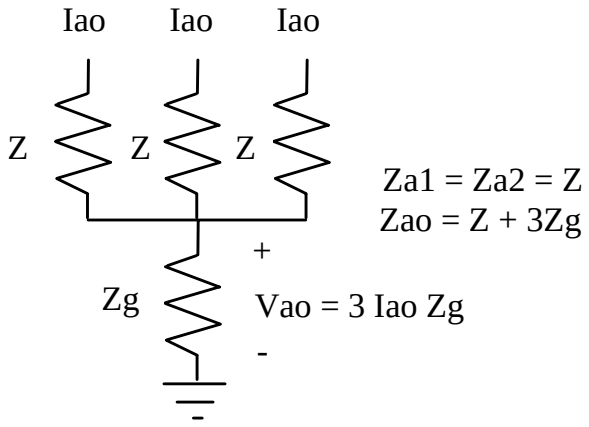
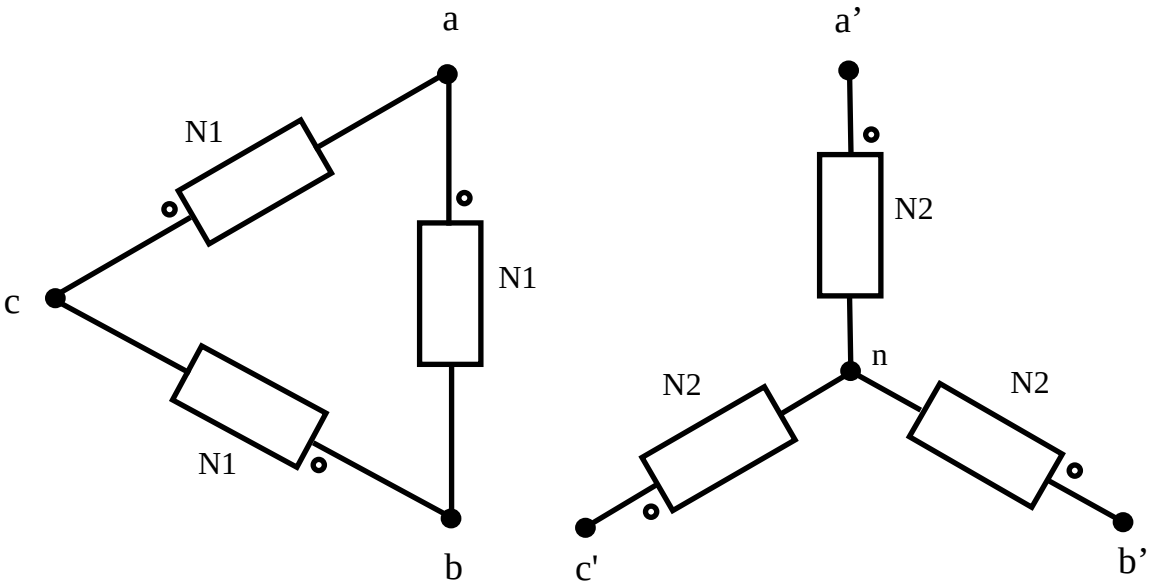
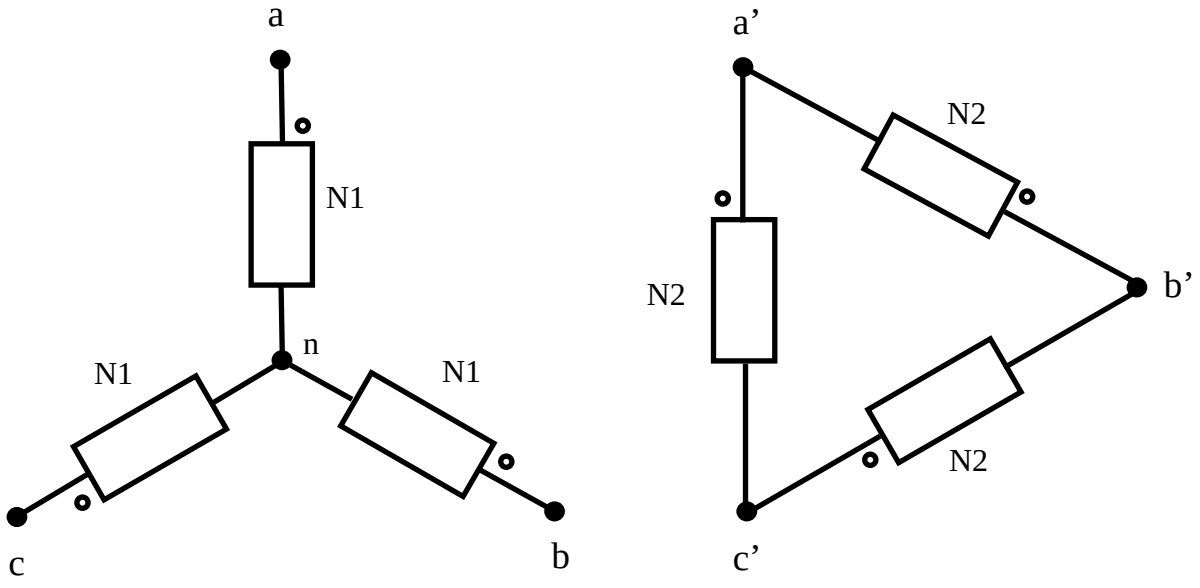


Figure 7. Effect of Grounding Impedance on Sequence Impedances



LV **HV**



LV **HV**

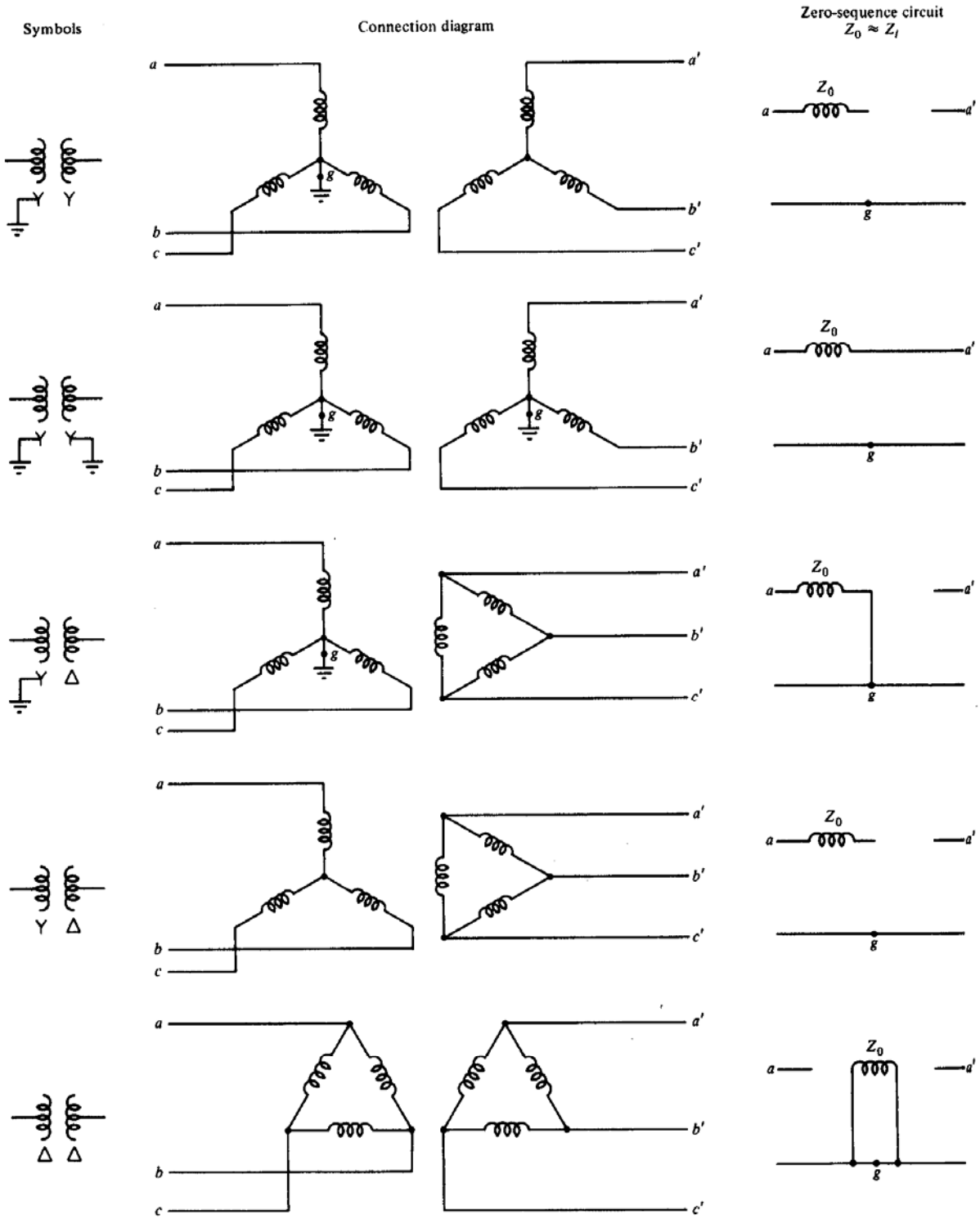


Figure 12.20 Zero-sequence equivalent circuits. (Adapted from W. D. Stevenson, Jr., *Elements of Power Systems Analysis,* 4th ed., McGraw-Hill Book Company, New York, 1982.)

1. Single-Phase Per Unit System

Advantages of the per unit system:

1. Transformers can be replaced by their equivalent series impedances.

2. Equipment impedances can be easily estimated since their per unit impedances lie within a relatively narrow range.

Define four quantities in per unit of their respective base values:

$$V_{pu} = \frac{V}{V_{base}} \frac{volts}{volts} , \qquad I_{pu} = \frac{I}{I_{base}} \frac{amps}{amps} ,$$

$$S_{pu} = \frac{S}{S_{base}} \frac{voltamps}{voltamps} , \qquad Z_{pu} = \frac{Z}{Z_{base}} \frac{ohms}{ohms} .$$

The relationships among the bases are

$$S_{base} = V_{base} \bullet I_{base} \text{ , and } Z_{base} = \frac{V_{base}}{I_{base}} .$$

Once two base variables are specified, the other two base variables may be calculated.

A convenient relation, derived from the two above equations, is

$$S_{base} = V_{base} \bullet I_{base} = V_{base} \bullet \frac{V_{base}}{Z_{base}} = \frac{V_{base}^2}{Z_{base}} .$$

Consider a transformer with an $N_1 : N_2$ turns ratio and series impedance, reflected on side 1, equal Z_{L1} . Z_{L1} can be reflected to side 2 using

$$Z_{L2} = \left[\frac{N_2}{N_1} \right]^2 Z_{L1} .$$

Let side 1 and side 2 have base values designated by subscripts S1 and S2. Then

$$Z_{base1} = \frac{V_{base1}^2}{S_{base1}} , \qquad Z_{base2} = \frac{V_{base2}^2}{S_{base2}} ,$$

Expressing the transformer impedance on the two respective bases yields

$$Z_{L1PU} = \frac{Z_{L1} S_{base1}}{V_{base1}^2} \ , \qquad Z_{L2PU} = \frac{Z_{L2} S_{base2}}{V_{base2}^2} \ .$$

If $S_{B1} = S_{B2}$, the two above equations may be combined so that

$$Z_{L2PU} = \left[\frac{N_2}{N_1}\right]^2 \left[\frac{V_{base1}}{V_{base2}}\right]^2 Z_{L1PU} \ .$$

Substituting the relation between Z_{L1} and Z_{L2} yields

$$Z_{L2PU} = \left[\frac{N_2}{N_1}\right]^2 \left[\frac{V_{base1}}{V_{base2}}\right]^2 Z_{L1PU} \ .$$

Therefore, if $\frac{V_{base2}}{V_{base1}} = \frac{N_2}{N_1}$, then $Z_{L2PU} = Z_{L1PU}$.

Hence, if a common voltampere base is chosen on both sides of the transformer, and if the voltage bases are chosen so that they vary according to the transformer turns ratio, then the per unit series impedance of the transformer is the same value on both sides.

When analyzing a circuit with many transformers, a common voltampere base should be chosen throughout the circuit, and a voltage base should be chosen at one location. The voltage base must vary across the circuit according to the transformer turns ratios.

When analyzing a circuit in per unit, if the bases are chosen according to the above rules, transformers can be replaced by their equivalent per unit series impedances, and their turns can be ignored.

A manufacturer usually provides the impedance of a transformer on the transformer's rated voltage and power bases. However, when solving a power network circuit, the power and voltage bases must vary according to the above rules, and they may not equal the manufacturer-specified bases. Per unit impedances, specified on one base, may be converted to a new base as follows:

Given $Z_{PU,old} = \frac{Z}{Z_{base,old}}$, on bases $V_{base,old}$ and $S_{base,old}$, $Z_{PU,new}$ on new bases $V_{base,new}$ and $S_{base,new}$ is

$$Z_{PU,new} = \frac{Z_{ohms}}{Z_{base,new}} = \frac{Z_{PU,old} Z_{base,old}}{Z_{base,new}} = Z_{PU,old} \left[\frac{V_{base,old}^2}{S_{base,old}}\right]\left[\frac{S_{base,new}}{V_{base,new}^2}\right] \ .$$

2. Three-Phase Per Unit System

The same advantages apply to a three-phase system if the following rules are obeyed:

1. A common three-phase voltampere base is used throughout the system, where

$$S_{base,3\Phi} = 3S_{base,1\Phi} \ .$$

2. Once selected at a point in the network, the three-phase voltage base must vary according to the line-to-line transformer turns ratios.

Convenient formulas relating single-phase to three-phase bases are given below.

$$S_{base,1\phi} = V_{base,line-neutral} \bullet I_{base} \ ,$$

$$S_{base,3\phi} = 3S_{base,1\phi} \ ,$$

$$Z_{base} = \frac{V_{base,line-neutral}^2}{S_{base,1\phi}} = \frac{\left[V_{base,line-line} / \sqrt{3}\right]^2}{S_{base,3\phi} / 3} = \frac{V_{base,line-line}^2}{S_{base,3\phi}} \ .$$

PU System

Single Phase

$V_{B,LN}$ $\quad V_{pu} = \frac{V}{V_B}$, etc

I_B $\quad I_{pu} = \frac{I}{I_B}$

$Z_B = \frac{V_{B,LN}}{I_B}$ $\quad Z_{pu} = \frac{Z}{Z_B}$

$S_{B_{1\phi}} = V_{B,LN} I_B$ $\quad S_{pu} = \frac{S}{S_B}$

Pick two (usually $V_{B,LN}$, $S_{B1\phi}$), then other two known

Handy Formula

$$Z_B = \frac{V_{B,LN}}{I_B} = \frac{V_{B,LN}}{S_{B,1\phi}/V_{B,LN}}$$

$$Z_B = \frac{(V_{B,LN})^2}{S_{B,1\phi}}$$

Three Phase

$V_{B,LN}$ $\quad (V_{B,LL} = \sqrt{3}\, V_{B,LN})$

I_B

$$Z_B = \frac{V_{B,LN}}{I_B} = \frac{V_{B,LL}}{\sqrt{3}\, I_B}$$

$$S_{B_{3\phi}} = 3 S_{B_{1\phi}} = 3 V_{B,LN} I_B$$

$$= \sqrt{3}\, V_{B,LL} I_B$$

Handy Formula

$$Z_B = \frac{V_{B,LN}}{I_B} = \frac{V_{B,LN}}{S_{B,1\phi}/V_{B,LN}}$$

$$= \frac{(V_{B,LN})^2}{S_{B,1\phi}} = \frac{\left(\frac{V_{B,LL}}{\sqrt{3}}\right)^2}{S_{B,1\phi}}$$

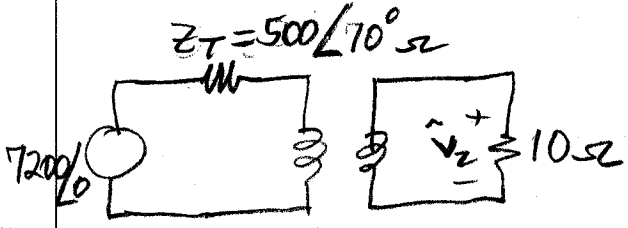
$$Z_B = \frac{V_{B,LL}^2}{3 S_{B,1\phi}} = \frac{V_{B,LL}^2}{S_{B,3\phi}}$$

As shown with transformer models, Z_{Ω} reflects by square of line-to-line turns ratio. So, if V_B varies by this ratio, then Z_{pu} will be same on both sides, if S_B same on both sides.

A plus – impedances of equipment tend to fall in narrow ranges when expressed in pu on equipment S_B.

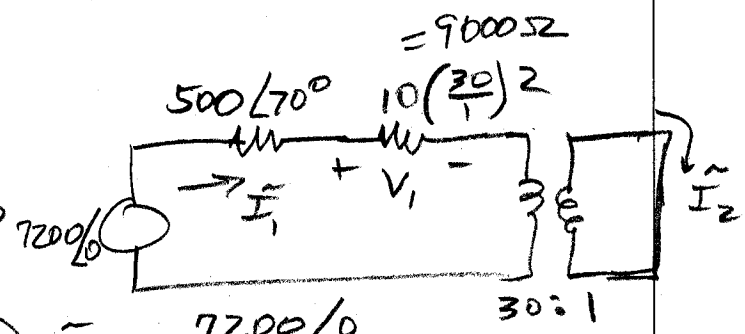
13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Sample Transformer Problem



7200:240
5KVA

$\Longrightarrow$



(1) $$\tilde{I}_1 = \frac{7200\angle 0}{171 + j470 + 9000}$$

$$\tilde{I}_1 = 0.784\angle -2.9^\circ\ A$$

$$\left.\begin{aligned} \hat{I}_2 &= (30)\tilde{I}_1 \\ \hat{V}_2 &= \frac{\hat{V}_1}{30} \end{aligned}\right] \quad \tilde{V}_2 \tilde{I}_2^* = \hat{V}_1 \tilde{I}_1^*$$

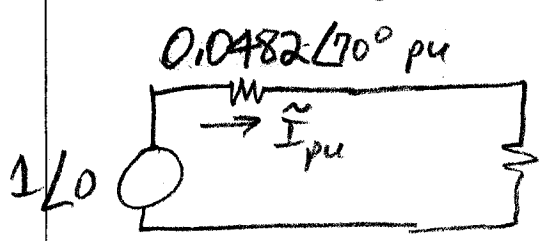
In pu

Let $V_{B1} = 7200$, $V_{B2} = V_{B1}\left(\frac{240}{7200}\right) = 240$

$S_B = 5000$, $I_{B1} = \frac{5000}{7200} = 0.694$

$Z_{B1} = \frac{(V_{B1})^2}{S_B} = 10368\ \Omega$ $\quad Z_{B2} = \frac{(240)^2}{5000} = 11.52$

900:1 RATIO



$$\tilde{I}_{pu} = \frac{1\angle 0}{0.0165 + j0.0453 + 0.868} =$$

$$\tilde{I}_{pu} = \frac{1\angle 0}{0.886\angle 2.93} = 1.13\angle -2.93$$

$$\tilde{I}_1 = \tilde{I}_{pu} I_{B1} = (1.13\angle -2.93)(0.694) = 0.784\angle -2.9\ A$$

Now, must follow rules on V_B varying by line-to-line turns ratio, S_B same throughout.

For individual equipment, impedance often given on equipment base. Must convert

$$Z_\Omega = Z_{pu} Z_B = Z_{pu,OLD} Z_{B,OLD} = Z_{pu,NEW} Z_{B,NEW}$$

so, $$Z_{pu,NEW} = Z_{pu,OLD} \frac{Z_{B,OLD}}{Z_{B,NEW}}$$

But $$Z_{B,OLD} = \frac{V_{BLL,OLD}^2}{S_{B3\phi,OLD}}, \quad Z_{B,NEW} = \frac{V_{BLL,NEW}^2}{S_{B3\phi,NEW}}$$

so $$\boxed{Z_{pu,NEW} = Z_{pu,OLD} \left[\frac{S_{B3\phi,NEW}}{S_{B3\phi,OLD}}\right] \left[\frac{V_{BLL,OLD}^2}{V_{BLL,NEW}^2}\right]}$$

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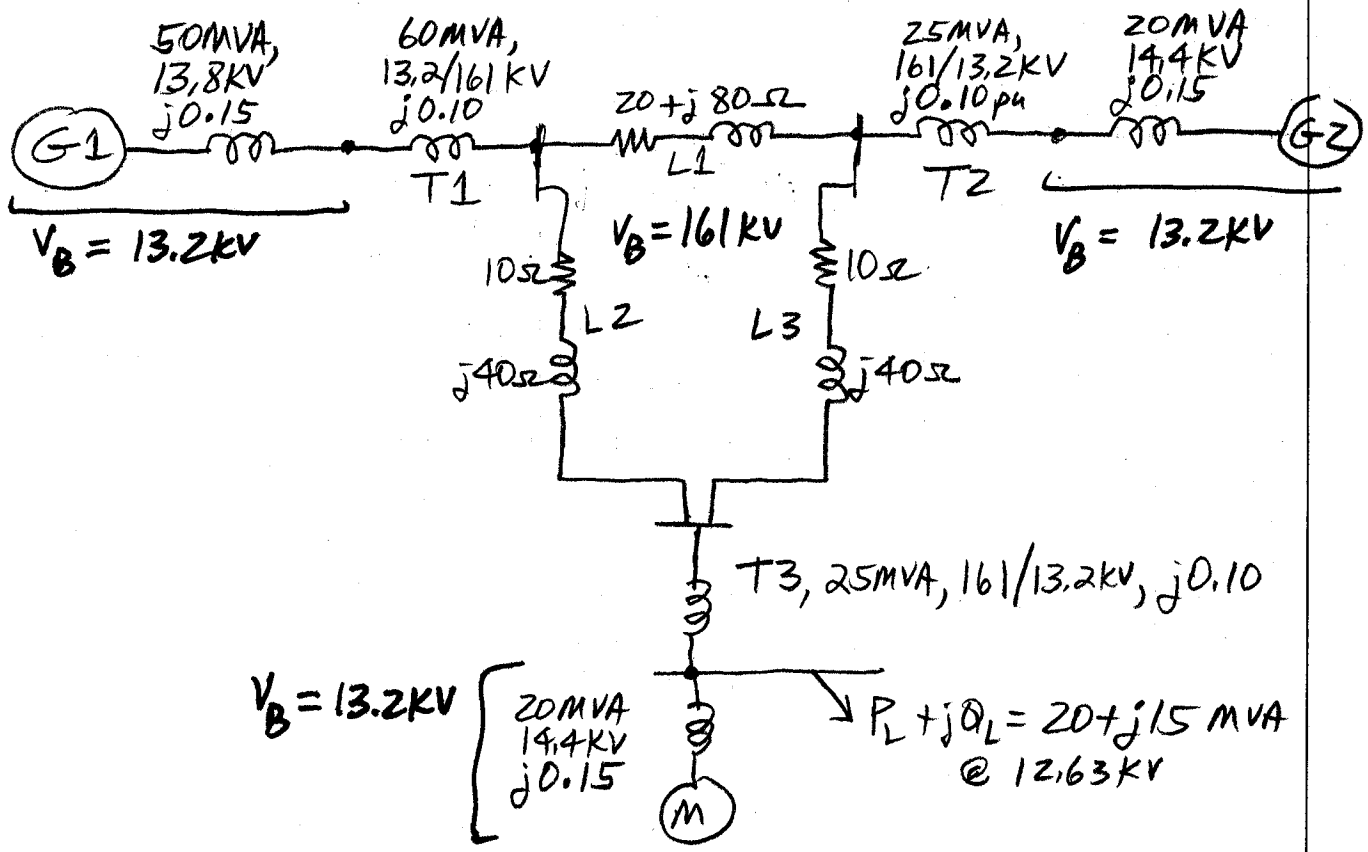
5.19 Use 161 kV base voltage in transmission lines (wasn't stated in problem), and 100 MVA



- Setting $V_B = 161$ kV in the transmission lines, the bases as you step across a transformer follow the line-to-line turns ratio
- Now, component by component

$$Z_{G1} = j0.15\left(\frac{100}{50}\right)\left(\frac{13.8}{13.2}\right)^2 = j0.328$$

$$Z_{T1} = j0.10\left(\frac{100}{60}\right)\left(\frac{13.2}{13.2}\right)^2 = j0.1667$$

For transmission lines, $Z_B = \frac{161^2}{100} = 259\,\Omega$

$$Z_{L1} = \frac{20+j80}{259} = 0.0772 + j0.309$$

$$Z_{L2} = Z_{L3} = \frac{10+j40}{259} = 0.0386 + j0.1544$$

$$Z_{T2} = j0.10\left(\frac{100}{25}\right)\left(\frac{13.2}{13.2}\right)^2 = j0.40$$

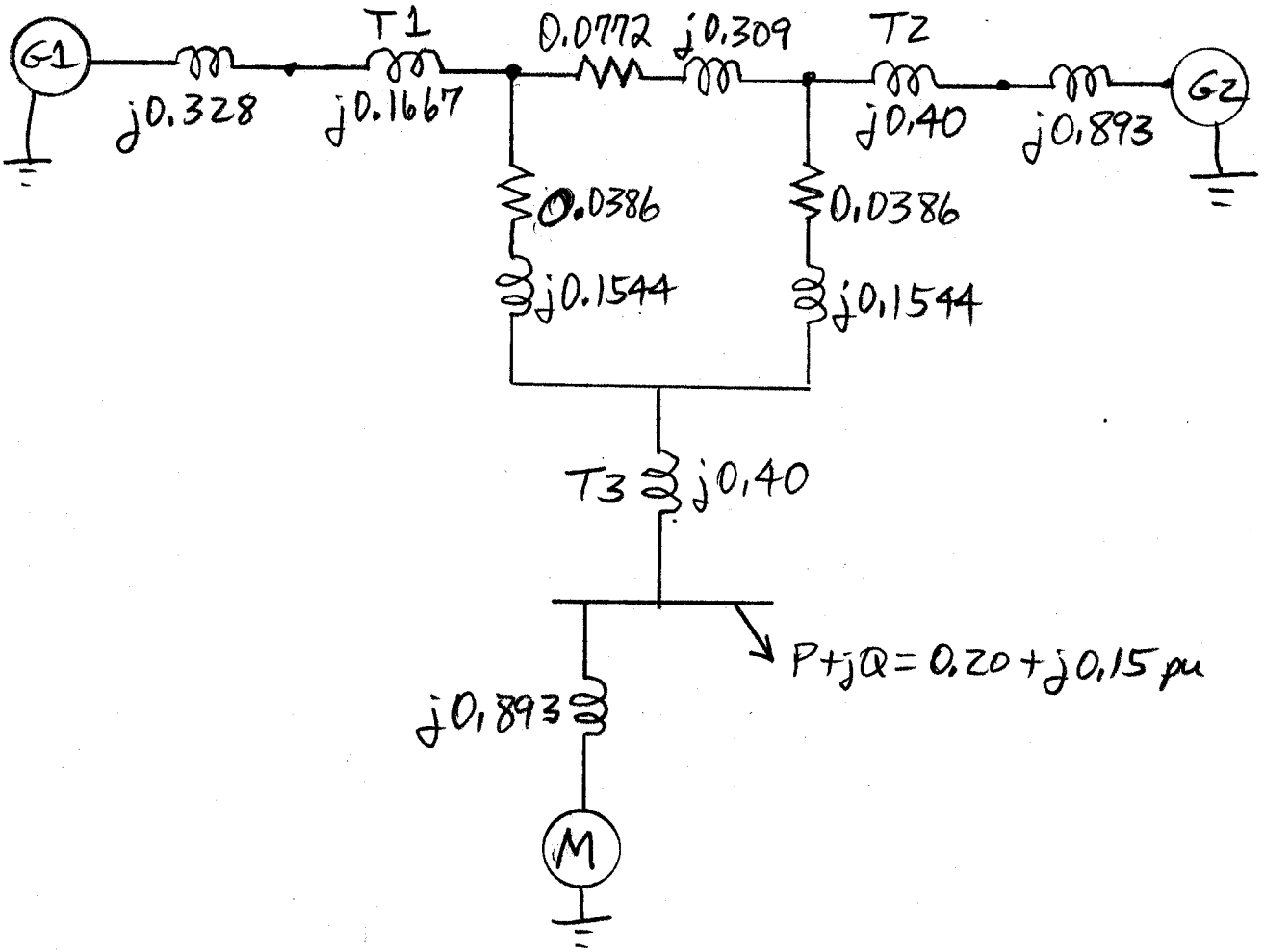
5.19, cont.

$$Z_{G2} = j0.15\left(\frac{100}{20}\right)\left(\frac{14.4}{13.2}\right)^2 = j0.893$$

$$Z_{T3} = j0.10\left(\frac{100}{25}\right)\left(\frac{13.2}{13.2}\right)^2 = j0.40$$

$$Z_M = j0.15\left(\frac{100}{20}\right)\left(\frac{14.4}{13.2}\right)^2 = j0.893$$

For load, we assume constant power, so $P + jQ = 0.20 + j0.15$



G1, G2, M are represented as voltage sources. Since the network is loaded with $P + jQ$, then we can set the V/S at one gen, but the other gen & motor voltages must be solved by a load flow program where gen & motor P, Q are known

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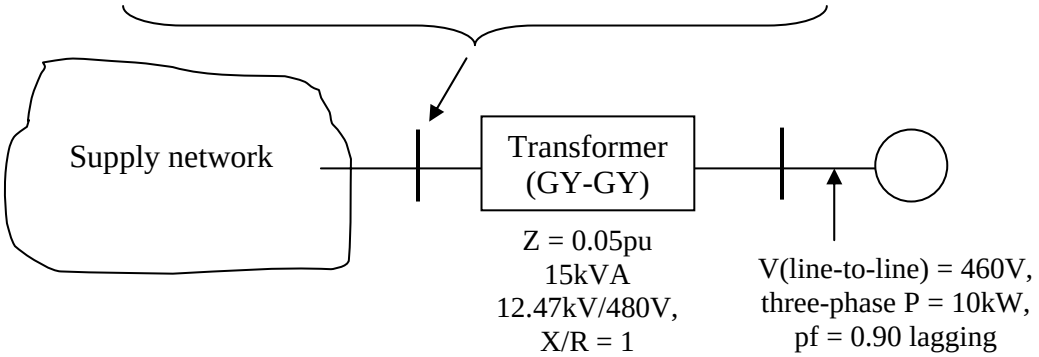
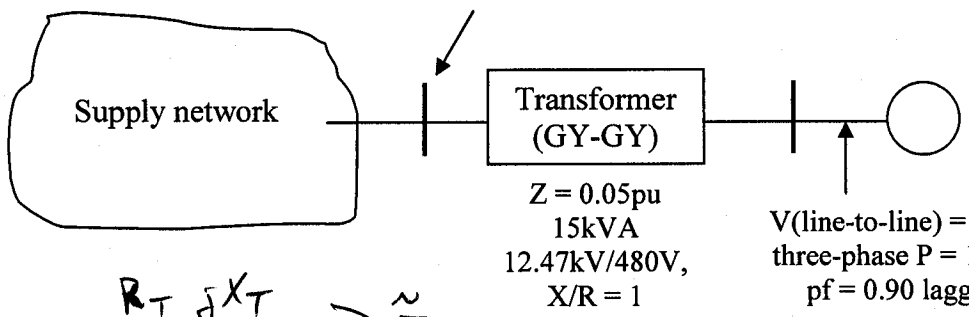
Find the magnitude of the line-to-line voltages on the 12.47kV side of the transformer.



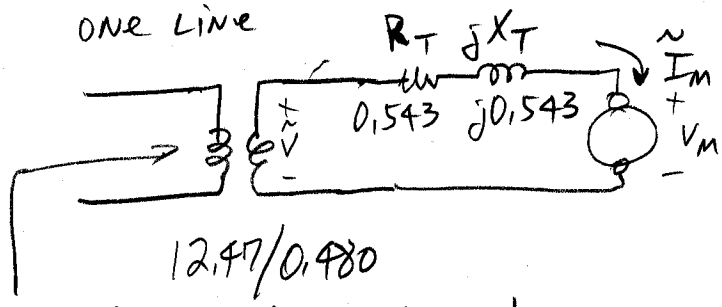
EE369, Final Exam, 2-5pm, Dec. 9, WRW102. Name: KEY

Six equally-weighted problems. Three sheets of notes permitted. Show all work.

Problem 3. Find the magnitude of the line-to-line voltages on the 12.47kV side of the transformer.



ONE LINE

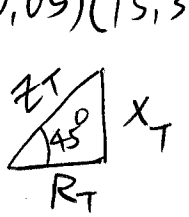


Want line-to-line voltage here

$R_T + jX_T$ is transformer impedance on 480V side

$$Z_B = \frac{(480)^2}{15000} = 15.36\,\Omega$$

$$Z_T = (0.05)(15.36) = 0.768\,\Omega$$



$$\frac{X_T}{R_T} = 1,$$

so 45°, so

$$R_T = X_T = \frac{Z_T}{\sqrt{2}}$$

$$R_T = X_T = 0.543$$

$$V_{M_{LN}} I_M^* = S_{M_{1\phi}}, \quad I_M = \frac{S_{M_{1\phi}}^*}{V_{M_{LN}}^*}$$

$$I_M = \frac{\left[\frac{10000}{3(0.9)} \angle \cos^{-1}(0.90)\right]^*}{\frac{460}{\sqrt{3}} \angle 0} = 13.95 \angle -25.8^\circ \text{ A}$$

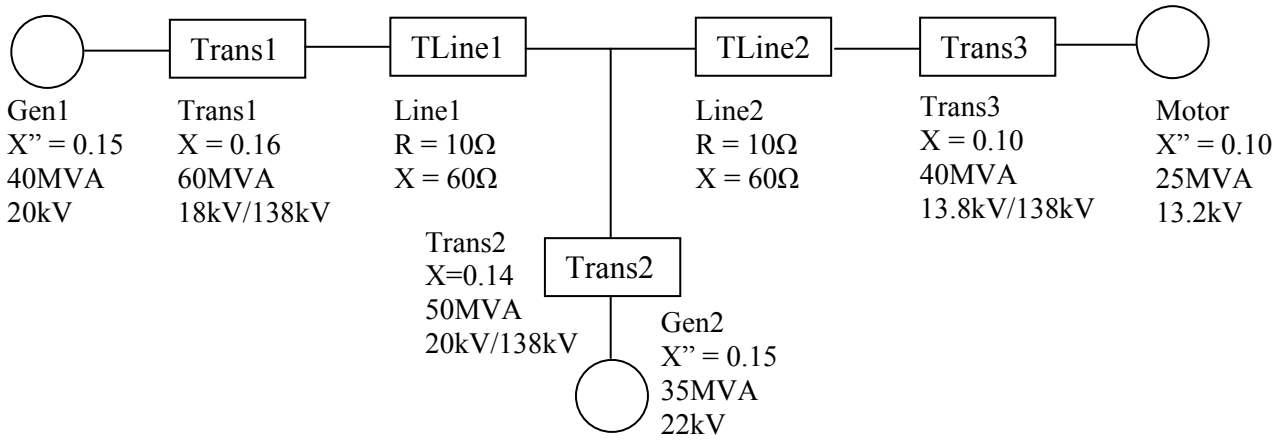
$$\tilde{V} = \tilde{V}_M + \tilde{I}_M Z_T = \frac{460}{\sqrt{3}} \angle 0 + (13.95 \angle -25.8)(0.768 \angle 45^\circ)$$

$$= 266 + 10.71 \angle -19.2 = 266 + 10.11 - j3.52 = 276 - j3.52$$

$\tilde{V} = 276 \angle -0.73$, so on the 12.47kV side,

$$|\tilde{V}_{LL}| = (\sqrt{3})(276)\left(\frac{12.47}{0.480}\right) = \boxed{12419 \text{ V}}$$

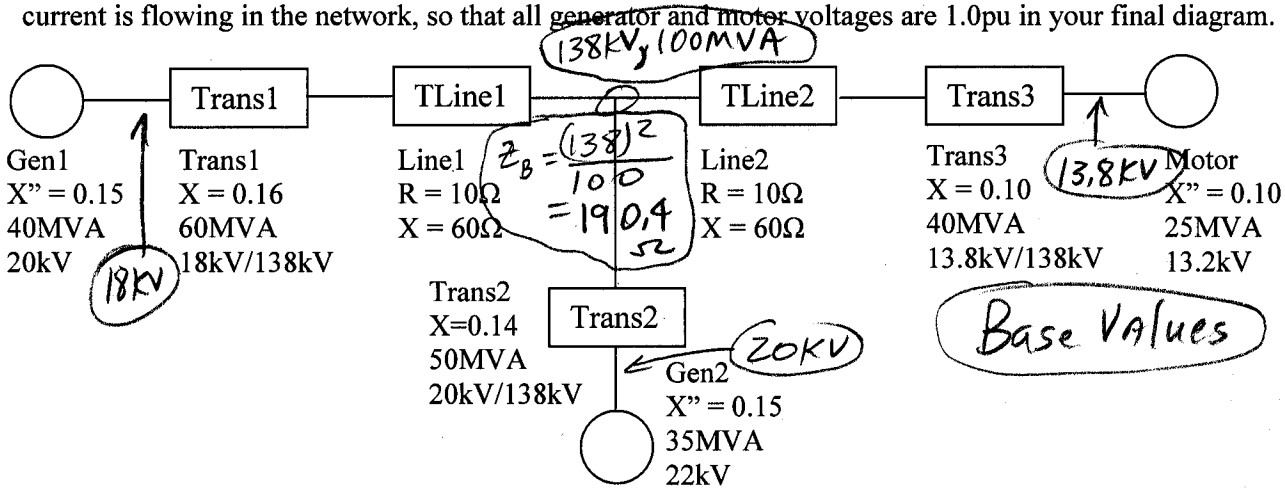
Information for a small power system is shown below. Per unit values are given on the equipment bases. Using a 138kV, 100MVA base in the transmission lines, draw the per unit diagram. Assume that no current is flowing in the network, so that all generator and motor voltages are 1.0pu in your final diagram.

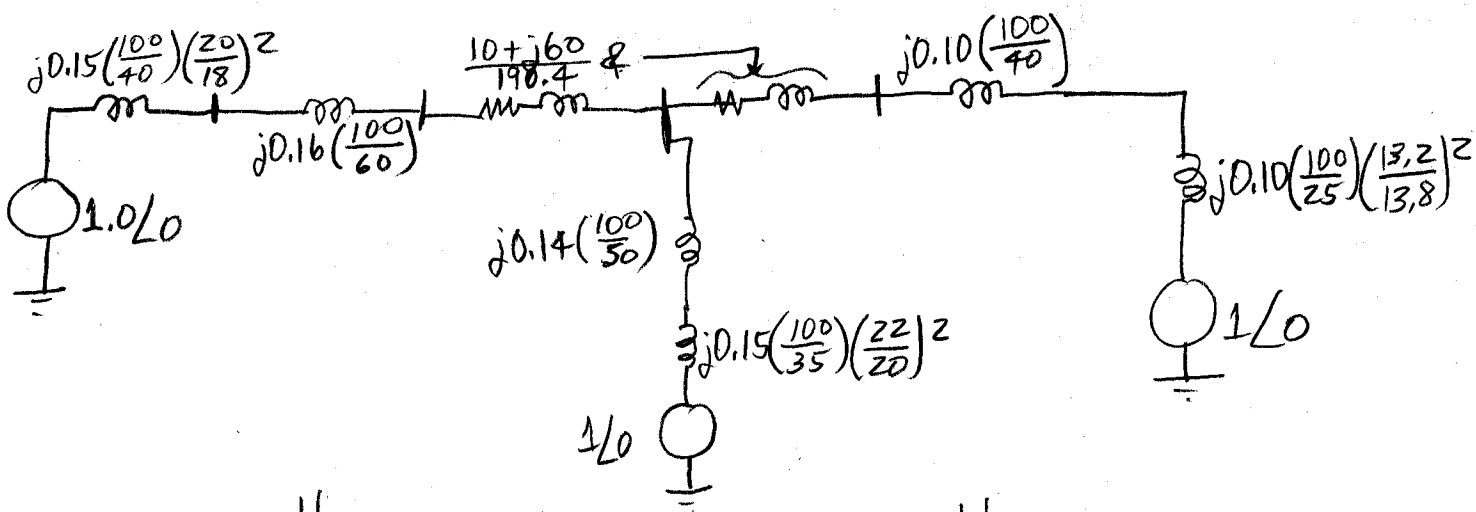


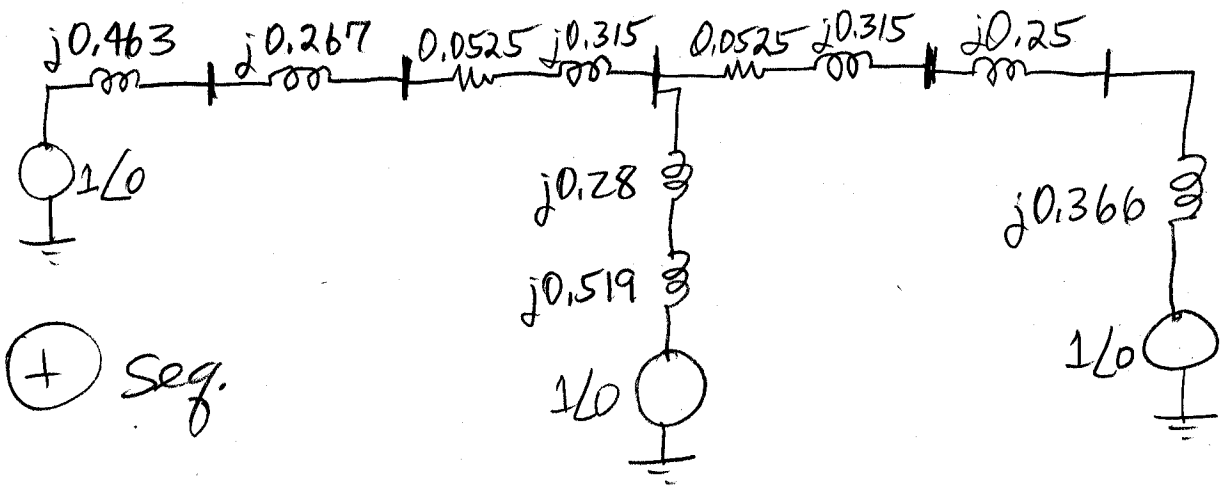
EE369, Test 5, Oct. 1, 2004. Name: KEY

One Sheet of Notes Permitted. Show all Work of This Page

Information for a small power system is shown below. Per unit values are given on the equipment bases. Using a 138kV, 100MVA base in the transmission lines, draw the per unit diagram. Assume that no current is flowing in the network, so that all generator and motor voltages are 1.0pu in your final diagram.







6.15 The one-line diagram of an unloaded power system is shown in Fig. 6.31. Reactances of the two sections of transmission line are shown on the diagram. The generators and transformers are rated as follows:

Generator 1:	20 MVA, 13.8 kV, $X'' = 0.20$ per unit
Generator 2:	30 MVA, 18 kV, $X'' = 0.20$ per unit
Generator 3:	30 MVA, 20 kV, $X'' = 0.20$ per unit
Transformer T_1:	25 MVA, 220Y/13.8Δ kV, $X = 10\%$
Transformer T_2:	Single-phase units each rated 10 MVA, 127/18 kV, $X = 10\%$
Transformer T_3:	35 MVA, 220Y/22Y kV, $X = 10\%$

Draw the impedance diagram with all reactances marked in per unit and with letters to indicate points corresponding to the one-line diagram. Choose a base of 50 MVA, 13.8 kV in the circuit of generator 1.

6.16 Draw the impedance diagram for the power system shown in Fig. 6.32. Mark impedances in per unit. Neglect resistance, and use a base of 50 kVA, 138 kV in the 40-Ω line. The ratings of the generators, motors, and transformers are:

Should be 50 MVA

Generator 1:	20 MVA, 18 kV, $X'' = 20\%$
Generator 2:	20 MVA, 18 kV, $X'' = 20\%$
Synchronous motor 3:	30 MVA, 13.8 kV, $X'' = 20\%$
Three-phase Y-Y transformers:	20 MVA, 138Y/20Y kV, $X = 10\%$
Three-phase Y-Δ transformers:	15 MVA, 138Y/13.8Δ kV, $X = 10\%$

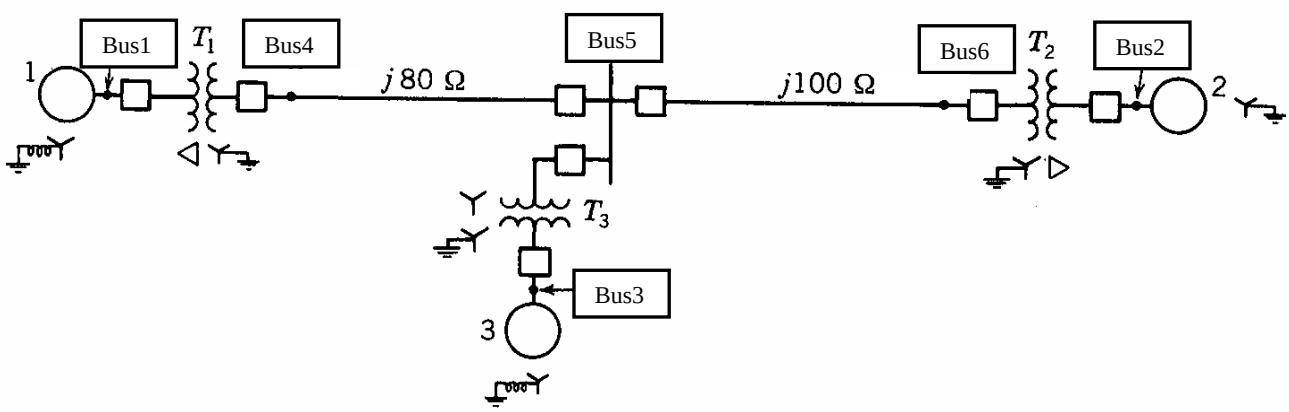


Figure 6.31 One-line diagram for Prob. 6.15.

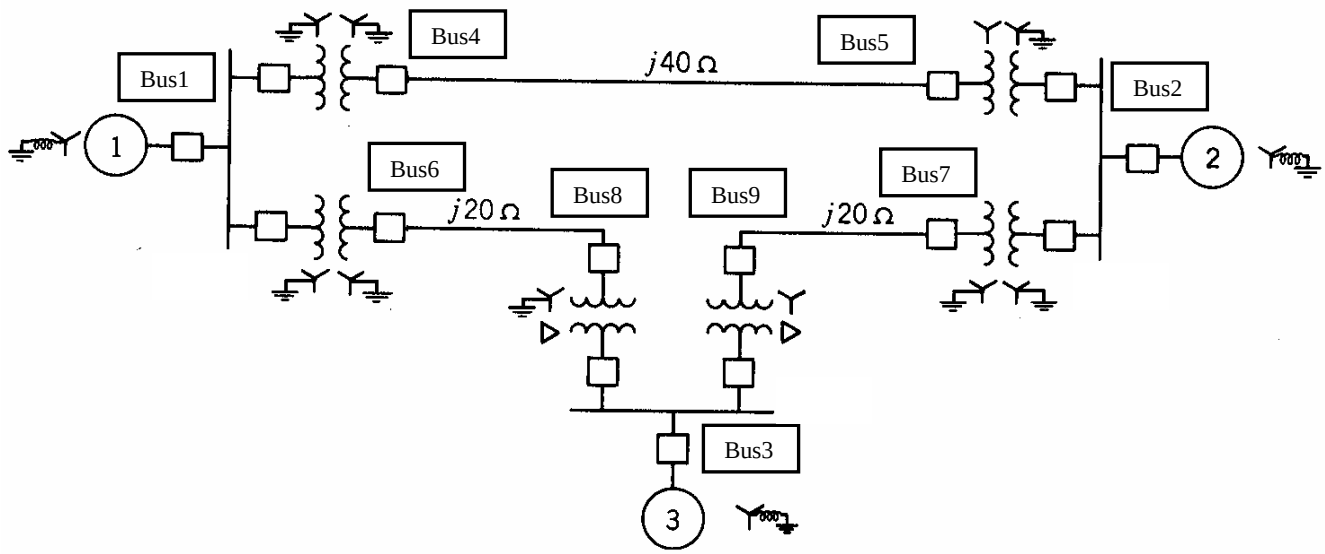


Figure 6.32 One-line diagram for Prob. 6.16.

11.11 Draw the negative- and zero-sequence impedance networks for the power system of Prob. 6.15. Mark the values of all reactances in per unit on a base of 50 MVA, 13.8 kV in the circuit of generator 1. Letter the networks to correspond to the one-line diagram. The neutrals of generators 1 and 3 are connected to ground through current-limiting reactors having a reactance of 5%, each on the base of the machine to which it is connected. Each generator has negative- and zero-sequence reactances of 20 and 5%, respectively, on its own rating as base. The zero-sequence reactance of the transmission line is 210 Ω from *B* to *C* and 250 Ω from *C* to *E*.

11.12 Draw the negative- and zero-sequence impedance networks for the power system of Prob. 6.16. Choose a base of 50 MVA, 138 kV in the 40-Ω transmission line, and mark all reactances in per unit. The negative-sequence reactance of each synchronous machine is equal to its subtransient reactance. The zero-sequence reactance of each machine is 8% based on its own rating. The neutrals of the machines are connected to ground through current-limiting reactors having a reactance of 5%, each on the base of the machine to which it is connected. Assume that the zero-sequence reactances of the transmission lines are 300% of their positive-sequence reactances.

Short Circuit Calculation

(Based upon Grainger/Stevenson Problem 6.15/11.11)

SC Problem 1. A three-phase balanced fault, with $Z_F = 0$, occurs at Bus 4. Determine

a. I_{4a}^{F} (in per unit and in amps)
b. Phasor abc line-to-neutral voltages at the terminals of Gen 1
c. Phasor abc currents flowing out of Gen 1 (in per unit and in amps)

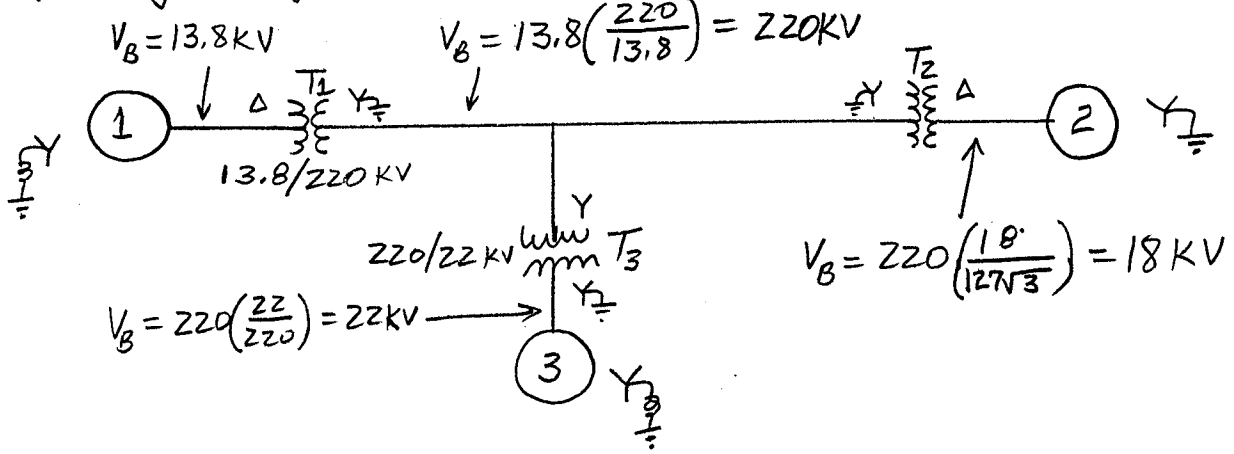
SC Problem 2. Repeat Problem 1, again with $Z_F = 0$, but with a single-phase-to-ground fault at Bus 4, phase a.

In both Problems 1 and 2, make sure that you add in the transformer phase shifts, assuming that pre-fault Van in the transmission line has reference phase angle 0.

Problem 6.15, page 164, Stevenson

Step 1: Establish the base values throughout the network.

You are given that $P_B = 50\,MVA$, $V_B = 13.8\,KV$ (line-to-line) at generator 1. Therefore, $P_B = 50\,MVA$ everywhere in the network, and V_B is adjusted by the line-to-line voltage ratings of the transformers, beginning at generator 1.



Step 2: Convert all impedances to per-unit values

$$Z_{pu} = \frac{Z}{Z_B}, \quad Z_B = \frac{V_{B(KV)}^2}{P_{B(MVA)}}, \text{ where } V_B \text{ is line-to-line}$$

For change of bases, $Z_{pu}^{NEW} = Z_{pu}^{OLD}\left[\frac{P_B^{NEW}}{P_B^{OLD}}\right]\left[\frac{V_B^{OLD}}{V_B^{NEW}}\right]^2$

Generator 1 – $Z_{pu}^{NEW} = j0.20\left[\frac{50}{20}\right]\left[\frac{13.8}{13.8}\right]^2 = j0.50$

Generator 2 – $Z_{pu}^{NEW} = j0.20\left[\frac{50}{30}\right]\left[\frac{18}{18}\right]^2 = j0.3333$

Generator 3 – $Z_{pu}^{NEW} = j0.20\left[\frac{50}{30}\right]\left[\frac{20}{22}\right]^2 = j0.2755$

6.15, cont

Transformer 1 — $Z_{pu}^{new} = j0.10\left[\frac{50}{25}\right] = j0.20$

Transformer 2 — $Z_{pu}^{new} = j0.10\left[\frac{50}{30}\right] = j0.1667$

Transformer 3 — $Z_{pu}^{new} = j0.10\left[\frac{50}{35}\right] = j0.1429$

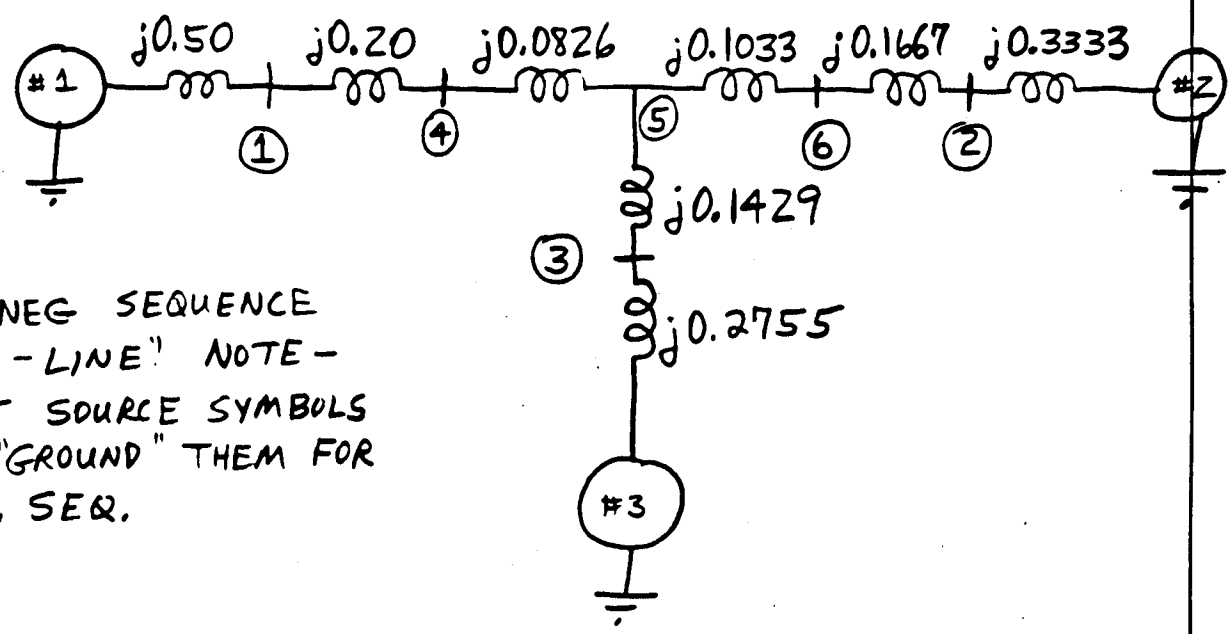
Line Segment B-C — $Z_{BASE} = \frac{(220)^2}{50} = 968\,\Omega$

$$Z_{pu} = \frac{j80}{968} = j0.0826\,\Omega$$

Line Segment C-E — $Z_{BASE} = 968\,\Omega$

$$Z_{pu} = \frac{j100}{968} = j0.1033\,\Omega$$

Then, the "one-line" diagram for pos/neg sequence is



POS/NEG SEQUENCE "ONE-LINE" NOTE - OMIT SOURCE SYMBOLS AND "GROUND" THEM FOR NEG. SEQ.

Compute some pos/neg short ckt. impedances. Treat the generators as perfect voltage sources in series with their subtransient impedances.

Grady 2007, p. 102

6.15, cont

$$Z_{55}^{+} = (j0.50 + j0.20 + j0.0826) \| (j0.1429 + j0.2755) \| (j0.1033 + j0.1667 + j0.3333)$$

$$= j0.783 \| j0.418 \| j0.603$$

$$\frac{1}{Z_{55}^{+}} = \frac{1}{j0.783} + \frac{1}{j0.418} + \frac{1}{j0.603}, \quad Z_{55}^{+} = 0.188 \text{ pu}$$

$$Z_{11}^{+} = (j0.50) \| \left[\underbrace{(j0.20 + j0.083)}_{j0.283} + \underbrace{(j0.143 + j0.275)}_{j0.418} \| \underbrace{(j0.103 + j0.167 + j0.333)}_{j0.603} \right]$$

$$Z_{11}^{+} = j0.257 \text{ pu}$$

For Computer Simulations

From-Bus	To-Bus	Branch Impedance
1	0	j0.50
1	4	j0.20
4	5	j0.0826
5	3	j0.1429
3	0	j0.2755
5	6	j0.1033
6	2	j0.1667
2	0	j0.3333

Admittance Matrix

	1	2	3	4	5	6
1	$\frac{1}{j0.50} + \frac{1}{j0.20}$	0	0	$\frac{-1}{j0.20}$	0	0
2	0	$\frac{1}{j0.1667} + \frac{1}{j0.3333}$	0	0	0	$\frac{-1}{j0.1667}$
3	0	0	$\frac{1}{j0.2755} + \frac{1}{j0.1429}$	0	$\frac{-1}{j0.1429}$	0
4	$\frac{-1}{j0.20}$	0	0	$\frac{1}{j0.20} + \frac{1}{j0.0826}$	$\frac{-1}{j0.0826}$	0
5	0	0	$\frac{-1}{j0.1429}$	$\frac{-1}{j0.0826}$	$\frac{1}{j0.0826} + \frac{1}{j0.1429} + \frac{1}{j0.1033}$	$\frac{-1}{j0.1033}$
6	0	$\frac{-1}{j0.1667}$	0	0	$\frac{-1}{j0.1033}$	$\frac{1}{j0.1033} + \frac{1}{j0.1667}$

Problem 11.11, page 304, Stevenson

Modify Problem 6.15 to include the zero sequence network. Step 1 same as 6.15.

Step 2: Convert all impedances to per-unit values

Generator 1 – Generator itself $Z_{pu}^{NEW} = j0.05\left[\frac{50}{20}\right]\left[\frac{13.8}{13.8}\right]^2 = j0.125$

Grounding reactor $Z_{pu} = j0.05\left[\frac{50}{20}\right]\left[\frac{13.8}{13.8}\right]^2 = j0.125$

Generator 2 – Generator itself $Z_{pu}^{NEW} = j0.05\left[\frac{50}{30}\right]\left[\frac{18}{18}\right]^2 = j0.083$

No grounding reactor, $Z_{pu} = 0$

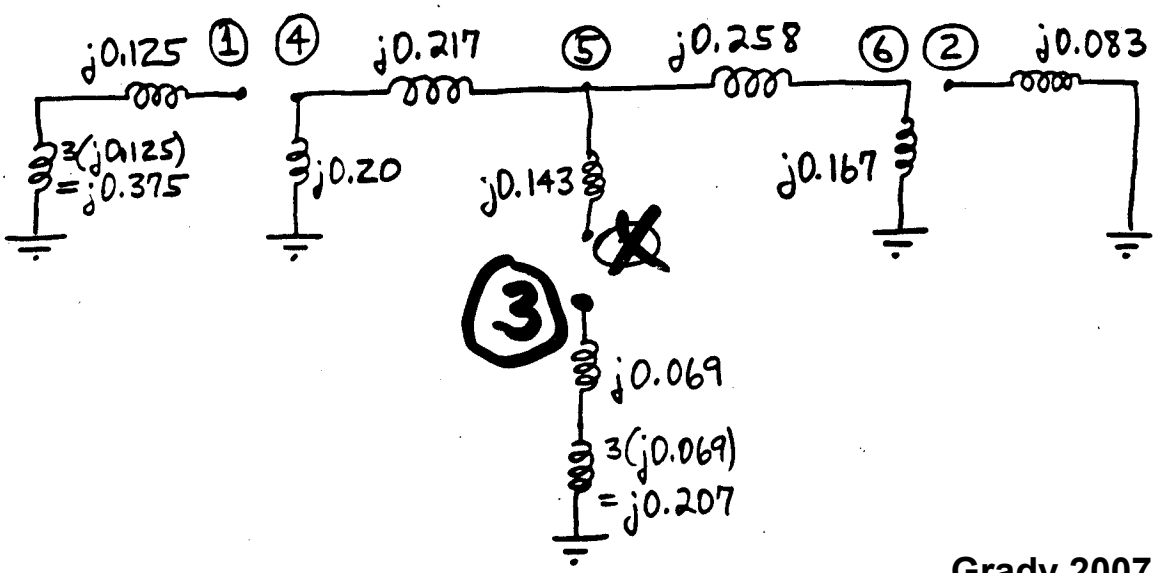
Generator 3 – Generator itself $Z_{pu}^{NEW} = j0.05\left[\frac{50}{30}\right]\left[\frac{20}{22}\right]^2 = j0.069$

Grounding Reactor $Z_{pu} = j0.069$ ← same

Transformers 1, 2, 3. Same as pos/neg sequence, but following the connection rules.

Line Segment B–C $Z_{pu} = \frac{j210}{968} = j0.217$

Line Segment C–E $Z_{pu} = \frac{j250}{968} = j0.258$



Problem 11.11, cont

$$Z_{55}^{0} = (j0.20 + j0.217) \| (j0.258 + j0.167)$$
$$= j0.417 \| j0.425$$
$$= j0.210 \text{ pu}$$

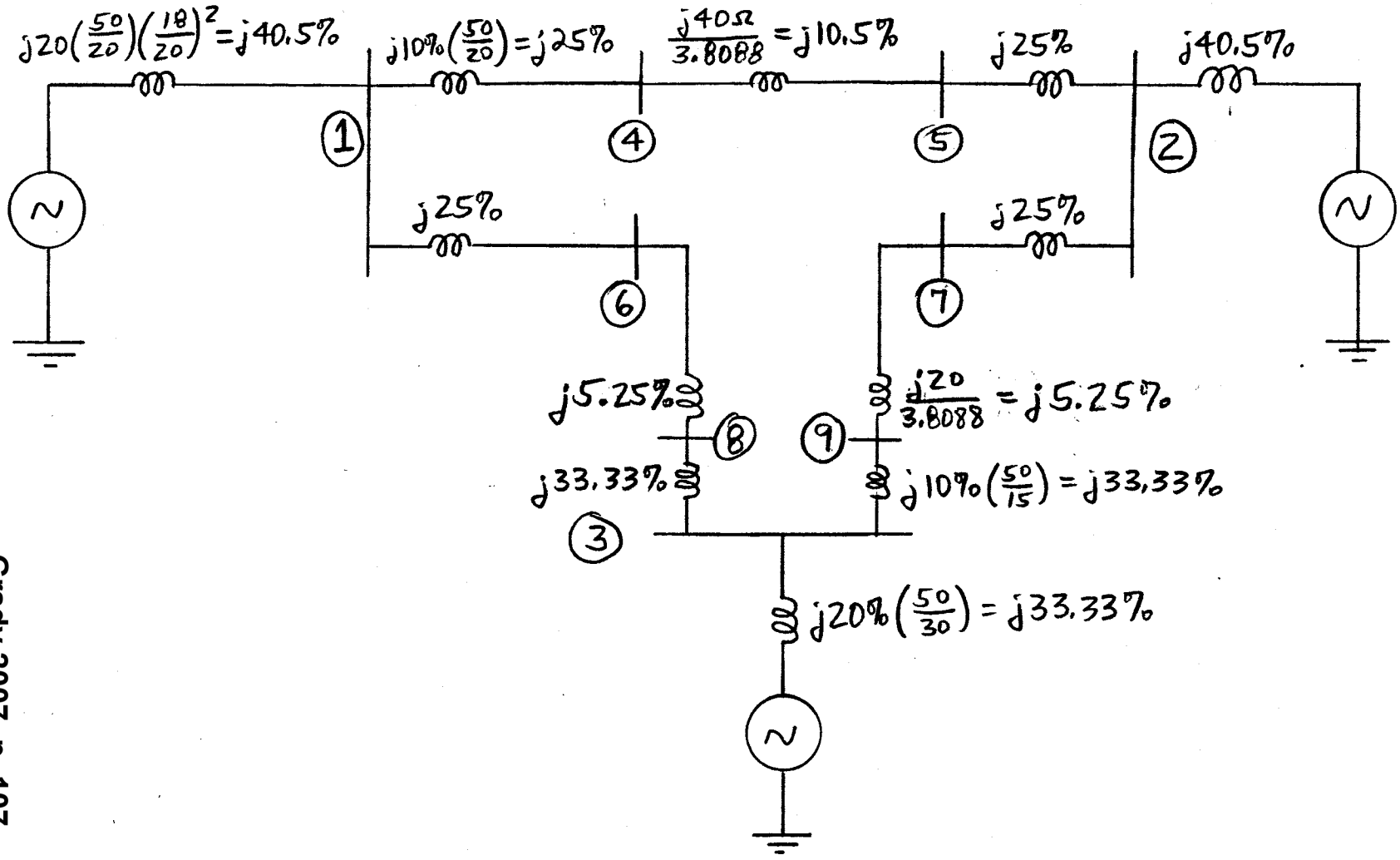
$$Z_{11}^{0} = j0.125 + j0.375 = j0.50 \text{ pu}$$

$$Z_{44}^{0} = j0.20 \| (j0.217 + j0.258 + j0.167) = j0.152 \text{ pu}$$

For Computer Simulations

From Bus	To Bus	Branch Impedance
1	0	j0.50
4	0	j0.20
4	5	j0.217
~~5~~	~~3~~	~~j0.143~~
5	6	j0.258
6	0	j0.167
2	0	j0.083
3	0	j0.069 + j0.207

Stevenson 6.16, cont — Positive Sequence One Line



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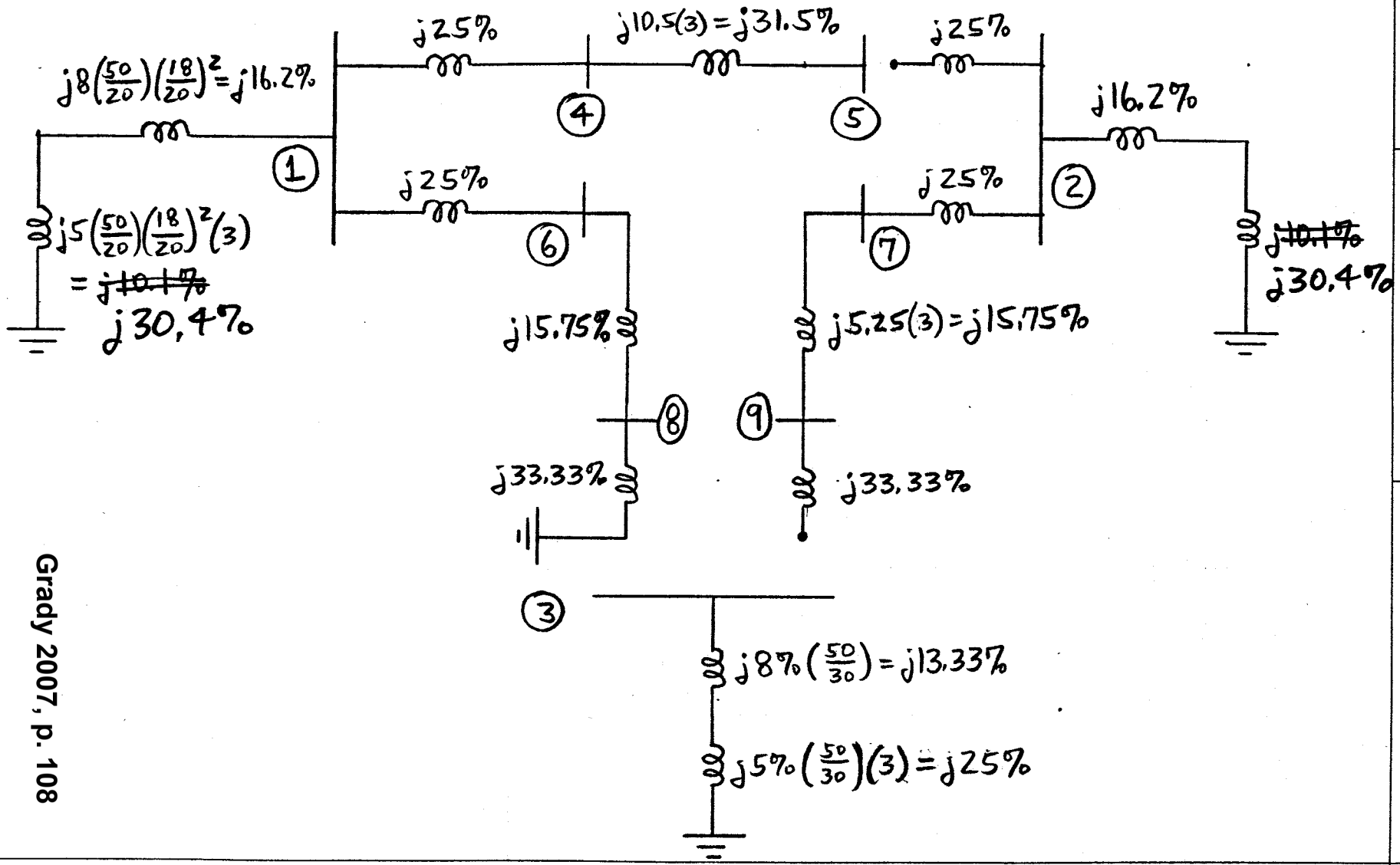
Stevenson 6.16, cont - Zero Sequence One Line



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SC1. $I_{4,a}^{F} = \dfrac{V_{4,a}^{PRE}}{Z_{44,1} + Z_F}$. $V_{4,a}^{PRE} = 1.0\angle 0$, $Z_F = 0$

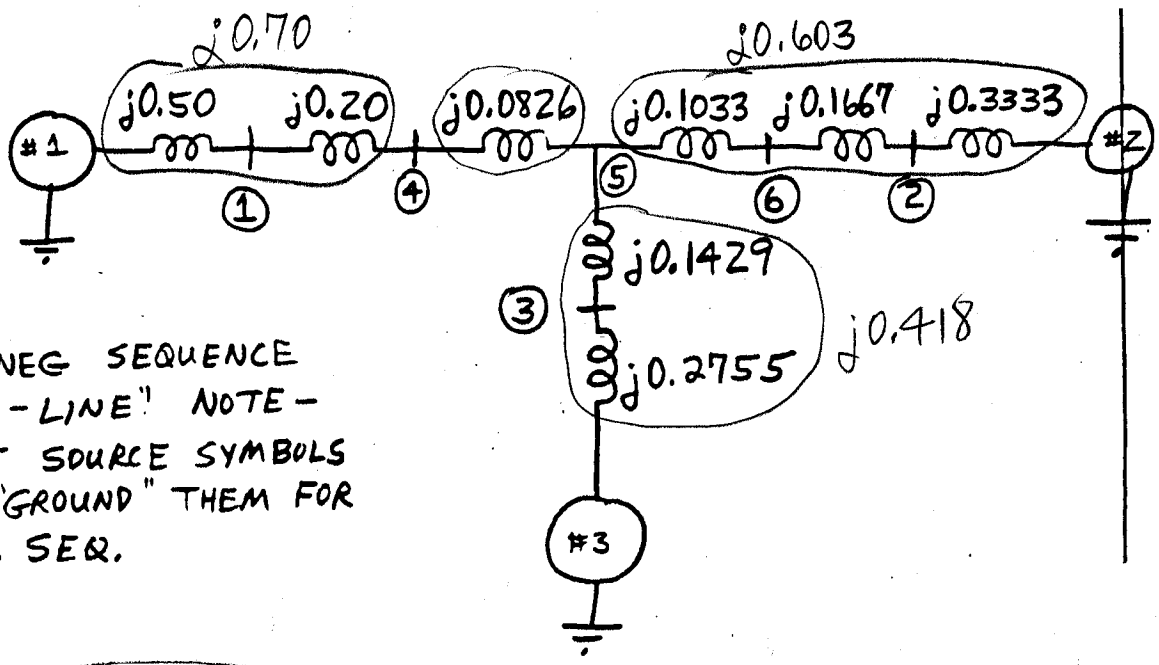
$$Z_{44,1} = j0.70 \,||\, \{j0.0826 + j0.603 \,||\, j0.418\}$$

$$= j0.70 \,||\, \{j0.0826 + j0.247\}$$

$$\boxed{Z_{44,1} = j0.70 \,||\, j0.3296 = j0.224\ pu}$$

$$Z_{14,1} = \frac{\partial V_1}{\partial I_4} = \underbrace{\frac{\partial V_4}{\partial I_4}}_{Z_{44}} \cdot \underbrace{\frac{\partial V_1}{\partial V_4}}_{\text{Voltage Divider}} = j0.224\left[\frac{j0.50}{j0.50 + j0.20}\right]$$

$$\boxed{Z_{14,1} = j0.224\left[\frac{5}{7}\right] = j0.160}$$



POS/NEG SEQUENCE "ONE-LINE" NOTE - OMIT SOURCE SYMBOLS AND "GROUND" THEM FOR NEG. SEQ.

$$I_{4,a}^{F} = \frac{1.0\angle 0}{j0.224} = -j4.46\,pu \qquad I_{base} = \frac{50\,MVA/3}{220\,kV/\sqrt{3}} = 131.2\,A$$

$$I_{4,a}^{F} = 4.46\angle -90, \quad \underbrace{I_{4,1}^{F} = I_{4,a}^{F}, \quad I_{4,0}^{F} = 0, \quad I_{4,2}^{F} = 0}_{\text{Balanced 3}\phi\ \text{case}}$$

$$\boxed{I_{4,a}^{F} = 585\angle -90\ A, \quad I_{4,b}^{F} = 585\angle 150\,A, \quad I_{4,c}^{F} = 585\angle 30\,A}$$

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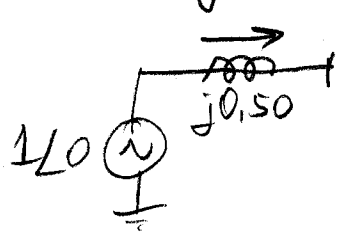
The network equation

$V^F_{1,012} = V^{PRE}_{1,012} - Z_{14,012} I^F_{4,012}$. We have only ⊕ seq,

$V^F_{1,1} = 1.0\angle 0 - (j0.160)(4.46\angle -90^\circ)$

$V^F_{1,1} = 1.0\angle 0 - 0.714 = 0.286\angle 0$ pu

For the generator current, use ⊕ seq. diagram



$I^F_{gen,1} = \dfrac{1\angle 0 - 0.286\angle 0}{j0.50} = -j1.428$ pu

Now, account for phase shift due to the ΔY transformer

$V^F_{1,1} = 0.286\angle -30^\circ$ pu → $V^F_{1,a} = 0.286\angle -30^\circ$. $V^F_{1,b}$ & $V^F_{1,c}$ lag and lead $V^F_{1,a}$ by 120°, respectively.

$I^F_{gen,1} = 1.428\angle -120^\circ$ pu

$I^F_{gen,a} = 1.428\angle -120^\circ$ pu

$I^F_{gen,b} = 1.428\angle +120^\circ$ pu

$I^F_{gen,c} = 1.428\angle 0$ pu

$I_{Base} = \dfrac{50\text{MVA}/3}{13.8\text{kV}/\sqrt{3}} = 2092\text{A}$, so $I^F_{gen,a} = 2987\text{A}\angle -120^\circ$

$I^F_{gen,b} = 2987\angle 120^\circ$

$I^F_{gen,c} = 2987\angle 0$

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SC2. $I_{4,a}^F = \dfrac{3V_{4,a}^{PRE}}{Z_{44,0} + Z_{44,1} + Z_{44,2} + 3Z_F}$. $Z_{44,1} = Z_{44,2} = j0.224$ pu, $Z_F = 0$

$Z_{44,0} = j0.20 \| j0.642 = j0.1525$ pu

$$I_{4,a}^F = \frac{3\angle 0}{j0.1525 + 2(j0.224)} = \frac{3}{j0.601} = -j5.00 \text{ pu}$$

$$I_{4,b}^F = I_{4,c}^F = 0$$

$I_{BASE} = 131.2$A, so

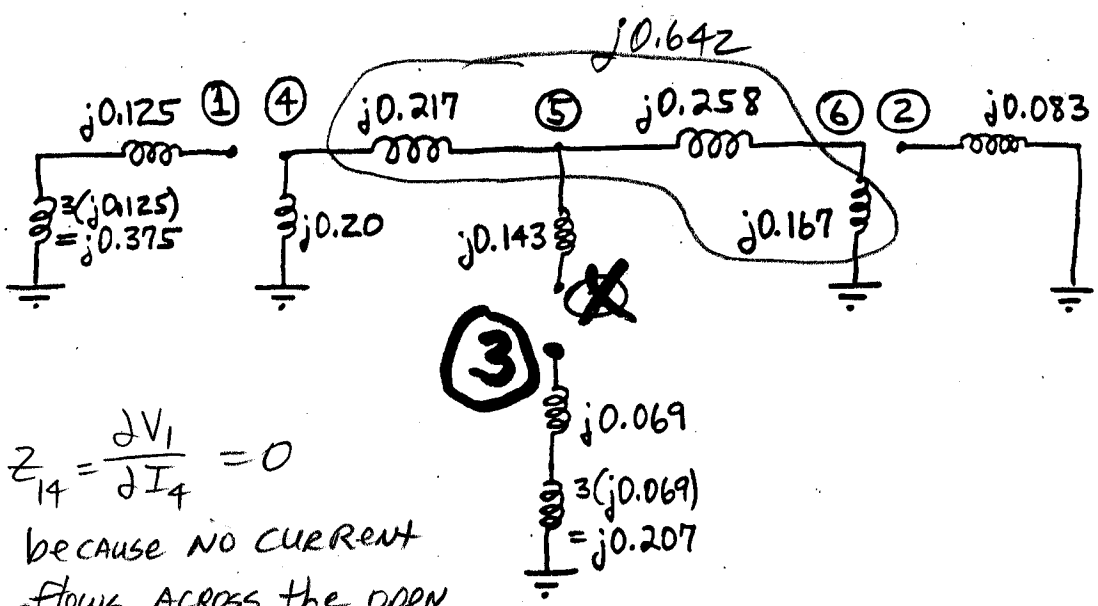
$\boxed{I_{4,a}^F = -j656\text{A}}$

(bigger than 3Ø fault!)

The network equation

$$V_{1,012}^F = V_{1,012}^{PRE} - Z_{14,012} I_{4,012}^F$$

$$\begin{bmatrix} I_{4,0}^F \\ I_{4,1}^F \\ I_{4,2}^F \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} -j5.00 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -j1.667 \\ -j1.667 \\ -j1.667 \end{bmatrix}$$



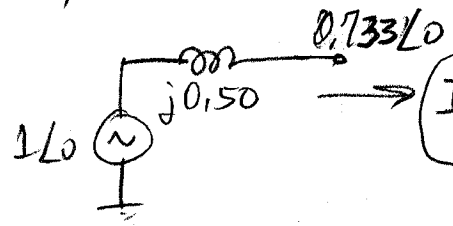
$Z_{14} = \dfrac{\partial V_1}{\partial I_4} = 0$

because no current flows across the open segment between 1 & 4

$$\begin{bmatrix} V_{1,0}^F \\ V_{1,1}^F \\ V_{1,2}^F \end{bmatrix} = \begin{bmatrix} 0 \\ 1\angle 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & j0.160 & 0 \\ 0 & 0 & j0.160 \end{bmatrix} \begin{bmatrix} -j1.667 \\ -j1.667 \\ -j1.667 \end{bmatrix}$$

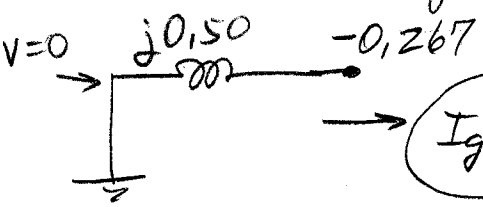
$$\begin{bmatrix} V_{1,0}^F \\ V_{1,1}^F \\ V_{1,2}^F \end{bmatrix} = \begin{bmatrix} 0 \\ 1 - 0.267 \\ -0.267 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.733 \\ -0.267 \end{bmatrix}$$

Now, from the (+) seq network



$$\rightarrow I_{gen,1}^F = \frac{1-0.733}{j0.50} = -j0.534\,pu$$

From the (−) seq network



$$\rightarrow I_{gen,2}^F = \frac{0-(-0.267)}{j0.50} = \frac{0.267}{j0.50} = -j0.534$$

Of course, no zero sequence current can flow out of the generator terminal because it is open-circuited in zero sequence. $I_{gen,0}^F = 0$

Now, employ the phase shifts.

$$V_{1,0}^F = 0$$
$$V_{1,1}^F = 0.733\angle -30^\circ$$
$$V_{1,2}^F = -0.267\angle 30^\circ = 0.267\angle -150^\circ$$

$$I_{gen,0}^F = 0$$
$$I_{gen,1}^F = -j0.534\angle -30^\circ = 0.534\angle -120^\circ$$
$$I_{gen,2}^F = -j0.534\angle +30^\circ = 0.534\angle -60^\circ$$

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Converting to a-b-c

$$\begin{bmatrix} V_{1,a}^F \\ V_{1,b}^F \\ V_{1,c}^F \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.733\angle -30^\circ \\ 0.267\angle -150^\circ \end{bmatrix} = \begin{bmatrix} 0.733\angle -30 + 0.267\angle -150 \\ 0.733\angle -150^\circ + 0.267\angle -30 \\ 0.733\angle 90^\circ + 0.267\angle 90^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 0.635 - j0.367 - 0.231 - j0.134 \\ -0.635 - j0.367 + 0.231 - j0.134 \\ j0.733 + j0.267 \end{bmatrix} = \begin{bmatrix} 0.404 - j0.501 \\ -0.404 - j0.501 \\ j1.00 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} V_{1a}^F \\ V_{1b}^F \\ V_{1c}^F \end{bmatrix} = \begin{bmatrix} 0.644\angle -51.1 \\ 0.644\angle -128.9 \\ 1.00\angle 90^\circ \end{bmatrix} \text{ pu}}$$

$$\begin{bmatrix} I_{gen,a}^F \\ I_{gen,b}^F \\ I_{gen,c}^F \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.534\angle -120 \\ 0.534\angle -60 \end{bmatrix} = \begin{bmatrix} 0.534\angle -120^\circ + 0.534\angle -60^\circ \\ 0.534\angle 120 + 0.534\angle 60 \\ 0.534\angle 0 + 0.534\angle -180 \end{bmatrix}$$

$$= \begin{bmatrix} -0.267 - j0.463 + 0.267 - j0.463 \\ -0.267 + j0.463 + 0.267 + j0.463 \\ 0.534 - 0.534 \end{bmatrix}$$

$$= \begin{bmatrix} -j0.926 \\ +j0.926 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} I_{gen,a}^F \\ I_{gen,b}^F \\ I_{gen,c}^F \end{bmatrix} = \begin{bmatrix} 0.926\angle -90^\circ \\ 0.926\angle +90^\circ \\ 0 \end{bmatrix} \text{ pu.}}$$

Multiply by $I_{base} = 2092A$ to get Amps

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:Six Bus Stevenson Short Circuit Study Case

:Bus Data

:Bus :Number :(I)	Bus Name (A)	Bus Type (I)	Linear P Generation (%) (F)	Linear Q Generation (%) (F)	Linear P Load (%) (F)	Linear Q Load (%) (F)	Desired Voltage (pu) (F)	Shunt Reactive Q Load (%) (F)	Maximum Q Generation (%) (F)	Minimum Q Generation (%) (F)	Bus Control Area (I)	Remote-Controlled Bus No. (I)	Connection Type for Shunt Reactive Q Load (I)	Positive Sequence Subtransient R (pu) (F)	Positive Sequence Subtransient X (pu) (F)	Negative Sequence Subtransient R (pu) (F)	Negative Sequence Subtransient X (pu) (F)	Zero Sequence Subtransient R (pu) (F)	Zero Sequence Subtransient X (pu) (F)	Connection Type for Subtransient R and X (I)	Grounding Impedance R for Subtransient Impedance (pu) (F)	Grounding Impedance X for Subtransient Impedances (pu) (F)
1	gen#1	1	0	0	0	0	1	0	0	0	1	0	0	0	0.5	0	0.5	0	0.125	1	0	0.125
2	gen#2	3	0	0	0	0	0	0	0	0	1	0	0	0	0.3333	0	0.3333	0	0.0833	1	0	0
3	gen#3	3	0	0	0	0	0	0	0	0	1	0	0	0	0.2755	0	0.2755	0	0.0689	1	0	0.0689
4	bus#4	3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
5	bus#5	3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
6	bus#6	3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0

:Six Bus Stevenson Short Circuit Study Case

:Line and Transformer Data

:From :Bus :(I)	To Bus (I)	Circuit Number (I)	Positive Sequence R (pu) (F)	Positive Sequence X (pu) (F)	Pos/Neg Sequence Charging (%) (F)	Rating (%) (F)	Minimum Tap (pu) (F)	Maximum Tap (pu) (F)	Tap Step Size (pu) (F)	Fixed Tap (pu) (F)	Phase Shift (Degrees) (F)	Voltage-Cont. Bus Number (I)	Voltage-Cont. Bus Side (I)	Desired Voltage at Voltage Cont. Bus (pu) or Desired Phase Shift (%) (F)	Zero Sequence R (pu) (F)	Zero Sequence X (pu) (F)	Zero Sequence Charging (%) (F)	Connect. Type for Trans. and Shunt Elements (I)	Series Grounding R (pu) (F)	Series Grounding X (pu) (F)
1	4	1	0	0.2	0	0	0	0	0	1	-30	0	0	0	0	0.2	0	8	0	0
2	6	1	0	0.1667	0	0	0	0	0	1	-30	0	0	0	0	0.1667	0	8	0	0
3	5	1	0	0.1429	0	0	0	0	0	1	0	0	0	0	0	0.1429	0	2	0	0
4	5	1	0	0.0826	0	0	0	0	0	0	0	0	0	0	0	0.217	0	0	0	0
6	5	1	0	0.1033	0	0	0	0	0	0	0	0	0	0	0	0.258	0	0	0	0

:Nine Bus Stevenson Short Circuit Study Case

:Bus Data

:Bus :Number :(I)	Bus Name (A)	Bus Type (I)	Linear P Generation (%) (F)	Linear Q Generation (%) (F)	Linear P Load (%) (F)	Linear Q Load (%) (F)	Desired Voltage (pu) (F)	Shunt Reactive Q Load (%) (F)	Maximum Q Generation (%) (F)	Minimum Q Generation (%) (F)	Bus Control Area (I)	Remote-Controlled Bus No. (I)	Connection Type for Shunt Reactive Q Load (I)	Positive Sequence Subtransient R (pu) (F)	Positive Sequence Subtransient X (pu) (F)	Negative Sequence Subtransient R (pu) (F)	Negative Sequence Subtransient X (pu) (F)	Zero Sequence Subtransient R (pu) (F)	Zero Sequence Subtransient X (pu) (F)	Connection Type for Subtransient R and X (I)	Grounding Impedance R for Subtransient Impedance (pu) (F)	Grounding Impedance X for Subtransient Impedances (pu) (F)
1	gen#1	1	0	0	0	0	1	0	0	0	1	0	0	0	0.405		0.405		0.162			0.101
2	gen#2	3	0	0	0	0	0	0	0	0	1	0	0	0	0.405		0.405		0.162			0.101
3	mot#3	3	0	0	0	0	0	0	0	0	1	0	0	0	0.3333		0.3333		0.1333			0.0833
4	bus#4	3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
5	bus#5	3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
6	bus#6	3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
7	bus#7	3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
8	bus#8	3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
9	bus#9	3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0

:Nine Bus Stevenson Short Circuit Study Case

:Line and Transformer Data

:From :Bus :(I)	To Bus (I)	Circuit Number (I)	Positive Sequence R (pu) (F)	Positive Sequence X (pu) (F)	Pos/Neg Sequence Charging (%) (F)	Rating (%) (F)	Minimum Tap (pu) (F)	Maximum Tap (pu) (F)	Tap Step Size (pu) (F)	Fixed Tap (pu) (F)	Phase Shift (Degrees) (F)	Voltage-Cont. Bus Number (I)	Voltage-Cont. Bus Side (I)	Desired Voltage at Voltage Cont. Bus (pu) or Desired Phase Shift (%) (F)	Zero Sequence R (pu) (F)	Zero Sequence X (pu) (F)	Zero Sequence Charging (%) (F)	Connect. Type for Trans. and Shunt Elements (I)	Series Grounding R (pu) (F)	Series Grounding X (pu) (F)
1	4	1	0	0.25	0	0	0	0	0	1	0	0	0	0	0	0.25	0	1	0	0
1	6	1	0	0.25	0	0	0	0	0	1	0	0	0	0	0	0.25	0	1	0	0
2	5	1	0	0.25	0	0	0	0	0	1	0	0	0	0	0	0.25	0	2	0	0
2	7	1	0	0.25	0	0	0	0	0	1	0	0	0	0	0	0.25	0	1	0	0
3	8	1	0	0.3333	0	0	0	0	0	1	-30	0	0	0	0	0.3333	0	8	0	0
3	9	1	0	0.3333	0	0	0	0	0	1	-30	0	0	0	0	0.3333	0	9	0	0
4	5	1	0	0.105	0	0	0	0	0	0	0	0	0	0	0	0.315	0	0	0	0
6	8	1	0	0.0525	0	0	0	0	0	0	0	0	0	0	0	0.1575	0	0	0	0
7	9	1	0	0.0525	0	0	0	0	0	0	0	0	0	0	0	0.1575	0	0	0	0

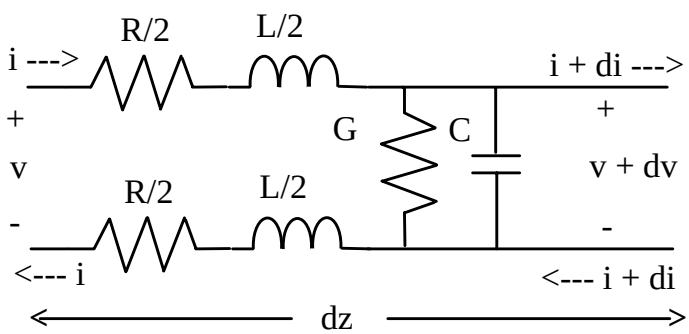
Grady 2007, p. 114

Transmission Lines

Inductance and capacitance calculations for transmission lines. GMR, GMD, L, and C matrices, effect of ground conductivity. Underground cables.

1. Equivalent Circuit for Transmission Lines (Including Overhead and Underground)

The power system model for transmission lines is developed from the conventional distributed parameter model, shown in Figure 1.



R, L, G, C per unit length

Figure 1. Distributed Parameter Model for Transmission Line

Once the values for distributed parameters resistance R, inductance L, conductance G, and capacitance are known (units given in per unit length), then either "long line" or "short line" models can be used, depending on the electrical length of the line.

Assuming for the moment that R, L, G, and C are known, the relationship between voltage and current on the line may be determined by writing Kirchhoff's voltage law (KVL) around the outer loop in Figure 1, and by writing Kirchhoff's current law (KCL) at the right-hand node.

KVL yields

$$-v + \frac{Rdz}{2} i + \frac{Ldz}{2} \frac{\partial i}{\partial t} + v + dv + \frac{Rdz}{2} i + \frac{Ldz}{2} \frac{\partial i}{\partial t} = 0 .$$

This yields the change in voltage per unit length, or

$$\frac{\partial v}{\partial z} = -Ri - L \frac{\partial i}{\partial t} ,$$

which in phasor form is

$$\frac{\partial \tilde{V}}{\partial z} = -(R + j\omega L)\tilde{I} .$$

KCL at the right-hand node yields

$$-i + i + di + Gdz(v + dv) + Cdz\frac{\partial(v + dv)}{\partial t} = 0 \ .$$

If *dv* is small, then the above formula can be approximated as

$$di = -(Gdz)v - Cdz\frac{\partial v}{\partial t} \text{ , or } \frac{\partial i}{\partial z} = -Gv - C\frac{\partial v}{\partial t} \text{ , which in phasor form is}$$

$$\frac{\partial \tilde{I}}{\partial z} = -(G + j\omega C)\tilde{V} \ .$$

Taking the partial derivative of the voltage phasor equation with respect to z yields

$$\frac{\partial^2 \tilde{V}}{\partial z^2} = -(R + j\omega L)\frac{\partial \tilde{I}}{\partial z} \ .$$

Combining the two above equations yields

$$\frac{\partial^2 \tilde{V}}{\partial z^2} = (R + j\omega L)(G + j\omega C)\tilde{V} = \gamma^2 \tilde{V} \text{ , where } \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta \text{ , and}$$

where γ, α, and β are the propagation, attenuation, and phase constants, respectively.

The solution for $\tilde{V}$ is

$$\tilde{V}(z) = Ae^{\gamma z} + Be^{-\gamma z} \ .$$

A similar procedure for solving $\tilde{I}$ yields

$$\tilde{I}(z) = \frac{-Ae^{\gamma z} + Be^{-\gamma z}}{Z_o} \ ,$$

where the characteristic or "surge" impedance Z_o is defined as

$$Z_o = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}} \ .$$

Constants *A* and *B* must be found from the boundary conditions of the problem. This is usually accomplished by considering the terminal conditions of a transmission line segment that is d meters long, as shown in Figure 2.

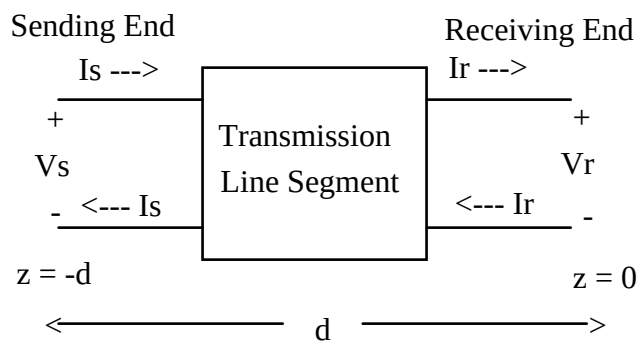


Figure 2. Transmission Line Segment

In order to solve for constants *A* and *B*, the voltage and current on the receiving end is assumed to be known so that a relation between the voltages and currents on both sending and receiving ends may be developed.

Substituting $z = 0$ into the equations for the voltage and current (at the receiving end) yields

$$\tilde{V}_R = A + B, \tilde{I}_R = \frac{-(A-B)}{Z_o} \ .$$

Solving for *A* and *B* yields

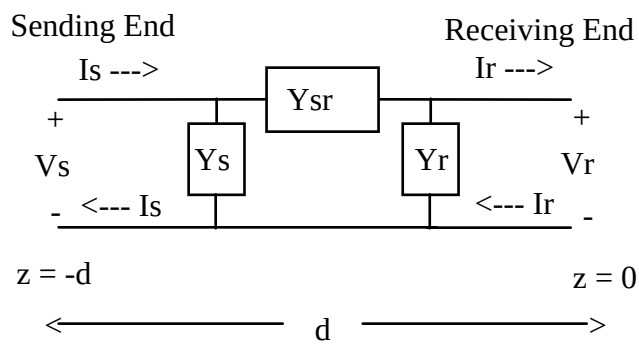
$$A = \frac{\tilde{V}_R - Z_o I_R}{2}, B = \frac{\tilde{V}_R + Z_o I_R}{2} \ .$$

Substituting into the $\tilde{V}(z)$ and $\tilde{I}(z)$ equations yields

$$\tilde{V}_S = \tilde{V}_R \cosh(\gamma d) + Z_0 \tilde{I}_R \sinh(\gamma d) \ ,$$

$$\tilde{I}_S = \frac{\tilde{V}_R}{Z_o} \sinh(\gamma d) + \tilde{I}_R \cosh(\gamma d) \ .$$

A pi equivalent model for the transmission line segment can now be found, in a similar manner as it was for the off-nominal transformer. The results are given in Figure 3.



$$Y_S = Y_R = \frac{\tanh\left(\frac{\gamma d}{2}\right)}{Z_o}\ ,\ Y_{SR} = \frac{1}{Z_o \sinh(\gamma d)}\ ,\ Z_o = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}}\ ,\ \gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

R, *L*, *G*, *C* per unit length

Figure 3. Pi Equivalent Circuit Model for Distributed Parameter Transmission Line

Shunt conductance *G* is usually neglected in overhead lines, but it is not negligible in underground cables.

For electrically "short" overhead transmission lines, the hyperbolic pi equivalent model simplifies to a familiar form. Electrically short implies that d < 0.05λ, where wavelength $\lambda = \frac{3(10^8)m/s}{f\sqrt{\varepsilon_r}\,Hz}$ = 5000 km @ 60 Hz, or 6000 km @ 50 Hz. Therefore, electrically short overhead lines have d < 250 km @ 60 Hz, and d < 300 km @ 50 Hz. For underground cables, the corresponding distances are less since cables have somewhat higher relative permittivities (i.e. $\varepsilon_r \approx 2.5$).

Substituting small values of γd into the hyperbolic equations, and assuming that the line losses are negligible so that $G = R = 0$, yields

$$Y_S = Y_R = \frac{j\omega Cd}{2}\ \text{, and } Y_{SR} = \frac{1}{j\omega Ld}\ .$$

Then, including the series resistance yields the conventional "short" line model shown in Figure 4, where half of the capacitance of the line is lumped on each end.

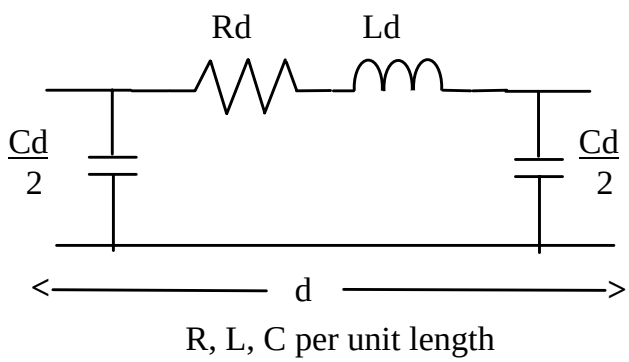


Figure 4. Standard Short Line Pi Equivalent Model for a Transmission Line

2. Capacitance of Overhead Transmission Lines

Overhead transmission lines consist of wires that are parallel to the surface of the earth. To determine the capacitance of a transmission line, first consider the capacitance of a single wire over the earth. Wires over the earth are typically modeled as line charges ρ_l Coulombs per meter of length, and the relationship between the applied voltage and the line charge is the capacitance.

A line charge in space has a radially outward electric field described as

$$\overline{E} = \frac{q_l}{2\pi\varepsilon_o r}\hat{a}_r \text{ Volts per meter .}$$

This electric field causes a voltage drop between two points at distances $r = a$ and $r = b$ away from the line charge. The voltage is found by integrating electric field, or

$$V_{ab} = \int_{r=a}^{r=b} \overline{E} \bullet r\hat{a}_r = \frac{q_l}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right) \text{ V.}$$

If the wire is above the earth, it is customary to treat the earth's surface as a perfect conducting plane, which can be modeled as an equivalent image line charge $-q_l$ lying at an equal distance below the surface, as shown in Figure 5.

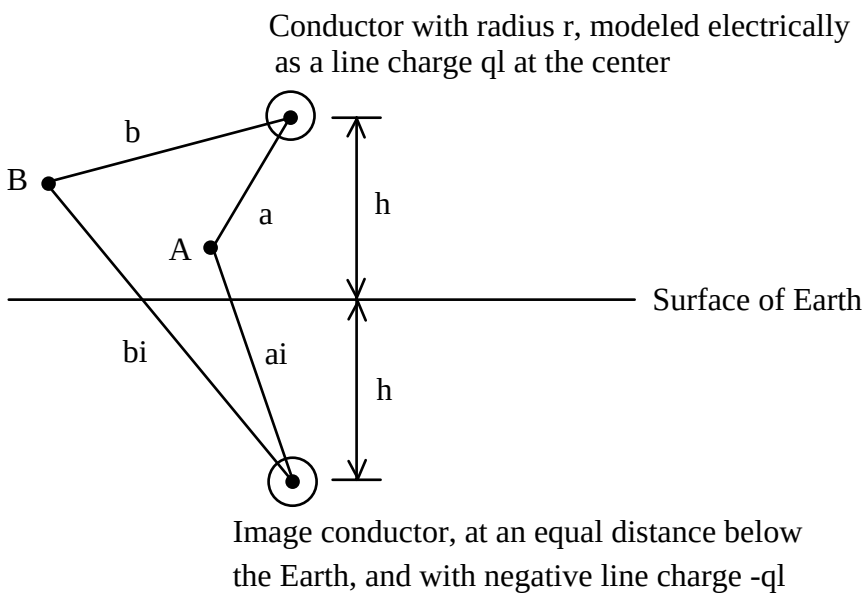


Figure 5. Line Charge q_l at Center of Conductor Located h Meters Above the Earth

From superposition, the voltage difference between points A and B is

$$V_{ab} = \int_{r=a}^{r=b} \overline{E}_\rho \bullet \hat{a}_r + \int_{r=ai}^{r=bi} \overline{E}_{\rho i} \bullet \hat{a}_r = \frac{q_l}{2\pi\varepsilon_o}\left[\ln\left(\frac{b}{a}\right) - \ln\left(\frac{bi}{ai}\right)\right] = \frac{q_l}{2\pi\varepsilon_o}\ln\left(\frac{b \bullet ai}{a \bullet bi}\right) .$$

If point B lies on the earth's surface, then from symmetry, $b = bi$, and the voltage of point A with respect to ground becomes

$$V_{ag} = \frac{q_l}{2\pi\varepsilon_o}\ln\left(\frac{ai}{a}\right) .$$

The voltage at the surface of the wire determines the wire's capacitance. This voltage is found by moving point A to the wire's surface, corresponding to setting $a = r$, so that

$$V_{rg} \approx \frac{q_l}{2\pi\varepsilon_o}\ln\left(\frac{2h}{r}\right) \text{ for } h >> r.$$

The exact expression, which accounts for the fact that the equivalent line charge drops slightly below the center of the wire, but still remains within the wire, is

$$V_{rg} = \frac{q_l}{2\pi\varepsilon_o}\ln\left(\frac{h + \sqrt{h^2 + r^2}}{r}\right) .$$

The capacitance of the wire is defined as $C = \dfrac{\rho_l}{V_{rg}}$ which, using the approximate voltage formula above, becomes

$$C = \frac{2\pi\varepsilon_o}{\ln\left(\dfrac{2h}{r}\right)} \text{ Farads per meter of length.}$$

When several conductors are present, then the capacitance of the configuration must be given in matrix form. Consider phase a-b-c wires above the earth, as shown in Figure 6.

Three Conductors Represented by Their Equivalent Line Charges

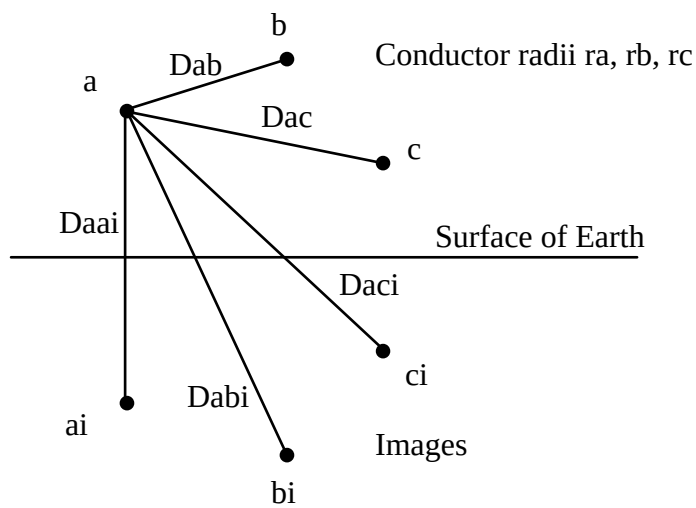


Figure 6. Three Conductors Above the Earth

Superposing the contributions from all three line charges and their images, the voltage at the surface of conductor a is given by

$$V_{ag} = \frac{1}{2\pi\varepsilon_o}\left[q_a \ln\frac{D_{aai}}{r_a} + q_b \ln\frac{D_{abi}}{D_{ab}} + q_c \ln\frac{D_{aci}}{D_{ac}} \right] .$$

The voltages for all three conductors can be written in generalized matrix form as

$$\begin{bmatrix} V_{ag} \\ V_{bg} \\ V_{cg} \end{bmatrix} = \frac{1}{2\pi\varepsilon_o}\begin{bmatrix} p_{aa} & p_{ab} & p_{ac} \\ p_{ba} & p_{bb} & p_{bc} \\ p_{ca} & p_{cb} & p_{cc} \end{bmatrix}\begin{bmatrix} q_a \\ q_b \\ q_c \end{bmatrix} \text{, or } V_{abc} = \frac{1}{2\pi\varepsilon_o} P_{abc} Q_{abc} \ ,$$

where

$p_{aa} = \ln \frac{D_{aai}}{r_a}$, $p_{ab} = \ln \frac{D_{abi}}{D_{ab}}$, etc., and

r_a is the radius of conductor a,

D_{aai} is the distance from conductor a to its own image (i.e. twice the height of conductor a above ground),

D_{ab} is the distance from conductor a to conductor b,

$D_{abi} = D_{bai}$ is the distance between conductor a and the image of conductor b (which is the same as the distance between conductor b and the image of conductor a), etc.

A Matrix Approach for Finding C

From the definition of capacitance, $Q = CV$, then the capacitance matrix can be obtained via inversion, or

$$C_{abc} = 2\pi\varepsilon_o P_{abc}^{-1} \ .$$

If ground wires are present, the dimension of the problem increases proportionally. For example, in a three-phase system with two ground wires, the dimension of the *P* matrix is 5 x 5. However, given the fact that the line-to-ground voltage of the ground wires is zero, equivalent 3 x 3 *P* and *C* matrices can be found by using matrix partitioning and a process known as Kron reduction. First, write the *V* = *PQ* equation as follows:

$$\begin{bmatrix} V_{ag} \\ V_{bg} \\ V_{cg} \\ - \\ V_{vg} = 0 \\ V_{wg} = 0 \end{bmatrix} = \frac{1}{2\pi\varepsilon_o} \begin{bmatrix} P_{abc}(3\text{x}3) & | & P_{abc,vw}(3\text{x}2) \\ - & & - \\ P_{vw,abc}(2\text{x}3) & | & P_{vw}(2\text{x}2) \end{bmatrix} \begin{bmatrix} q_a \\ q_b \\ q_c \\ - \\ q_v \\ q_w \end{bmatrix} ,$$

or

$$\begin{bmatrix} V_{abc} \\ V_{vw} \end{bmatrix} = \frac{1}{2\pi\varepsilon_o} \begin{bmatrix} P_{abc} & P_{abc,vw} \\ P_{vw,abc} & P_{vw} \end{bmatrix} \begin{bmatrix} Q_{abc} \\ Q_{vw} \end{bmatrix} ,$$

where subscripts *v* and *w* refer to ground wires w and v, and where the individual *P* matrices are formed as before. Since the ground wires have zero potential, then

$$\left[\frac{0}{0}\right] = \frac{1}{2\pi\varepsilon_o}\left[P_{vw,abc}Q_{abc} + P_{vw}Q_{vw}\right],$$

so that

$$Q_{vw} = -P_{vw}^{-1}\left[P_{vw,abc}Q_{abc}\right].$$

Substituting into the V_{abc} equation above, and combining terms, yields

$$V_{abc} = \frac{1}{2\pi\varepsilon_o}\left[P_{abc}Q_{abc} - P_{abc,vw}P_{vw}^{-1}P_{vw,abc}Q_{abc}\right] = \frac{1}{2\pi\varepsilon_o}\left[P_{abc} - P_{abc,vw}P_{vw}^{-1}P_{vw,abc}\right]Q_{abc} \ ,$$

or

$$V_{abc} = \frac{1}{2\pi\varepsilon_o}\left[P_{abc}^{'}\right]Q_{abc} \ \text{, so that}$$

$$Q_{abc} = C_{abc}^{'}V_{abc} \ \text{, where } C_{abc}^{'} = 2\pi\varepsilon_o\left[P_{abc}^{'}\right]^{-1}.$$

Therefore, the effect of the ground wires can be included into a 3 x 3 equivalent capacitance matrix.

An alternative way to find the equivalent 3 x 3 capacitance matrix $C_{abc}^{'}$ is to

- obtain the 5 x 5 *C* matrix by inverting the 5 x 5 *P*, and then
- Kron reduce the 5 x 5 *C* directly.

Computing 012 Capacitances from Matrices

Once the 3 x 3 $C_{abc}^{'}$ matrix is found by either of the above two methods, 012 capacitances can be determined by averaging the diagonal terms, and averaging the off-diagonal terms of, $C_{abc}^{'}$ to produce

$$C_{abc}^{avg} = \begin{bmatrix} C_S & C_M & C_M \\ C_M & C_S & C_S \\ C_M & C_M & C_S \end{bmatrix}.$$

C_{abc}^{avg} has the special symmetric form for diagonalization into 012 components, which yields

$$C_{012}^{avg} = \begin{bmatrix} C_S + 2C_M & 0 & 0 \\ 0 & C_S - C_M & 0 \\ 0 & 0 & C_S - C_M \end{bmatrix}.$$

The Approximate Formulas for 012 Capacitances

Asymmetries in transmission lines prevent the P and C matrices from having the special form that allows their diagonalization into decoupled positive, negative, and zero sequence impedances. Transposition of conductors can be used to nearly achieve the special symmetric form and, hence, improve the level of decoupling. Conductors are transposed so that each one occupies each phase position for one-third of the lines total distance. An example is given below in Figure 7, where the radii of all three phases are assumed to be identical.

a b c then a c b then b a c then

b c a then c a b then c b a

where each configuration occupies one-sixth of the total distance

Figure 7. Transposition of A-B-C Phase Conductors

For this mode of construction, the average P matrix (or Kron reduced P matrix if ground wires are present) has the following form:

$$P_{abc}^{avg} = \frac{1}{6}\begin{bmatrix} p_{aa} & p_{ab} & p_{ac} \\ \bullet & p_{bb} & p_{bc} \\ \bullet & \bullet & p_{cc} \end{bmatrix} + \frac{1}{6}\begin{bmatrix} p_{aa} & p_{ac} & p_{ab} \\ \bullet & p_{cc} & p_{bc} \\ \bullet & \bullet & p_{bb} \end{bmatrix} + \frac{1}{6}\begin{bmatrix} p_{bb} & p_{ab} & p_{bc} \\ \bullet & p_{aa} & p_{ac} \\ \bullet & \bullet & p_{cc} \end{bmatrix} +$$

$$\begin{bmatrix} p_{cc} & p_{ac} & p_{bc} \\ \bullet & p_{aa} & p_{ab} \\ \bullet & \bullet & p_{bb} \end{bmatrix} + \frac{1}{6}\begin{bmatrix} p_{bb} & p_{bc} & p_{ab} \\ \bullet & p_{cc} & p_{ac} \\ \bullet & \bullet & p_{aa} \end{bmatrix} + \frac{1}{6}\begin{bmatrix} p_{cc} & p_{bc} & p_{ac} \\ \bullet & p_{bb} & p_{ab} \\ \bullet & \bullet & p_{aa} \end{bmatrix},$$

where the individual p terms are described previously. Note that these individual P matrices are symmetric, since $D_{ab} = D_{ba}, p_{ab} = p_{ba}$, etc. Since the sum of natural logarithms is the same as the logarithm of the product, P becomes

$$P_{abc}^{avg} = \begin{bmatrix} p_S & p_M & p_M \\ p_M & p_S & p_M \\ p_M & p_M & p_S \end{bmatrix},$$

where

$$p_s = \frac{P_{aa} + P_{bb} + P_{cc}}{3} = \ln \frac{\sqrt[3]{D_{aai} D_{bbi} D_{cci}}}{\sqrt[3]{r_a r_b r_c}} \ ,$$

and

$$p_M = \frac{P_{ab} + P_{ac} + P_{bc}}{3} = \ln \frac{\sqrt[3]{D_{abi} D_{aci} D_{bci}}}{\sqrt[3]{D_{ab} D_{ac} D_{bc}}} \ .$$

Since P_{abc}^{avg} has the special property for diagonalization in symmetrical components, then transforming it yields

$$P_{012}^{avg} = \begin{bmatrix} p_0 & 0 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & p_2 \end{bmatrix} = \begin{bmatrix} p_S + 2p_M & 0 & 0 \\ 0 & p_S - p_M & 0 \\ 0 & 0 & p_S - p_M \end{bmatrix} ,$$

where

$$p_s - p_M = \ln \frac{\sqrt[3]{D_{aai} D_{bbi} D_{cci}}}{\sqrt[3]{r_a r_b r_c}} - \ln \frac{\sqrt[3]{D_{abi} D_{aci} D_{bci}}}{\sqrt[3]{D_{ab} D_{ac} D_{bc}}} = \ln \frac{\sqrt[3]{D_{aai} D_{bbi} D_{cci}} \sqrt[3]{D_{ab} D_{ac} D_{bc}}}{\sqrt[3]{r_a r_b r_c} \sqrt[3]{D_{abi} D_{aci} D_{bci}}} \ .$$

Inverting P_{012}^{avg} and multiplying by $2\pi\varepsilon_o$ yields the corresponding 012 capacitance matrix

$$C_{012}^{avg} = \begin{bmatrix} C_0 & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix} = 2\pi\varepsilon_o \begin{bmatrix} \frac{1}{p_0} & 0 & 0 \\ 0 & \frac{1}{p_1} & 0 \\ 0 & 0 & \frac{1}{p_2} \end{bmatrix} = 2\pi\varepsilon_o \begin{bmatrix} \frac{1}{p_S + 2p_M} & 0 & 0 \\ 0 & \frac{1}{p_S - p_M} & 0 \\ 0 & 0 & \frac{1}{p_S - p_M} \end{bmatrix} .$$

When the a-b-c conductors are closer to each other than they are to the ground, then

$$D_{aai} D_{bbi} D_{cci} \approx D_{abi} D_{aci} D_{bci} \ ,$$

yielding the conventional approximation

$$p_1 = p_2 = p_S - p_M = \ln \frac{\sqrt[3]{D_{ab} D_{ac} D_{bc}}}{\sqrt[3]{r_a r_b r_c}} = \ln \frac{GMD_{1,2}}{GMR_{1,2}} \ ,$$

where $GMD_{1,2}$ and $GMR_{1,2}$ are the geometric mean distance (between conductors) and geometric mean radius, respectively, for both positive and negative sequences. Therefore, the positive and negative sequence capacitances become

$$C_1 = C_2 = \frac{2\pi\varepsilon_o}{p_S - p_M} = \frac{2\pi\varepsilon_o}{\ln\dfrac{GMD_{1,2}}{GMR_{1,2}}} \text{ Farads per meter.}$$

For the zero sequence term,

$$p_0 = p_s + 2p_M = \ln\frac{\sqrt[3]{D_{aai}D_{bbi}D_{cci}}}{\sqrt[3]{r_a r_b r_c}} + 2\ln\frac{\sqrt[3]{D_{abi}D_{aci}D_{bci}}}{\sqrt[3]{D_{ab}D_{ac}D_{bc}}} =$$

$$\ln\sqrt[3]{\frac{(D_{aai}D_{bbi}D_{cci})(D_{abi}D_{aci}D_{bci})^2}{(r_a r_b r)(D_{ab}D_{ac}D_{bc})^2}} \ .$$

Expanding yields

$$p_0 = 3\ln\sqrt[9]{\frac{(D_{aai}D_{bbi}D_{cci})(D_{abi}D_{aci}D_{bci})^2}{(r_a r_b r)(D_{ab}D_{ac}D_{bc})^2}} =$$

$$3\ln\sqrt[9]{\frac{(D_{aai}D_{bbi}D_{cci})(D_{abi}D_{aci}D_{bci})(D_{bai}D_{cai}D_{cbi})}{(r_a r_b r)(D_{ab}D_{ac}D_{bc})(D_{ba}D_{ca}D_{cb})}} \ ,$$

or

$$p_0 = 3\ln\frac{GMD_0}{GMR_0} \ ,$$

where

$$GMD_0 = \sqrt[9]{(D_{aai}D_{bbi}D_{cci})(D_{abi}D_{aci}D_{bci})(D_{bai}D_{cai}D_{cbi})} \ ,$$

$$GMR_0 = \sqrt[9]{(r_a r_b r_c)(D_{ab}D_{ac}D_{bc})(D_{ba}D_{ca}D_{cb})} \ .$$

The zero sequence capacitance then becomes

$$C_0 = \frac{2\pi\varepsilon_o}{p_S + 2p_M} = \frac{1}{3}\frac{2\pi\varepsilon_o}{\ln\dfrac{GMD_0}{GMR_o}} \text{ Farads per meter,}$$

which is one-third that of the entire a-b-c bundle by because it represents the average contribution of only one phase.

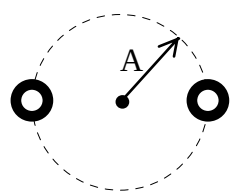
Bundled Phase Conductors

If each phase consists of a symmetric bundle of N identical individual conductors, an equivalent radius can be computed by assuming that the total line charge on the phase divides equally among the N individual conductors. The equivalent radius is

$$r_{eq} = \left[NrA^{N-1} \right]^{\frac{1}{N}} ,$$

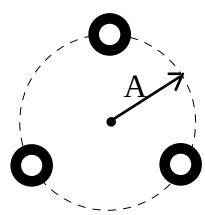
where r is the radius of the individual conductors, and A is the bundle radius of the symmetric set of conductors. Three common examples are shown below in Figure 8.

Double Bundle, Each Conductor Has Radius r



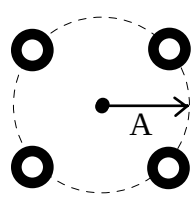
$$r_{eq} = \sqrt{2rA}$$

Triple Bundle, Each Conductor Has Radius r



$$r_{eq} = \sqrt[3]{3rA^2}$$

Quadruple Bundle, Each Conductor Has Radius r



$$r_{eq} = \sqrt[4]{4rA^3}$$

Figure 8. Equivalent Radius for Three Common Types of Bundled Phase Conductors

3. Inductance

The magnetic field intensity produced by a long, straight current carrying conductor is given by Ampere's Circuital Law to be

$$H_\phi = \frac{I}{2\pi r} \text{ Amperes per meter,}$$

where the direction of $\overline{H}$ is given by the right-hand rule.

Magnetic flux density is related to magnetic field intensity by permeability μ as follows:

$$\overline{B} = \mu\overline{H} \text{ Webers per square meter,}$$

and the amount of magnetic flux passing through a surface is

$$\Phi = \int \overline{B} \bullet d\overline{s} \text{ Webers,}$$

where the permeability of free space is $\mu_o = 4\pi\left(10^{-7}\right)$.

Two Parallel Wires in Space

Now, consider a two-wire circuit that carries current I, as shown in Figure 9.

Two current-carying wires with radii r

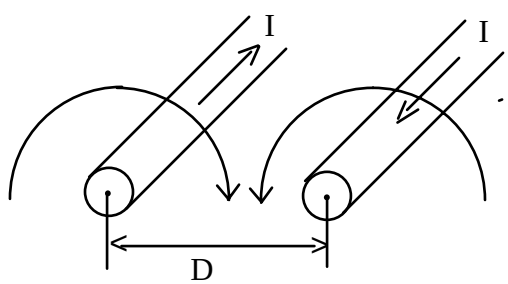


Figure 9. A Circuit Formed by Two Long Parallel Conductors

The amount of flux linking the circuit (i.e. passes between the two wires) is found to be

$$\Phi = \int_r^{D-r} \frac{\mu_o I}{2\pi x} dx + \int_r^{D-r} \frac{\mu_o I}{2\pi x} dx = \frac{\mu_o I}{\pi} \ln \frac{D-r}{r} \text{ Henrys per meter length.}$$

From the definition of inductance,

$$L = \frac{N\Phi}{I} \ ,$$

where in this case $N = 1$, and where $N >> r$, the inductance of the two-wire pair becomes

$$L = \frac{\mu_o}{\pi} \ln \frac{D}{r} \quad \text{Henrys per meter length.}$$

A round wire also has an internal inductance, which is separate from the external inductance shown above. The internal inductance is shown in electromagnetics texts to be

$$L_{int} = \frac{\mu_{int}}{8\pi} \quad \text{Henrys per meter length.}$$

For most current-carrying conductors, $\mu_{int} = \mu_o$ so that L_{int} = 0.05μH/m. Therefore, the total inductance of the two-wire circuit is the external inductance plus twice the internal inductance of each wire (i.e. current travels down and back), so that

$$L_{tot} = \frac{\mu_o}{\pi} \ln \frac{D}{r} + 2\frac{\mu_o}{8\pi} = \frac{\mu_o}{\pi}\left[\ln \frac{D}{r} + \frac{1}{4}\right] = \frac{\mu_o}{\pi}\left[\ln \frac{D}{r} + \ln\left(e^{\frac{1}{4}}\right)\right] = \frac{\mu_o}{\pi} \ln \frac{D}{re^{-\frac{1}{4}}} \ .$$

It is customary to define an effective radius

$$r_{eff} = re^{-\frac{1}{4}} = 0.7788r \ ,$$

and to write the total inductance in terms of it as

$$L_{tot} = \frac{\mu_o}{\pi} \ln \frac{D}{r_{eff}} \quad \text{Henrys per meter length.}$$

Wire Parallel to Earth's Surface

For a single wire of radius r, located at height h above the earth, the effect of the earth can be described by an image conductor, as it was for capacitance calculations. For a perfectly conducting earth, the image conductor is located h meters below the surface, as shown in Figure 10.

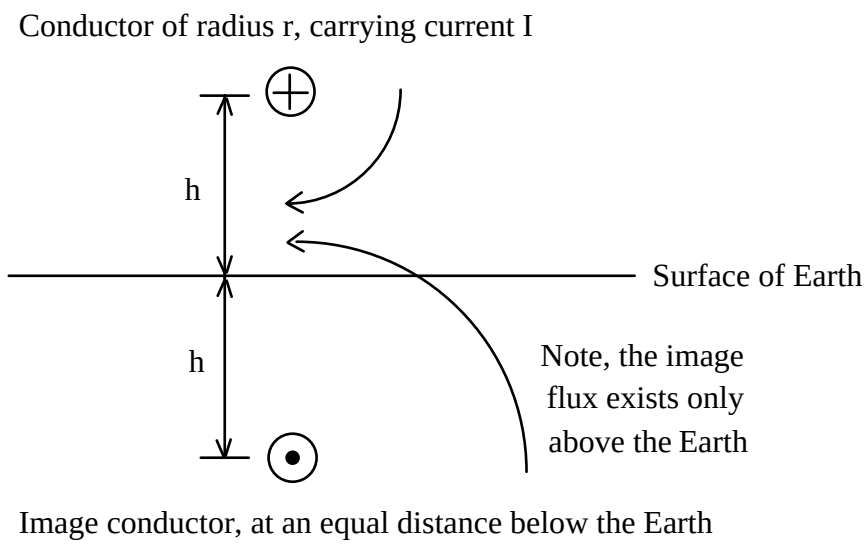


Figure 10. Current-Carrying Conductor Above Earth

The total flux linking the circuit is that which passes between the conductor and the surface of the earth. Summing the contribution of the conductor and its image yields

$$\Phi = \frac{\mu_o I}{2\pi}\left[\int_r^h \frac{dx}{x} + \int_h^{2h-r} \frac{dx}{x}\right] = \frac{\mu_o I}{2\pi}\ln\left[\frac{h(2h-r)}{rh}\right] = \frac{\mu_o I}{2\pi}\ln\left[\frac{(2h-r)}{r}\right] .$$

For $2h >> r$, a good approximation is

$$\Phi = \frac{\mu_o I}{2\pi}\ln\frac{2h}{r} \text{ Webers per meter length,}$$

so that the external inductance per meter length of the circuit becomes

$$L_{ext} = \frac{\mu_o}{2\pi}\ln\frac{2h}{r} \text{ Henrys per meter length.}$$

The total inductance is then the external inductance plus the internal inductance of one wire, or

$$L_{tot} = \frac{\mu_o}{2\pi}\ln\frac{2h}{r} + \frac{\mu_o}{8\pi} = \frac{\mu_o}{2\pi}\left[\ln\frac{2h}{r} + \frac{1}{4}\right] = \frac{\mu_o}{2\pi}\ln\frac{2h}{re^{-\frac{1}{4}}} ,$$

or, using the effective radius definition from before,

$$L_{tot} = \frac{\mu_o}{2\pi}\ln\frac{2h}{r_{eff}} \text{ Henrys per meter length.}$$

Bundled Conductors

The bundled conductor equivalent radii presented earlier apply for inductance as well as for capacitance. The question now is "what is the internal inductance of a bundle?" For N bundled conductors, the net internal inductance of a phase per meter must decrease as $\frac{1}{N}$ because the internal inductances are in parallel. Considering a bundle over the Earth, then

$$L_{tot} = \frac{\mu_o}{2\pi}\ln\frac{2h}{r_{eq}} + \frac{\mu_o}{8\pi N} = \frac{\mu_o}{2\pi}\left[\ln\frac{2h}{r_{eq}} + \frac{1}{4N}\right] = \frac{\mu_o}{2\pi}\left[\ln\frac{2h}{r_{eq}} + \frac{1}{N}\ln\left(e^{\frac{1}{4}}\right)\right] = \frac{\mu_o}{2\pi}\ln\left(\frac{2h}{r_{eq}e^{-\frac{1}{4N}}}\right).$$

Factoring in the expression for the equivalent bundle radius r_{eq} yields

$$r_{eq}e^{-\frac{1}{4N}} = \left[NrA^{N-1}\right]^{\frac{1}{N}} \bullet e^{-\frac{1}{4N}} = \left[Nre^{-\frac{1}{4}}A^{N-1}\right]^{\frac{1}{N}} = \left[Nr_{eff}A^{N-1}\right]^{\frac{1}{N}}$$

Thus, r_{eff} remains $re^{-\frac{1}{4}}$, no matter how many conductors are in the bundle.

The Three-Phase Case

For situations with multiples wires above the Earth, a matrix approach is needed. Consider the capacitance example given in Figure 6, except this time compute the external inductances, rather than capacitances. As far as the voltage (with respect to ground) of one of the a-b-c phases is concerned, the important flux is that which passes between the conductor and the Earth's surface. For example, the flux "linking" phase a will be produced by six currents: phase a current and its image, phase b current and its image, and phase c current and its image, and so on. Figure 11 is useful in visualizing the contribution of flux "linking" phase a that is caused by the current in phase b (and its image).

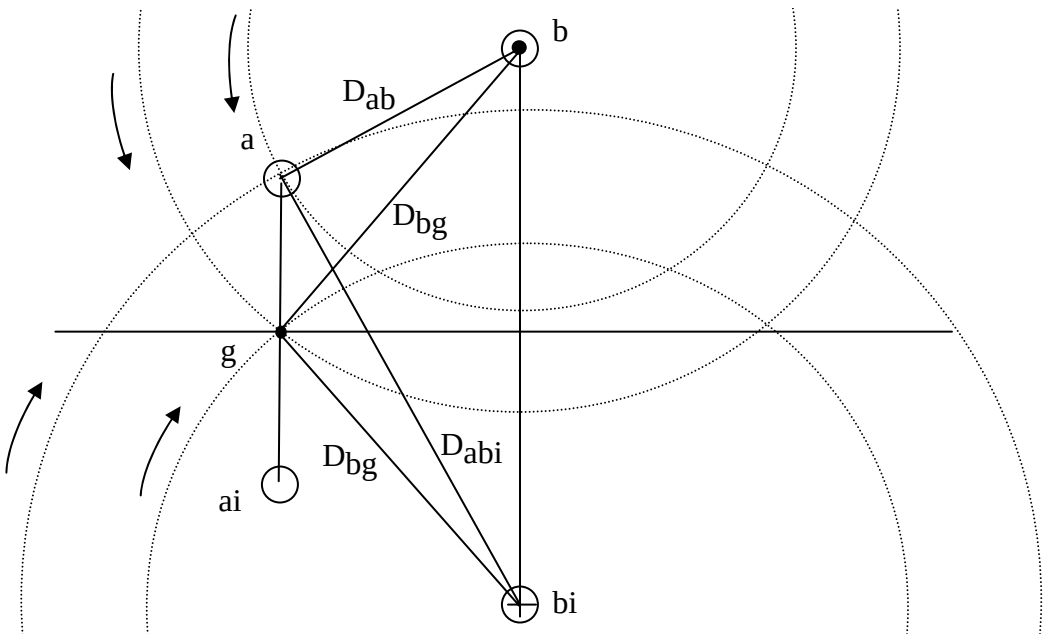


Figure 11. Flux Linking Phase a Due to Current in Phase b and Phase b Image

The linkage flux is

$$\Phi_a \text{(due to } I_b \text{ and } I_b \text{ image)} = \frac{\mu_o I_b}{2\pi}\ln\frac{D_{bg}}{D_{ab}} + \frac{\mu_o I_b}{2\pi}\ln\frac{D_{abi}}{D_{bg}} = \frac{\mu_o I_b}{2\pi}\ln\frac{D_{abi}}{D_{ab}} \ .$$

Considering all phases, and applying superposition, yields the total flux

$$\Phi_a = \frac{\mu_o I_a}{2\pi}\ln\frac{D_{aai}}{r_a} + \frac{\mu_o I_b}{2\pi}\ln\frac{D_{abi}}{D_{ab}} + \frac{\mu_o I_c}{2\pi}\ln\frac{D_{aci}}{D_{ac}} \ .$$

Note that D_{aai} corresponds to 2h in Figure 10. Performing the same analysis for all three phases, and recognizing that $N\Phi = LI$, where N = 1 in this problem, then the inductance matrix is developed using

$$\begin{bmatrix}\Phi_a\\ \Phi_b\\ \Phi_c\end{bmatrix} = \frac{\mu_o}{2\pi}\begin{bmatrix}\ln\frac{D_{aai}}{r_a} & \ln\frac{D_{abi}}{D_{ab}} & \ln\frac{D_{aci}}{D_{ac}}\\ \ln\frac{D_{bai}}{D_{ba}} & \ln\frac{D_{bbi}}{r_b} & \ln\frac{D_{bci}}{D_{bc}}\\ \ln\frac{D_{cai}}{D_{ca}} & \ln\frac{D_{cbi}}{D_{cb}} & \ln\frac{D_{cci}}{r_c}\end{bmatrix}\begin{bmatrix}I_a\\ I_b\\ I_c\end{bmatrix} \text{, or } \Phi_{abc} = L_{abc}I_{abc} \ .$$

A comparison to the capacitance matrix derivation shows that the same matrix of natural logarithms is used in both cases, and that

$$L_{abc} = \frac{\mu_o}{2\pi}P_{abc} = \frac{\mu_o}{2\pi}\bullet 2\pi\varepsilon_o \bullet C_{abc}^{-1} = \mu_o\varepsilon C_{abc}^{-1} \ .$$

This implies that the product of the L and C matrices is a diagonal matrix with $\mu_o\varepsilon$ on the diagonal, providing that the earth is assumed to be a perfect conductor and that the internal inductances of the wires are ignored.

If the circuit has ground wires, then the dimension of L increases accordingly. Recognizing that the flux linking the ground wires is zero (because their voltages are zero), then L can be Kron reduced to yield an equivalent 3 x 3 matrix L'_{abc} .

To include the internal inductance of the wires, replace actual conductor radius r with r_{eff} .

Computing 012 Inductances from Matrices

Once the 3 x 3 L'_{abc} matrix is found, 012 inductances can be determined by averaging the diagonal terms, and averaging the off-diagonal terms, of L'_{abc} to produce

$$L_{abc}^{avg} = \begin{bmatrix} L_S & L_M & L_M \\ L_M & L_S & L_S \\ L_M & L_M & L_S \end{bmatrix},$$

so that

$$L_{012}^{avg} = \begin{bmatrix} L_S + 2L_M & 0 & 0 \\ 0 & L_S - L_M & 0 \\ 0 & 0 & L_S - L_M \end{bmatrix}.$$

The Approximate Formulas for 012 Inductancess

Because of the similarity to the capacitance problem, the same rules for eliminating ground wires, for transposition, and for bundling conductors apply. Likewise, approximate formulas for the positive, negative, and zero sequence inductances can be developed, and these formulas are

$$L_1 = L_2 = \frac{\mu_o}{2\pi} \ln \frac{GMD_{1,2}}{GMR_{1,2}},$$

and

$$L_0 = 3\frac{\mu_o}{2\pi} \ln \frac{GMD_0}{GMR_0}.$$

It is important to note that the GMD and GMR terms for inductance differ from those of capacitance in two ways:

1. *GMR* calculations for inductance calculations should be made with $r_{eff} = re^{-\frac{1}{4}}$.

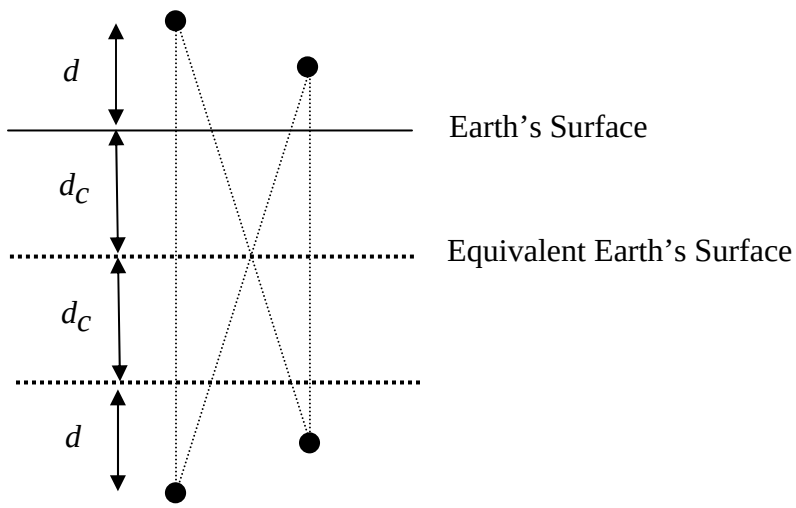
2. *GMD* distances for inductance calculations should include the equivalent complex depth for modeling finite conductivity earth (explained in the next section). This effect is ignored in capacitance calculations because the surface of the Earth is nominally at zero potential.

The Complex Depth Method for Modeling Imperfect Earth

The effect of the Earth's non-infinite conductivity should be included when computing inductances, especially zero sequence inductances. Because the Earth is not a perfect conductor, the image current does not actually flow on the surface of the Earth, but rather through a cross-section. The higher the conductivity, the narrower the cross-section.

The simplest method to account for finite conductivity is via a complex depth d_c, where the equivalent earth surface is assumed to be an additional d_c meters below the actual earth surface.

Using complex depth, if a conductor is d meters above earth, its image is $2(d + d_c)$ meters below the conductor.



The complex depth d_c is related to resistive skin depth δ by

$$d_c = \frac{\delta}{1+j} = \frac{\delta}{\sqrt{2}}\angle -45^o = \frac{\delta}{2} - j\frac{\delta}{2} \ ,$$

where

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \ .$$

At f = 60 Hz and $\sigma = 0.01\frac{1}{\Omega - m}$ (typical for limestone), the skin depth is 650 meters, so that

$$d_c = \frac{650}{2} - j\frac{650}{2} = 325 - j325 \text{ m.}$$

Skin depth in typical soils for 50 - 60 Hz varies from 500 to 2000 meters. The effect on positive/negative sequence inductances is not great, but the effect on zero sequence inductances is quite significant.

Since d_c is a frequency-dependent complex number, the self- and image-distance terms are complex, their natural logarithms are complex, and the L matrix contains complex numbers. The questions now are "how to take the natural log of a complex number and how to interpret the results?"

Begin with the natural log of a complex number $z = |z|e^{j\theta}$, which is

$$\ln(z) = \ln|z| + \ln(e^{j\vartheta}) = \ln|z| + j\theta \ , \ \theta \text{ in radians.}$$

Thus, for example, $\ln(325 - j325) = \ln(325) - j\frac{\pi}{4} = 5.78 - j0.785.$

To be consistent with the perfect Earth case (i.e., $\sigma \to \infty, \delta \to 0$), the real part of the natural logarithm must produce the inductive term. Then, the imaginary term yields the resistance of the Earth's current path, which must be added to the series resistances of the overhead conductors. In matrix form, the calculations are made accoding to

$$R_{abc}(\omega) + j\omega L_{abc}(\omega) = j\omega\frac{\mu_o}{2\pi}\begin{bmatrix} complex\ln\left(\frac{D_{aai}}{r_a}\right) & complex\ln\left(\frac{D_{abi}}{D_{ab}}\right) & complex\ln\left(\frac{D_{aci}}{D_{ac}}\right) \\ \bullet & complex\ln\left(\frac{D_{bbi}}{r_b}\right) & complex\ln\left(\frac{D_{bci}}{D_{bc}}\right) \\ \bullet & \bullet & complex\ln\left(\frac{D_{cci}}{r_c}\right) \end{bmatrix},$$

where complex depth d_c is included in the D terms. Because of the $j\omega$ multiplier, the real terms become positive, frequency-dependent resistances that account for the losses in the Earth. Off-diagonal resistances account for the fact that currents in neighboring phases contribute to the total voltage drop along the Earth, per meter, as seen by any given phase. The imaginary terms are inductive reactances.

4. Electric Field at Surface of Overhead Conductors

Ignoring all other charges, the electric field at a conductor's surface can be approximated by

$$E_r = \frac{q}{2\pi\varepsilon_o r} \ ,$$

where r is the radius. For overhead conductors, this is a reasonable approximation because the neighboring line charges are relatively far away. It is always important to keep the peak electric field at a conductor's surface below 30kV/cm to avoid excessive corono losses.

Going beyond the above approximation, the Markt-Mengele method provides a detailed procedure for calculating the maximum peak subconductor surface electric field intensity for three-phase lines with identical phase bundles. Each bundle has N symmetric subconductors of radius r. The bundle radius is A. The procedure is

1. Treat each phase bundle as a single conductor with equivalent radius

$$r_{eq} = \left[NrA^{N-1}\right]^{1/N} \ .$$

2. Find the *C(N x N)* matrix, including ground wires, using average conductor heights above ground. Kron reduce *C(N x N)* to *C(3 x 3)*. Select the phase bundle that will have the greatest peak line charge value (q_{lpeak}) during a 60Hz cycle by successively placing maximum line-to-ground voltage V_{max} on one phase, and – $V_{max}/2$ on each of the other two phases. Usually, the phase with the largest diagonal term in *C(3 by 3)* will have the greatest q_{lpeak}.

3. Assuming equal charge division on the phase bundle identified in Step 2, ignore equivalent line charge displacement, and calculate the average peak subconductor surface electric field intensity using

$$E_{avg,\,peak} = \frac{q_{lpeak}}{N} \bullet \frac{1}{2\pi\varepsilon_o r}$$

4. Take into account equivalent line charge displacement, and calculate the maximum peak subconductor surface electric field intensity using

$$E_{\max,\,peak} = E_{avg,\,peak}\left[1 + (N-1)\frac{r}{A}\right].$$

5. Resistance and Conductance

The resistance of conductors is frequency dependent because of the resistive skin effect. Usually, however, this phenomenon is small for 50 - 60 Hz. Conductor resistances are readily obtained from tables, in the proper units of Ohms per meter length, and these values, added to the equivalent-earth resistances from the previous section, to yield the *R* used in the transmission line model.

Conductance *G* is very small for overhead transmission lines and can be ignored.

6. Underground Cables

Underground cables are transmission lines, and the model previously presented applies. Capacitance *C* tends to be much larger than for overhead lines, and conductance *G* should not be ignored.

For single-phase and three-phase cables, the capacitances and inductances per phase per meter length are

$$C = \frac{2\pi\varepsilon_o\varepsilon_r}{\ln\frac{b}{a}} \text{ Farads per meter length,}$$

and

$$L = \frac{\mu_o}{2\pi} \ln \frac{b}{a} \text{ Henrys per meter length,}$$

where b and a are the outer and inner radii of the coaxial cylinders. In power cables, $\frac{b}{a}$ is typically e (i.e., 2.7183) so that the voltage rating is maximized for a given diameter.

For most dielectrics, relative permittivity $\varepsilon_r = 2.0 - 2.5$. For three-phase situations, it is common to assume that the positive, negative, and zero sequence inductances and capacitances equal the above expressions. If the conductivity of the dielectric is known, conductance G can be calculated using

$$G = C \frac{\sigma}{\varepsilon} \text{ Mhos per meter length.}$$

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ACL $\oint \bar{H}\cdot d\bar{l} = I_{enc}$, $\bar{B} = \mu \bar{H}$

$\bar{H}$: Magnetic field intensity A/m

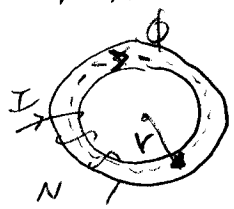
$\bar{B}$: Mag flux density

μ: permeability $\mu = \mu_0 \mu_r$, $\mu_0 = 4\pi\times10^{-7}$, $\mu_r = 1$ for air, $\mu_r = 100$ for steel

$$L I \equiv N\phi \qquad \phi = \int \bar{B}\cdot d\bar{A}$$

L: Inductance, N: turns linking flux, ϕ: Flux

Toroid



$$\oint H\cdot dl = NI$$

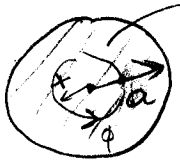
$$H_\phi 2\pi r = NI, \quad \frac{B_\phi}{\mu} 2\pi r = NI, \quad \frac{\phi}{\mu A}(2\pi r) = NI,$$

$$\phi = \frac{\mu A N I}{2\pi r}, \quad L \equiv \frac{N\phi}{I} = \frac{N\mu A N \cancel{I}}{2\pi r \cancel{I}} = \frac{N^2 \mu A}{2\pi r} \text{ H}$$

Work example toroid, using ruler to take measurements

Internal inductance of round wire

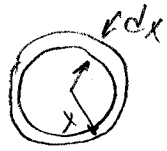
$$\underbrace{\tfrac{1}{2} L_{INT} I^2 = \tfrac{1}{2}\int BH\,dv}_{\text{energy concept}}$$



$J_{out} = \frac{I}{2\pi a^2}$ A/m² (uniform current density assumption)

$$W_{MAG\ per\ m\ length} = \frac{1}{2}\int BH\,dv = \frac{1}{2}\mu\int H^2 dv = \frac{1}{2}\mu \int_{VOL}\left[\frac{\overbrace{J\pi x^2}^{I}}{2\pi x}\right]^2 2\pi x\,dx$$

dv



$2\pi x\,dx$ per unit length

$$W_{MAG\ per\ m} = \frac{1}{2}\mu \frac{J^2\pi^2}{4\pi^2}\cdot 2\pi \int_{x=0}^{a} x^3 dx \qquad \frac{1}{4}x^4$$

$$= \frac{\mu}{8\pi^2}(J\pi a^2)^2 \cdot \frac{2\pi}{4} = \frac{\mu I^2}{16\pi} \equiv \frac{1}{2} L I^2$$

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$$\text{so } \boxed{L = \frac{\mu \cancel{I^2}}{16\pi}\left(\frac{2}{\cancel{I^2}}\right) = \frac{\mu}{8\pi} \text{ H/m}}$$

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For Cu, Al, $\mu = \mu_0 = 4\pi \times 10^{-7}$

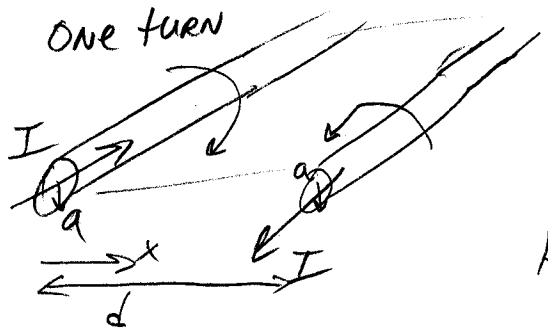
$$L_{int} = \frac{4\pi \cdot 10^{-7}}{8\pi} = 0.5 \cdot 10^{-7} = 0.05\ \mu H/m$$

So, 1 KM wire has 50 μH of internal inductance

Transmission Line

Two wires in space — far from ground

One turn



$$LI \equiv N\phi$$

For the left conductor

$$H_\phi = \frac{I}{2\pi x}$$

$$B_\phi = \mu_0 H_\phi$$

$d >> a$

ϕ (cutting surface between the conductors) $= \int B_\phi dx$ per m length

due to left. No internal wire consid.

$$\phi = \int_{x=a}^{d} \frac{\mu_0 I}{2\pi x} dx = \frac{\mu_0 I}{2\pi} \ln(x) \Big|_{x=a}^{d} = \frac{\mu_0 I}{2\pi} \ln \frac{d}{a}$$

ϕ (due to Right) is same, so $\phi_{TOT} = \frac{\mu_0 I}{\pi} \ln \frac{d}{a}$

Now, $\boxed{L_{ext} = \frac{N\phi_{TOT}}{I} = \frac{\mu_0 \cancel{I}}{\pi \cancel{I}} \ln \frac{d}{a} = \frac{\mu_0}{\pi} \ln \frac{d}{a}}$ H/m

For the two-wire pair

$$L_{TOT} = L_{ext} + 2L_{INT} = \frac{\mu_0}{\pi} \ln \frac{d}{a} + \frac{2\mu_0}{8\pi} \quad H/m$$

$$L_{TOT} = \frac{\mu_0}{\pi}\left[\ln \frac{d}{a} + \frac{1}{4}\right] = \frac{\mu_0}{\pi}\left[\ln \frac{d}{a} + \ln e^{1/4}\right] \quad H/m$$

$$= \frac{\mu_0}{\pi}\left[\ln \frac{d}{a} - \ln e^{-1/4}\right] = \frac{\mu_0}{\pi} \ln\left(\frac{d}{\underbrace{a e^{-1/4}}_{\text{equivalent radius}}}\right)$$

$e^{-1/4} = 0.7788$

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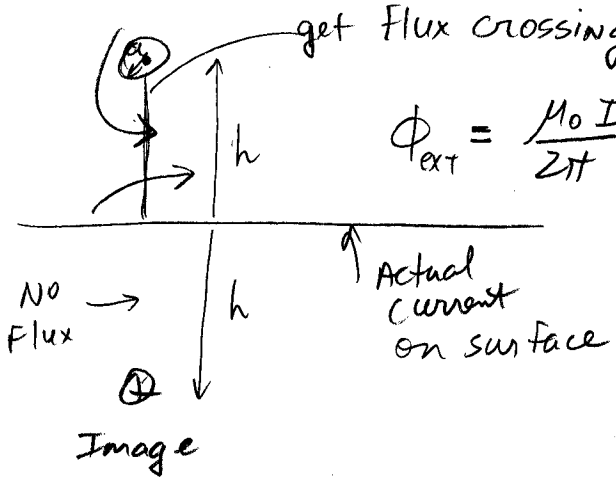
One wire over perfect earth



$$\phi_{ext} = \frac{\mu_0 I}{2\pi}\left[\ln\frac{h}{a} + \ln\frac{2h}{h}\right] = \frac{\mu_0 I}{2\pi}\left[\ln\frac{h}{a} + \ln 2\right] \quad H/m$$

$$= \frac{\mu_0 I}{2\pi}\ln\frac{2h}{a}$$

So $\boxed{L_{ext} = \frac{\mu_0}{2\pi}\ln\frac{2h}{a} \quad H/m}$

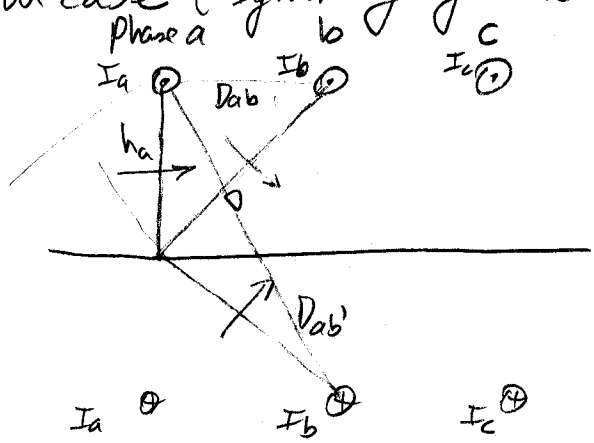
Note – $2h$ is distance of separation

Add the internal

$$L_{TOT} = L_{ext} + 1L_{INT} = \frac{\mu_0}{2\pi}\left[\ln\frac{2h}{a} + \frac{1}{4}\right] = \frac{\mu_0}{2\pi}\left[\ln\frac{2h}{a} + \ln e^{1/4}\right]$$

Again, $L_{TOT} = \frac{\mu_0}{2\pi}\ln\frac{2h}{ae^{-1/4}}$ H/m equiv.

Our case (Ignoring ground wires)



Flux crossing surface yields $N\frac{d\phi}{dt}$

$$\phi_a = \underbrace{\frac{\mu_0 I_a}{2\pi}\ln\frac{2h_a}{r_a}}_{\text{due to phase a \& image}} + \underbrace{\left[\frac{\mu_0 I_b}{2\pi}\ln\frac{D}{D_{ab}} + \frac{\mu_0 I_b}{2\pi}\ln\frac{D_{ab}'}{D}\right]}_{\frac{\mu_0}{2\pi}\ln\frac{D_{ab}'}{D_{ab}}}$$

h_a – height of a

r_a – radius of a

So, the pattern is

$$\phi_a = \frac{\mu_0 I_a}{2\pi}\ln\frac{2h_a}{r_a} + \frac{\mu_0}{2\pi} I_b \ln\frac{D_{ab}'}{D_{ab}} + \frac{\mu_0}{2\pi} I_c \ln\frac{D_{ac}'}{D_{ac}}$$

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Which Leads Us to A Matrix

$$\begin{bmatrix}\phi_a\\ \phi_b\\ \phi_c\end{bmatrix} = \frac{\mu_0}{2\pi}\begin{bmatrix}\ln\frac{2h_a}{r_a} & \ln\frac{D_{ab}'}{D_{ab}} & \ln\frac{D_{ac}'}{D_{ac}}\\ \ln\frac{D_{ba}'}{D_{ba}} & \ln\frac{2h_b}{r_b} & \ln\frac{D_{bc}'}{D_{bc}}\\ & & \ln\frac{2h_c}{r_c}\end{bmatrix}\begin{bmatrix}I_a\\ I_b\\ I_c\end{bmatrix} \quad H/m$$

symmetric

With <u>transposition</u>, A common Assumption is to Average

$$\begin{bmatrix}\phi_a\\ \phi_b\\ \phi_c\end{bmatrix} = \frac{\mu_0}{2\pi}\begin{bmatrix}S & M & M\\ M & S & M\\ M & M & S\end{bmatrix} H/m, \quad S = \frac{1}{3}\left[\ln\frac{2h_a}{r_a} + \ln\frac{2h_b}{r_b} + \ln\frac{2h_c}{r_c}\right]$$

$$M = \frac{1}{3}\left[\ln\frac{D_{ab}'}{D_{ab}} + \ln\frac{D_{ac}'}{D_{ac}} + \ln\frac{D_{bc}'}{D_{bc}}\right]$$

$$S = \frac{1}{3}\ln\frac{(2h_a)(2h_b)(2h_c)}{r_a r_b r_c} = \ln\frac{\sqrt[3]{(2h_a)(2h_b)(2h_c)}}{\sqrt[3]{r_a r_b r_c}}$$

GMD (phases to own images)

GMR_L

$$M = \ln\frac{\sqrt[3]{D_{ab}' D_{ac}' D_{bc}'}}{\sqrt[3]{D_{ab} D_{ac} D_{bc}}}$$

GMD (phases to other images)

GMD (phases to other phases)

For <u>balanced</u> Case

$$I_a = -I_b - I_c$$

$$\phi_a = \frac{\mu_0}{2\pi}\left[S I_a + M I_b + M I_c\right] = \frac{\mu_0}{2\pi}\left[S I_a + M(-I_a)\right]$$

$$\phi_a = \frac{\mu_0}{2\pi}(S - M) I_a$$

Likewise for ϕ_b, ϕ_c (phase shift because of I_a, I_b, I_c)

$$\phi_a = \frac{\mu_0 I_a}{2\pi}\left[\ln\frac{\sqrt[3]{(2h_a)(2h_b)(2h_c)}}{\sqrt[3]{r_a r_b r_c}} - \ln\frac{\sqrt[3]{D_{ab}' D_{ac}' D_{bc}'}}{\sqrt[3]{D_{ab} D_{ac} D_{bc}}}\right]$$

$$= \frac{\mu_0 I_a}{2\pi}\ln\frac{\sqrt[3]{(2h_a)(2h_b)(2h_c) D_{ab} D_{ac} D_{bc}}}{\sqrt[3]{r_a r_b r_c D_{ab}' D_{ac}' D_{bc}'}}$$

About 1

$$\phi_a = \frac{\mu_0 I_a}{2\pi}\ln\frac{\sqrt[3]{D_{ab} D_{ac} D_{bc}}}{\sqrt[3]{r_a r_b r_c}}.$$

Ditto for ϕ_b, ϕ_c

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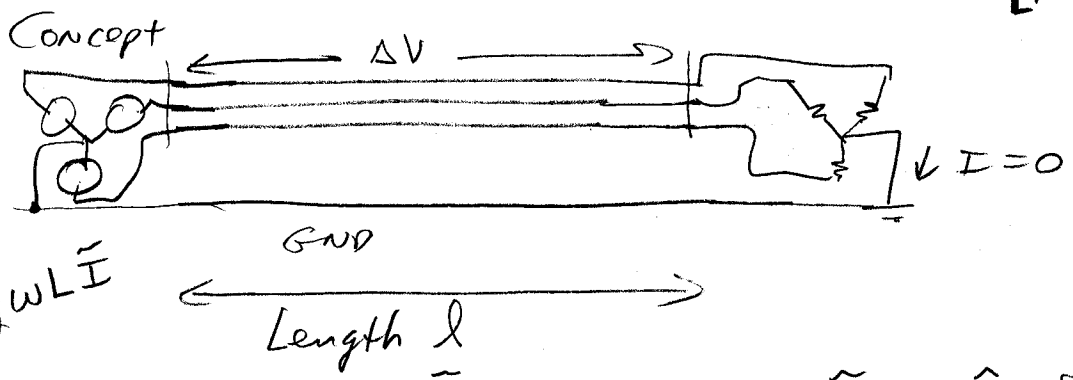
So we have a decoupling for ⊕ sequence, $\phi_a = f(I_a)$, $\phi_b = f(I_b)$, $\phi_c = f(I_c)$

$$L_+ = \frac{\mu_0}{2\pi} \ln \frac{\sqrt[3]{D_{ab} D_{ac} D_{bc}}}{\sqrt[3]{r_a r_b r_c}} \text{ H/m}$$

GMD_+

$GMR_{L+} = r_a$ usually (if $r_a = r_b = r_c$)

Concept



$$\Delta\tilde{V}_a = j\omega \frac{\mu_0 \tilde{I}_a}{2\pi} \ln \frac{GMD_+}{GMR_{L+}} \text{ V/M}, \quad \Delta\tilde{V}_b, \Delta\hat{V}_c \text{ follow suit}$$

For a single phase system, we would have

$$\Delta\tilde{V}_a = j\omega \frac{\mu_0}{2\pi} \ln \frac{2h_a}{r_a}$$

Now, how to include the internal inductance?

Phase a would have an additional $\frac{\mu_0}{8\pi}$ H/m

$$L_+ = \frac{\mu_0}{2\pi}\left[\ln \frac{GMD_+}{\sqrt[3]{r_a r_b r_c}} + \frac{1}{4}\right] \text{ H/m}$$

$$= \frac{\mu_0}{2\pi}\left[\ln \frac{GMD_+}{\sqrt[3]{r_a r_b r_c}} + \ln \frac{1}{4}\right] = \frac{\mu_0}{2\pi}\left[\ln \frac{GMD_+}{\sqrt[3]{\quad} \cdot e^{-1/4}}\right]$$

Modified GMR_{L+}

OR

$$= \frac{\mu_0}{2\pi} \ln \frac{GMD_+}{\sqrt[3]{r_a r_b r_c}\,\sqrt[3]{e^{-3/4}}} = \frac{\mu_0}{2\pi} \ln \frac{GMD_+}{\sqrt[3]{(r_a e^{-1/4})(r_b e^{-1/4})(r_c e^{-1/4})}} \text{ H/m}$$

What about R? For ⊕ seq, it's simply the resistance of one wire, so $R_+ = R_a$ Ω/m

Also, each wire has a r_{gmr} due to stranding

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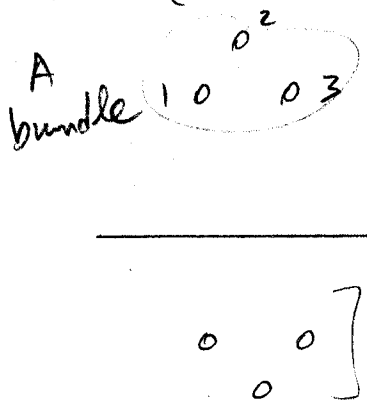
Bundled Conductors — One phase has N conductors

A bundle



D_{12}, D_{23}, D_{13}

D_{11^i}, D_{12^i}, etc

Symmetric bundle

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \frac{\mu_0}{2\pi} \begin{bmatrix} \ln\frac{2h_1}{r_1} & \ln\frac{D_{12^i}}{D_{12}} & \ln\frac{D_{13^i}}{D_{13}} \\ \ln\frac{D_{12^i}}{D_{12}} & \ln\frac{2h_2}{r_2} & \ln\frac{D_{23^i}}{D_{23}} \\ \ln\frac{D_{13^i}}{D_{13}} & \ln\frac{D_{23^i}}{D_{23}} & \ln\frac{2h_3}{r_3} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

Again, the distances to neighbor images are all about equal and the same as the distances to self images

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \frac{\mu_0}{2\pi} \begin{bmatrix} S & M & M \\ M & S & M \\ M & M & S \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$\approx \phi_{BUNDLE}$

Symmetric bundle, $\phi_1 \simeq \phi_2 \simeq \phi_3$, $I_1 = I_2 = I_3$

$$3\phi_{BUNDLE} = \frac{\mu_0}{2\pi}\left[S\underbrace{(I_1+I_2+I_3)}_{I_{BUNDLE}} + 2M\underbrace{(I_1+I_2+I_3)}_{I_{BUNDLE}}\right]$$

$$3\phi_{BUNDLE} = \underbrace{\frac{\mu_0}{2\pi}(S+2M)}_{\frac{\mu_0}{2\pi}\ln\frac{2h_{BUNDLE}}{r_{BUNDLE}}} I_{BUNDLE}$$

$$S+2M = \ln\frac{2h}{r_1} + 2\ln\frac{2h}{D_{12}} = \ln\left[\frac{2h(2h)^2}{r_1(D_{12})^2}\right] = \ln\left[\frac{(2h)^3}{r_1(D_{12})^2}\right]$$

$$= 3\ln\left[\frac{\sqrt[3]{(2h)(2h)(2h)}}{\sqrt[3]{r_1 D_{12} D_{12}}}\right], \text{ so } \phi_{BUNDLE} = \frac{\mu_0}{2\pi}\ln\left[\frac{\sqrt[3]{(2h)(2h)(2h)}}{\sqrt[3]{r_1 D_{12} D_{12}}}\right]$$

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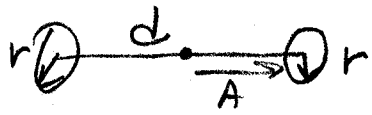
So the concept of a GMR for a bunde

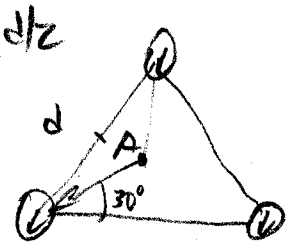


$$GMR = \sqrt{rd} = \sqrt{2rA}$$

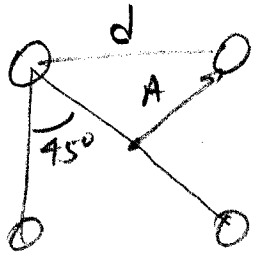
$A\cos 30^\circ = d/2$

$A\frac{\sqrt{3}}{2} = \frac{d}{2}$

$A = \frac{d}{\sqrt{3}}$



$$GMR = \sqrt[3]{rd^2} = \sqrt[3]{r(\sqrt{3}A)^2} = \sqrt[3]{3rA^2}$$



$$GMR = \sqrt[4]{r \cdot d \cdot d \cdot \sqrt{2}d} = \sqrt[4]{\sqrt{2}rd^3}$$

$A\cos 45^\circ = \frac{d}{2}$

$\frac{A}{\sqrt{2}} = \frac{d}{2}, \quad d = \frac{2A}{\sqrt{2}} = \sqrt{2}A$

$$GMR = \sqrt[4]{\sqrt{2}r(\sqrt{2}A)^3} = \sqrt[4]{4rA^3}$$

General Formula for symmetric bundle with N conductors and bundle radius A

$$GMR = \sqrt[N]{NrA^{N-1}}, \quad r \text{ for one conductor}$$

So, for a 3d bundled line

1. Get GMR for each bundle
2. Get D_{ab}, D_{bc}, D_{ac} from centers to centers
3. $\mathcal{L}_+ = \frac{\mu_0}{2\pi} \ln \frac{GMD_+}{GMR_+} = \frac{\mu_0}{2\pi} \ln \frac{\sqrt[3]{D_{ab}D_{ac}D_{bc}}}{GMR_+ \text{ (see above)}}$ H/m

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GMR_{L+}: use $re^{-1/4}$ instead of r

r_{GMR} for one conductor

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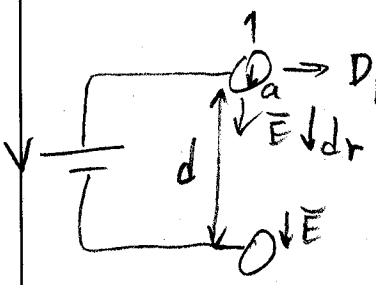
Two wires in space — Gauss' Law



$$D_r = \frac{q_\ell}{2\pi r}, \quad E_r = \frac{D_r}{\epsilon_0} = \frac{q_\ell}{2\pi\epsilon_0 r}$$

$$V = \int \bar{E}_r \cdot d\bar{r} = \frac{q_\ell}{2\pi\epsilon_0}\underbrace{\left[\int_{r=a}^{d}\frac{dr}{r} + \int_{r=a}^{d}\frac{dr}{r}\right]}_{\text{equal contributions}}$$

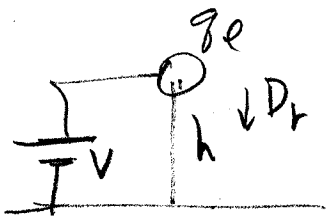
$$V = \frac{2q_\ell}{2\pi\epsilon_0}\ln\frac{d}{a} = \frac{q_\ell}{\pi\epsilon_0}\ln\frac{d}{a} \rightarrow q_\ell = \frac{V\pi\epsilon_0}{\ln d/a}$$

$$q_\ell \frac{C}{m} = (C\ F/m)\,V, \quad \boxed{C = \frac{\pi\epsilon_0}{\ln\frac{d}{a}}\ F/m}$$

Note, for L, $L_{ext} = \frac{\mu_0}{\pi}\ln\frac{d}{a}$, $\boxed{\underset{H/m}{L_{ext}}\,\underset{F/m}{C} = \mu_0\epsilon_0}$

$$\frac{1}{\sqrt{\mu_0\epsilon_0}} = 3\times10^8\ m/s = 300\ m/\mu sec$$

One wire over Earth



$$D_r = \frac{q_\ell}{2\pi r}, \quad E_r = \frac{q_\ell}{2\pi\epsilon_0 r},$$

$$V = \int_{r=a}^{h} E_r \cdot dr + \int_{r=h}^{2h} E_r\, dr$$

$$V = \frac{q_\ell}{2\pi\epsilon_0}\left[\ln\frac{h}{a} + \ln\frac{2h}{h}\right]$$

$$V = \frac{q_\ell}{2\pi\epsilon_0}\ln\frac{2h}{a},$$

so $C_\ell = \frac{q_\ell}{V} = \frac{2\pi\epsilon_0}{\ln\frac{2h}{a}}\ F/m$

Note, $L_{ext} = \frac{\mu_0}{2\pi}\ln\frac{2h}{a}$, $\boxed{\underset{H/m}{L_{ext}}\,\underset{F/m}{C} = \mu_0\epsilon_0}$

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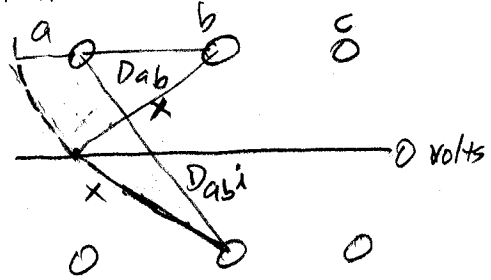
Three wires over Earth



$$V_{ag} = \underbrace{\frac{q_a}{2\pi\epsilon_0}\ln\frac{2h_a}{r_a}}_{\text{Due to } a \text{ \& } a'} + \frac{q_b}{2\pi\epsilon_0}\left[\ln\frac{x}{D_{ab}} + \ln\frac{D_{ab}'}{x}\right]$$

$+ q_c$ contribution

$$V_{ag} = \frac{1}{2\pi\epsilon_0}\left[\ln\frac{2h_a}{r_a}q_a + \ln\frac{D_{ab}'}{D_{ab}}q_b + \ln\frac{D_{ac}'}{D_{ac}}q_c\right]$$

Matrix

$$\begin{bmatrix} V_{ag} \\ V_{bg} \\ V_{cg} \end{bmatrix} = \frac{1}{2\pi\epsilon_0}\begin{bmatrix} \ln\frac{2h_a}{r_a} & \ln\frac{D_{ab}'}{D_{ab}} & \ln\frac{D_{ac}'}{D_{ac}} \\ & \ln\frac{2h_b}{r_b} & \ln\frac{D_{bc}'}{D_{bc}} \\ & & \ln\frac{2h_c}{r_c} \end{bmatrix}\begin{bmatrix} q_a \\ q_b \\ q_c \end{bmatrix}$$

symmetric

$$[V] = [C]^{-1}[Q], \quad \text{so} \quad [C_{3\times3}] = 2\pi\epsilon_0\left[\ln \text{Matrix}\right]^{-1}$$

before, $[L_{ext}]_{3\times3} = \frac{\mu_0}{2\pi}\left[\ln \text{Matrix}\right]$

$$[L_{ext}]_{3\times3}[C_{3\times3}] = \mu_0\epsilon_0\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} \mu_0\epsilon_0 & 0 & 0 \\ 0 & \mu_0\epsilon_0 & 0 \\ 0 & 0 & \mu_0\epsilon_0 \end{bmatrix}$$

So, you can use all the GMD, GMR rules from L, but use <u>actual</u> radius of conductors (no r_{gmr}) <u>and</u> no <u>$e^{-1/4}$</u> to get C

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Procedure for Computing Positive/Negative/Zero Sequence Line Constants for Transmission Lines in Air

POSITIVE/NEGATIVE SEQUENCE CALCULATIONS

Assumptions

Balanced, far from ground, ground wires ignored. Valid for identical single conductors per phase, or for identical symmetric phase bundles with N conductors per phase and bundle radius A.

Computation of positive/negative sequence capacitance

$$C_{+/-} = \frac{2\pi\varepsilon_o}{\ln \frac{GMD_{+/-}}{GMR_{C+/-}}} \text{ farads per meter,}$$

where

$$GMD_{+/-} = \sqrt[3]{D_{ab} \bullet D_{ac} \bullet D_{bc}} \text{ meters,}$$

where D_{ab}, D_{ac}, D_{bc} are

- distances between phase conductors if the line has one conductor per phase, or
- distances between phase bundle centers if the line has symmetric phase bundles,

and where

- $GMR_{C+/-}$ is the actual conductor radius r (in meters) if the line has one conductor per phase, or
- $GMR_{C+/-} = \sqrt[N]{N \bullet r \bullet A^{N-1}}$ if the line has symmetric phase bundles.

Computation of positive/negative sequence inductance

$$L_{+/-} = \frac{\mu_o}{2\pi} \ln \frac{GMD_{+/-}}{GMR_{L+/-}} \text{ henrys per meter,}$$

where $GMD_{+/-}$ is the same as for capacitance, and

- for the single conductor case, $GMR_{L+/-}$ is the conductor r_{gmr} (in meters), which takes into account both stranding and the $e^{-1/4}$ adjustment for internal inductance. If r_{gmr} is not given, then assume $r_{gmr} = re^{-1/4}$, and
- for bundled conductors, $GMR_{L+/-} = \sqrt[N]{N \bullet r_{gmr} \bullet A^{N-1}}$ if the line has symmetric phase bundles.

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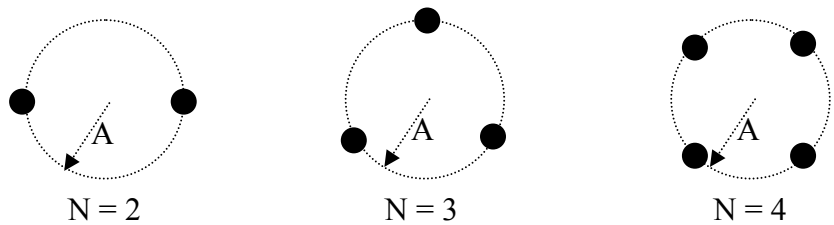
Procedure for Computing Positive/Negative/Zero Sequence Line Constants for Transmission Lines in Air

Computation of positive/negative sequence resistance

R is the 60Hz resistance of one conductor if the line has one conductor per phase. If the line has symmetric phase bundles, then divide the one-conductor resistance by N.

Some commonly-used symmetric phase bundle configurations

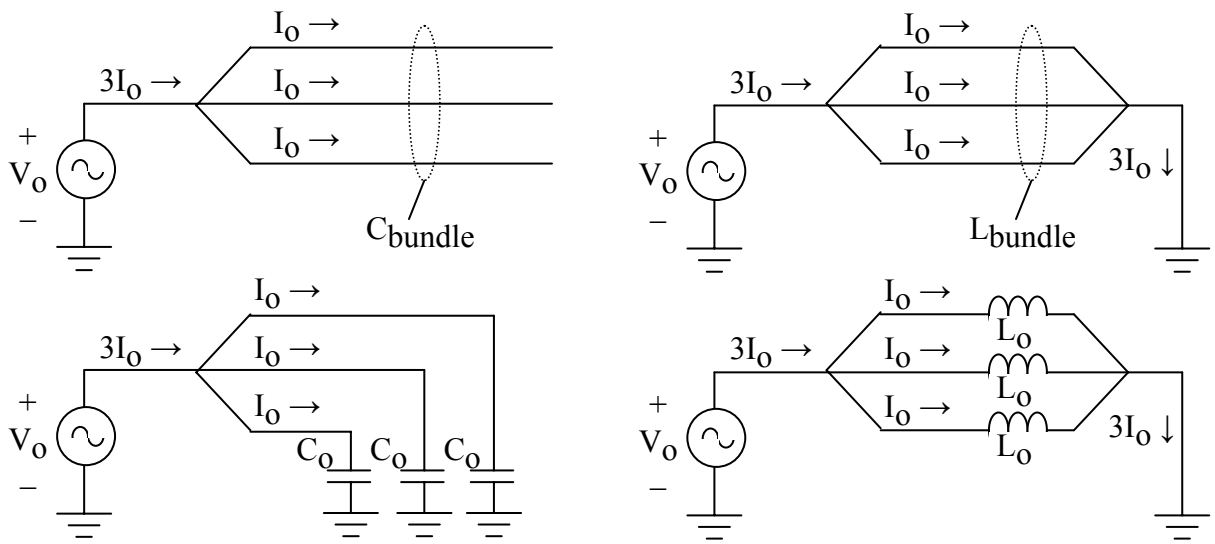


ZERO SEQUENCE CALCULATIONS

Assumptions

Ground wires are ignored. The a-b-c phases are treated as one bundle. If individual phase conductors are bundled, they are treated as single conductors using the bundle radius method. For capacitance, the Earth is treated as a perfect conductor. For inductance and resistance, the Earth is assumed to have uniform resistivity ρ. Conductor sag is taken into consideration, and a good assumption for doing this is to use an average conductor height equal to (1/3 the conductor height above ground at the tower, plus 2/3 the conductor height above ground at the maximum sag point).

The zero sequence excitation mode is shown below, along with an illustration of the relationship between bundle C and L and zero sequence C and L. Since the bundle current is actually $3I_O$, the zero sequence resistance and inductance are three times that of the bundle, and the zero sequence capacitance is one-third that of the bundle.



Computation of zero sequence capacitance

$$C_0 = \frac{1}{3} \bullet \frac{2\pi\varepsilon_o}{\ln \frac{GMD_{C0}}{GMR_{C0}}} \text{ farads per meter,}$$

where GMD_{C0} is the average height (with sag factored in) of the a-b-c bundle above perfect Earth. GMD_{C0} is computed using

$$GMD_{C0} = \sqrt[9]{D_{aa^i} \bullet D_{bb^i} \bullet D_{cc^i} \bullet D^2_{ab^i} \bullet D^2_{ac^i} \bullet D^2_{bc^i}} \text{ meters,}$$

where D_{aa^i} is the distance from a to a-image, D_{ab^i} is the distance from a to b-image, and so forth. The Earth is assumed to be a perfect conductor, so that the images are the same distance below the Earth as are the conductors above the Earth. Also,

$$GMR_{C0} = \sqrt[9]{GMR^3_{C+/-} \bullet D^2_{ab} \bullet D^2_{ac} \bullet D^2_{bc}} \text{ meters,}$$

where $GMR_{C+/-}$, D_{ab}, D_{ac}, and D_{bc} were described previously.

Computation of zero sequence inductance

$$L_0 = 3 \bullet \frac{\mu_o}{2\pi} \ln \frac{D_e}{GMR_{L0}} \text{ Henrys per meter,}$$

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where $D_e = 658.4\sqrt{\frac{\rho}{f}}$ meters (see Bergen), ρ is the resistivity of the Earth (a good assumption is 100Ω-m), and f is the frequency (Hz). D_e takes into account the fact that the Earth is resistive and that zero sequence currents flow deep in the Earth. In most cases, D_e is so large that the actual height of the conductors makes no difference in the calculations. For example, for 60Hz and ρ = 100Ω-m, D_e = 850m.

The geometric mean bundle radius is computed using

$$GMR_{L0} = \sqrt[9]{GMR_{L+/-}^3 \bullet D_{ab}^2 \bullet D_{ac}^2 \bullet D_{bc}^2} \text{ meters,}$$

where $GMR_{L+/-}$, D_{ab}, D_{ac}, and D_{bc} were shown previously.

Computation of zero sequence resistance

There are two components of zero sequence line resistance. First, the equivalent conductor resistance is the 60Hz resistance of one conductor if the line has one conductor per phase. If the line has symmetric phase bundles with N conductors per bundle, then divide the one-conductor resistance by N.

Second, the effect of resistive Earth is included by adding the following term to the conductor resistance:

$$3 \bullet 9.869 \bullet 10^{-7} f \text{ ohms per meter (see Bergen),}$$

where the multiplier of three is needed to take into account the fact that all three zero sequence currents flow through the Earth.

Electric Field at Surface of Overhead Conductors

Ignoring all other charges, the electric field at a conductor's surface can be approximated by

$$E_r = \frac{q}{2\pi\varepsilon_o r} ,$$

where r is the radius. For overhead conductors, this is a reasonable approximation because the neighboring line charges are relatively far away. It is always important to keep the peak electric field at a conductor's surface below 30kV/cm to avoid excessive corono losses.

Going beyond the above approximation, the **Markt-Mengele** method provides a detailed procedure for calculating the maximum peak subconductor surface electric field intensity for three-phase lines with identical phase bundles. Each bundle has N symmetric subconductors of radius r. The bundle radius is A. The procedure is

Procedure for Computing Positive/Negative/Zero Sequence Line Constants for Transmission Lines in Air

1. Treat each phase bundle as a single conductor with equivalent radius

$$r_{eq} = \left[NrA^{N-1} \right]^{1/N} .$$

2. Find the *C(N x N)* matrix, including ground wires, using average conductor heights above ground. Kron reduce *C(N x N)* to *C(3 x 3)*. Select the phase bundle that will have the greatest peak line charge value (q_{lpeak}) during a 60Hz cycle by successively placing maximum line-to-ground voltage V_{max} on one phase, and – $V_{max}/2$ on each of the other two phases. Usually, the phase with the largest diagonal term in *C(3 by 3)* will have the greatest q_{lpeak}.

3. Assuming equal charge division on the phase bundle identified in Step 2, ignore equivalent line charge displacement, and calculate the average peak subconductor surface electric field intensity using

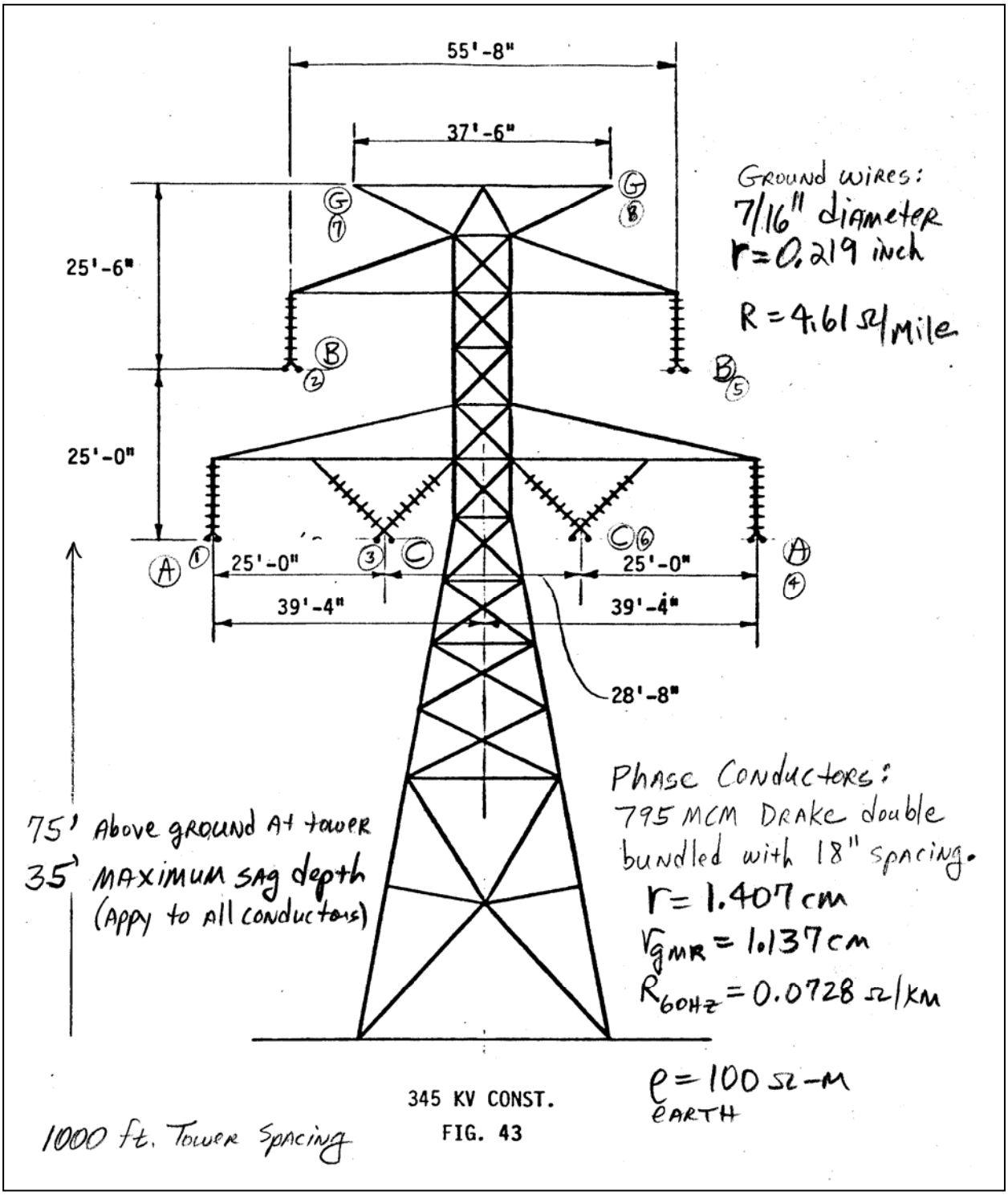
$$E_{avg,peak} = \frac{q_{lpeak}}{N} \bullet \frac{1}{2\pi\varepsilon_o r}$$

4. Take into account equivalent line charge displacement, and calculate the maximum peak subconductor surface electric field intensity using

$$E_{\max,peak} = E_{avg,peak} \left[1 + (N-1)\frac{r}{A} \right] .$$

Procedure for Computing Positive/Negative/Zero Sequence Line Constants for Transmission Lines in Air

Practice Problem – a 345kV double-circuit configuration that is commonly used in Texas

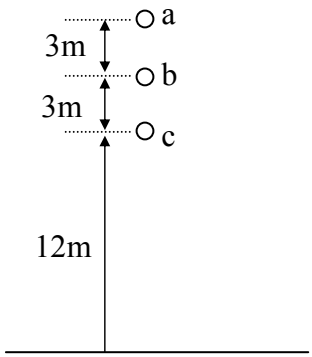


345 KV CONST.
FIG. 43

Grady 2007, p. 153

Consider the 69kV transmission line shown below. The phases have single conductors.
a. Find the positive sequence R, L, and C per meter for the transmission line shown.
b. Convert the above values to series (R + jX), and shunt MVAr, per km, using a base of 69kV, 100MVA.

Conductors have outer radius 1cm, and resistance 0.06 Ω per km



$\rho = 100\Omega\text{-m}$

EE369, Final Exam, 2-5pm, Dec. 9, WRW102. Name: KEY

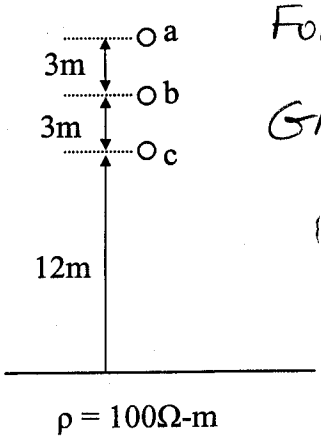
Six equally-weighted problems. Three sheets of notes permitted. Show all work.

Problem 4. Consider the 69kV, 60Hz transmission line shown below. The phases have single conductors.

a. Find the positive sequence R, L, and C per meter for the transmission line shown.

b. Convert the above values to series (R + jX), and shunt MVAr, per km, using a base of 69kV, 100MVA.

Conductors have outer radius 1cm, and resistance 0.06 Ω per km



For CAP, $r = 1\,cm$. For ind, $r_{eff} = re^{-1/4} = 0.7788\,cm$

$$GMD_{+/-} = (3\cdot 6\cdot 3\cdot 3\cdot 3\cdot 6)^{1/6} = 3.78\,m$$

$$C_{+/-} = \frac{2\pi\epsilon_0}{\ln\frac{GMD_{+/-}}{r}} = \frac{2\pi(8.854)}{\ln\frac{3.78}{0.01}}\,pF/m = 9.37\,pF/m$$

$$L_{+/-} = \frac{\mu_0}{2\pi}\ln\frac{GMD_{+/-}}{r_{eff}} = \frac{4\pi\times10^{-7}}{2\pi}\ln\frac{3.78}{0.007788} = 1.237\,\mu H/m$$

$$R_{+/-} = 0.06\,\Omega/km$$

So, $R_{+/-} = 6\times10^{-5}\,\Omega/m$

(a) $C_{+/-} = 9.37\,pF/m$

$L_{+/-} = 1.237\,\mu H/m$

(b) $Z_B = \frac{(69)^2}{100} = 47.61\,\Omega$. $R = \frac{0.06\,\Omega/km}{47.61} = 0.001260\,pu/km$

$j\omega L = j(120\pi)(1.237\times10^{-6})(1000) = j0.466\,\Omega/km = j0.00980\,pu/km$ ($j\omega L$)

$Q_{CAP} = 3V_{LN}^2\omega C = (3)\left(\frac{69000}{\sqrt{3}}\right)^2(120\pi)(9.37\times10^{-12})(1000) = 16.82\,KVAr/km$ (3φ)

$Q_{CAP\,pu} = \frac{16.82\times10^3}{100\times10^6} = 0.000168\,VAr/km$

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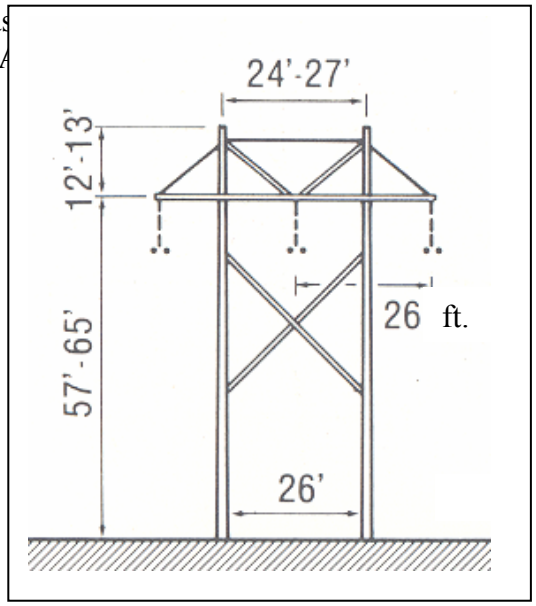
Each phase of the 345kV transmission configuration shown has
conductors, separated by 45.7cm. The characteristics of a 795A

$r = 1.407\text{cm}$, $r_{gmr} = 1.137\text{cm}$,
$R_{60Hz} = 0.0728\Omega/\text{km}$.

The conversion from feet to meters is 3.281 feet per meter.

a. Find the positive/negative sequence C, L, and R per meter of length

b. Draw the corresponding pi-equivalent circuit, in ohms @ 60Hz, for a 100km long line.



Grady 2007, p. 156

EE369/394J, Test #3, Sept. 17, 2004. Name KEY

One sheet of notes permitted. Show all steps.

Pos/Neg Seq.

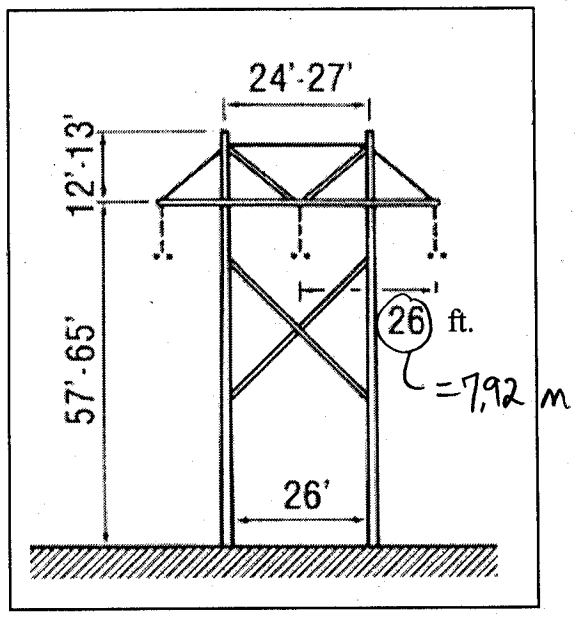
Each phase of the 345kV transmission configuration shown has two bundled 795ACSR conductors, separated by 45.7cm. The characteristics of a 795ACSR conductor are

$r = 1.407\text{cm}$, $r_{gmr} = 1.137\text{cm}$,
$R_{60Hz} = 0.0728\Omega/\text{km}$.

The conversion from feet to meters is 3.281 feet per meter.

a. Find the positive/negative sequence C, L, and R per meter of length

b. Draw the corresponding pi-equivalent circuit, in ohms @ 60Hz, for a 100km long line.



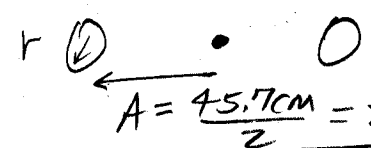
$$R_{60Hz} = \frac{0.0728}{2}\,\Omega/\text{km} = 0.0364\,\Omega/\text{km} = 36.4\,\mu\Omega/\text{m}$$ → 100KM has $3.64\,\Omega$

$$C_{+/-} = \frac{2\pi\epsilon_0}{\ln \frac{GMD_{+/-}}{GMR_{c+/-}}}\ \text{F/m}.$$

$$GMD_{+/-} = \sqrt[6]{(7.92)(15.84)(7.92)(7.92)(7.92)(15.84)}$$

$$GMD_{+/-} = 9.98\text{m}$$

For the bundle



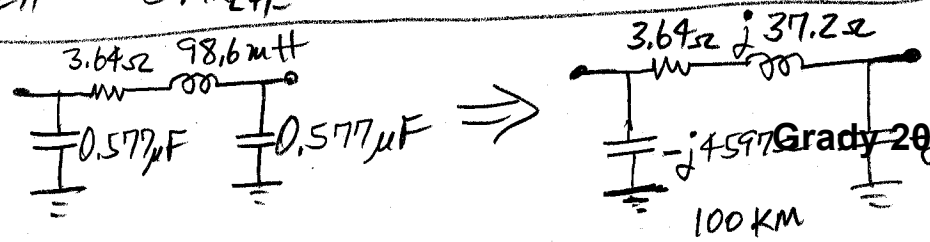
$$GMR_{c+/-} = \sqrt{2rA} = \sqrt{2(1.407)(22.85)} = 8.02\text{cm}$$

$$C_{+/-} = \frac{2\pi(8.854)}{\ln \frac{9.98}{0.0802}}\ \text{pF/m} = 11.53\ \text{pF/m}$$

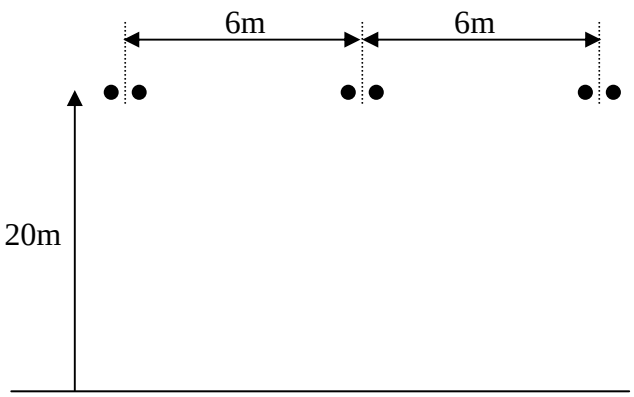
$$GMR_{L+/-} = \sqrt{2 r_{gmr} A} = \sqrt{2(1.137)(22.85)} = 7.21\text{cm}$$

$$L_{+/-} = \frac{\mu_0}{2\pi}\ln\frac{GMD_{+/-}}{GMR_{L+/-}} = (2\times10^{-7})\ln\frac{9.98}{0.0721} = 0.986\,\mu\text{H/m}$$

100KM



The a-b-c phase bundles of a three-phase, 345kV transmission line have the spacings and average heights above Earth as shown. Each conductor has outer radius 1cm, and the two conductors per phase are separated by 45cm. Ignoring ground wires, and making the "far from Earth" assumption, compute the positive/negative sequence inductance and capacitance per meter length (express your answer in μH/m and pF/m). Important - include the internal inductance in the μH/m calculation.

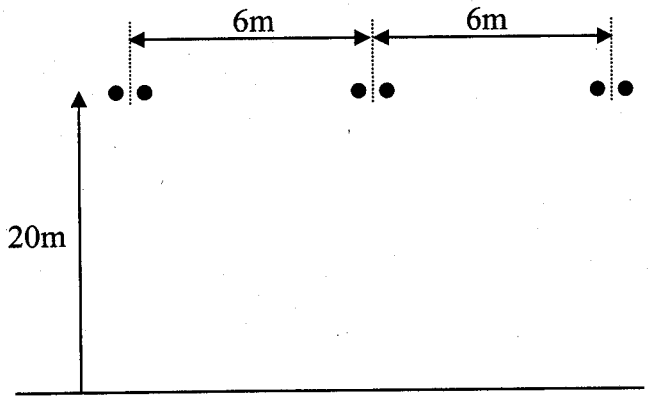


EE368, Test 1. March 3, 2006. One page of notes permitted. Name KEY

Show all work on these pages. Clearly show all steps needed to fully support your answers.

Four equally-weighted problems. a.

Problem 3. The a-b-c phase bundles of a three-phase, 345kV transmission line have the spacings and average heights above Earth as shown. Each conductor has outer radius 1cm, and the two conductors per phase are separated by 45cm. Ignoring ground wires, and making the "far from Earth" assumption, compute the positive/negative sequence inductance and capacitance per meter length (express your answer in μH/m and pF/m). Important - include the internal inductance in the μH/m calculation.



$$GMD_{+/-} = \sqrt[6]{(6)(12)(6)(6)(6)(12)} = \sqrt[3]{(6)(12)(6)} = 7.56\,m$$

For inductance, $\mathcal{L}_{+/-} = \frac{\mu_0}{2\pi} \ln \frac{GMD_{+/-}}{GMR_{L+/-}} = (2\times10^{-7})$

A = Bundle radius $= \frac{45cm}{2} = \frac{0.45m}{2} = 0.225m$

$$GMR_{L+/-} = \sqrt[N]{N r A^{N-1}} = \sqrt{(2)(0.01e^{-1/4})(0.225)} = 0.0592m$$

$$\boxed{\mathcal{L}_{+/-} = (2\times10^{-7}) \ln \frac{7.56}{0.0592} = 0.970\,\mu H/m}$$

For capacitance, $C_{+/-} = \frac{2\pi\epsilon_0}{\ln \frac{GMD_{+/-}}{GMR_{C+/-}}} = \frac{2\pi(8.854\times10^{-12})}{\ln \frac{7.56}{0.0671}}$

$$GMR_{C+/-} = \sqrt{(2)(0.01)(0.225)} = 0.0671m$$

$$\boxed{C_{+/-} = 11.78\,pF/m}$$

Version b,
5m between phases

$GMD_{+/-} = 6.30m$

$\mathcal{L}_{+/-} = 0.933\,\mu H/m$

$C_{+/-} = 12.25\,\mu F/m$

Version b
7m between phases

$GMD_{+/-} = 8.82$

$\mathcal{L}_{+/-} = 1.00\,\mu H/m$

$C_{+/-} = 11.40\,pF/m$

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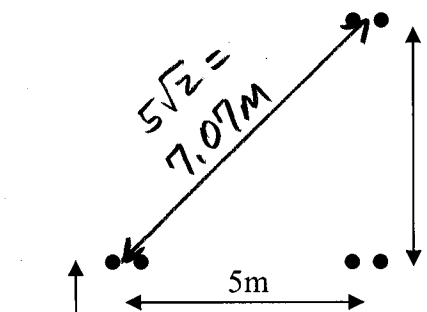
EE368, Spring 06, Final Exam. Name: KEY 4/5

Six equally-weighted problems. To receive proper credit, you must show all your work.

Problem 4. The a-b-c phase bundles of a three-phase, 138kV transmission line have the spacings and average heights above Earth as shown. Each conductor has outer radius 1cm, and the two conductors per phase are separated by 45cm. Ignoring ground wires, and making the "far from Earth" assumption, compute the positive/negative sequence inductance and capacitance per meter length (express your answer in μH/m and pF/m). Important - include the internal inductance in the μH/m calculation.



$$GMD_{+/-} = \sqrt[6]{(5)(5\sqrt{2})(5)(5)(5)(5\sqrt{2})} = \sqrt[6]{5^6(2)}$$

$$GMD_{+/-} = 5\sqrt[6]{2} = 5.61m$$

$$GMR_{C+/-} = \sqrt[N]{N r A^{N-1}} = \sqrt[2]{(2)(0.01)\left(\frac{0.450}{2}\right)^1}$$

$$GMR_{C+/-} = 0.0671\,m$$

$$C_{+/-} = \frac{2\pi\epsilon_0}{\ln\frac{GMD_{+/-}}{GMR_{C+/-}}} = \frac{2\pi(8.854\times10^{-12})}{\ln\frac{5.61}{0.0671}} = 12.6\ pF/m$$

For $L_{+/-}$, use $GMR_{L+/-} = \sqrt{(2)(0.01e^{-1/4})(0.45/2)} = 0.0592\,m$

$$L_{+/-} = \frac{\mu_0}{2\pi}\ln\frac{GMD_{+/-}}{GMR_{L+/-}} = 0.910\,\mu H/m$$

A 345kV, 300km, 60Hz three-phase overhead transmission line has positive/negative sequence L = 1μH/m and C = 11.5pF/m.

The receiving-end voltage and load are 345kV, 500MW, pf = 0.90 lagging. Use the ABCD matrix to determine the sending end

- voltage,
- current,
- P,
- and Q.

$$\begin{bmatrix} \tilde{V}_S \\ \tilde{I}_S \end{bmatrix} = \begin{bmatrix} \cos\beta x & jZ_o \sin\beta x \\ \dfrac{j\sin\beta x}{Z_o} & \cos\beta x \end{bmatrix} \begin{bmatrix} \tilde{V}_R \\ \tilde{I}_R \end{bmatrix}$$

EE368, Test 1. March 3, 2006. One page of notes permitted. Name KEY

Show all work on these pages. Clearly show all steps needed to fully support your answers.

Four equally-weighted problems. a.

Problem 4. A 345kV, 300km, 60Hz three-phase overhead transmission line has positive/negative sequence L = 1μH/m and C = 11.5pF/m.

The receiving-end voltage and load are 345kV, 500MW, pf = 0.90 lagging. Use the ABCD matrix to determine the sending end

- voltage,
- current,
- P,
- and Q.

$$\boxed{Z_0 = \sqrt{\frac{L}{C}} = 295\,\Omega}$$

$$\begin{bmatrix}\tilde{V}_S\\ \tilde{I}_S\end{bmatrix} = \begin{bmatrix}\cos\beta x & j\sin\beta x\\ \frac{j\sin\beta x}{Z_o} & \cos\beta x\end{bmatrix}\begin{bmatrix}\tilde{V}_R\\ \tilde{I}_R\end{bmatrix}$$

Z_0

$$\beta = \omega\sqrt{LC} = (120\pi)\sqrt{(10^{-6})(11.5\times10^{-12})} = \boxed{1.278\times10^{-6}\ \text{RAD/m}}$$

B

$$\beta x = (1.278\times10^{-6})(300\times10^{3}) = 0.383\ \text{RAD}\ \boxed{(21.9^\circ)}$$

$$\cos\beta x = 0.928,\quad \sin\beta x = 0.373$$

$$\tilde{V}_L\tilde{I}_L^* = S_L,\quad I_L = \left(\frac{S_{L\phi}}{V_{LN}}\right)^* = \left[\frac{\frac{500\times10^6}{3(0.9)}\angle{+25.8^\circ}}{\frac{345\times10^3}{\sqrt{3}} = 199.2\text{kV}}\right]^*$$

$$\boxed{I_L = 930\angle{-25.8^\circ}\ \text{A}}$$

$$\begin{bmatrix}V_s\\ I_s\end{bmatrix} = \begin{bmatrix}0.928 & j0.373(295)\\ \frac{j0.373}{295} & 0.928\end{bmatrix}\begin{bmatrix}\frac{345000}{\sqrt{3}}\angle 0\\ 930\angle{-25.8}\end{bmatrix}$$

Set up is sufficient

$$= \begin{bmatrix}185000 + 102400\angle 64.2\\ j252 + 863\angle{-25.8}\end{bmatrix} = \begin{bmatrix}185000 + 44600 + j92200\\ 777 - j124\end{bmatrix} = \begin{bmatrix}247000\angle 22\\ 787\angle{-9.1^\circ}\ \text{A}\end{bmatrix}$$

$$S_s = V_s I_s^* = 3(247000\angle 22^\circ)(787\angle{-9.1})^* = 583\text{MVA},\ pf = 0.856\ (P = 499\text{MW})$$

OK

Test b

For 400MW

$\tilde{I}_L = 744\angle{-25.8}$ A

$\tilde{V}_L = 199.2$kV

Test c

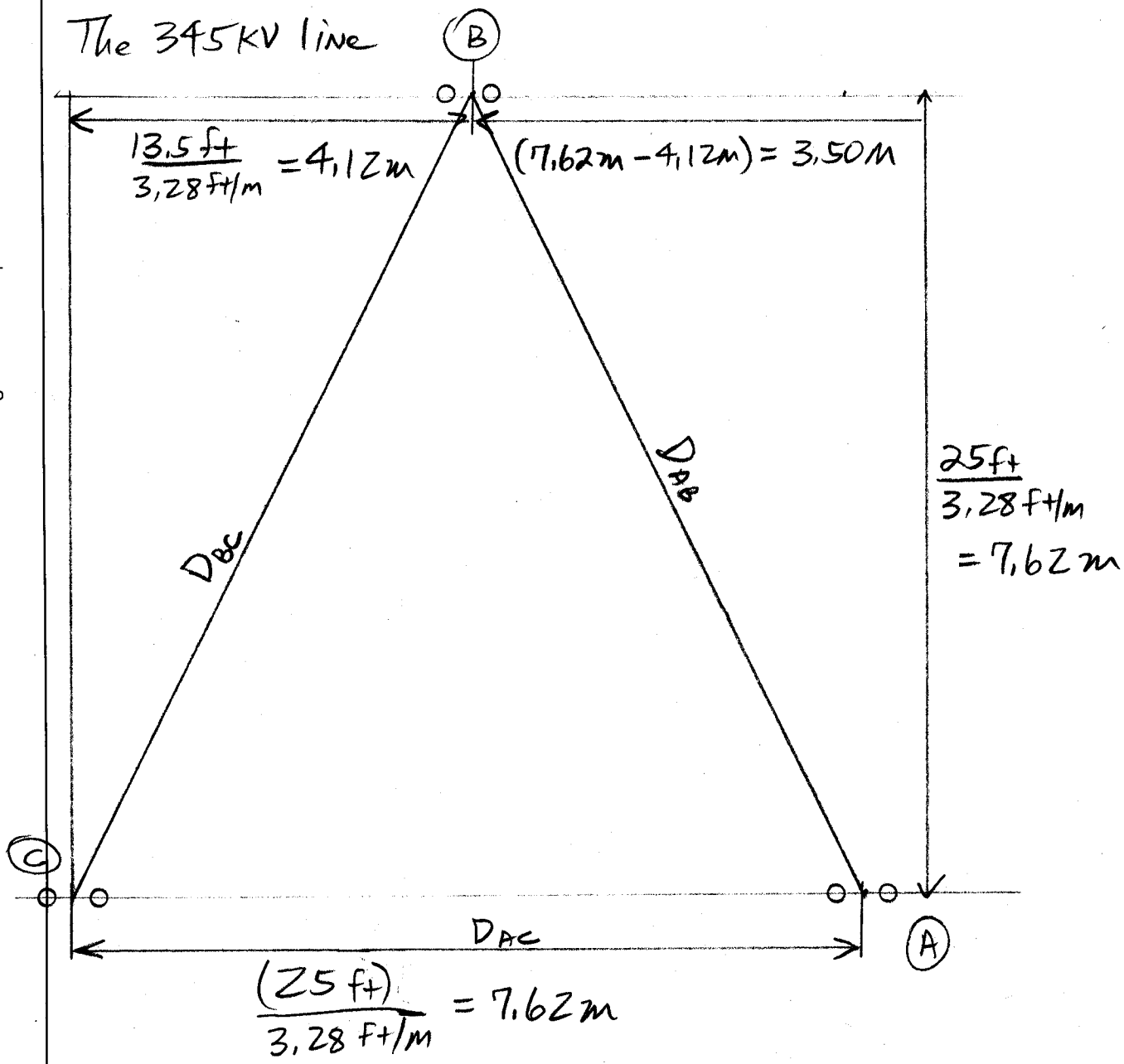
600MW

$\tilde{I}_L = 1116\angle{-25.8}$

$\tilde{V}_L = 199.2$kV

HW#3

The 345KV line

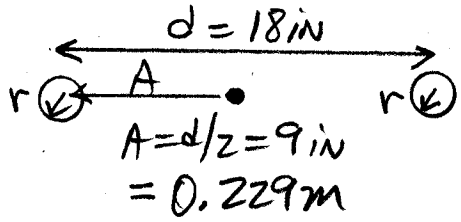


$D_{AC} = 7.62m$, $D_{BC} = \sqrt{4.12^2 + 7.62^2} = 8.66m$

$D_{AB} = \sqrt{7.62^2 + 3.50^2} = 8.39m$. $GMD_{+/-} = \sqrt[3]{D_{AB} D_{AC} D_{BC}}$

$GMD_{+/-} = \sqrt[3]{(7.62)(8.66)(8.39)} = 8.21\,m$

Now, the phase bundles



$r = 1.407cm$

$r_{gmr} = 1.137cm$

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STAEDTLER® No. 937 811E Engineer's Computation Pad

$$GMR_{C+/-} = \sqrt[2]{2rA^{2-1}} = \sqrt[2]{(2)(0.01407)(0.229)} = 0.0803\ m$$

$$C_{+/-} = \frac{2\pi\epsilon_0}{\ln \frac{GMD_{+/-}}{GMR_{C+/-}}} = \frac{2\pi(8.854)}{\ln \frac{8.21}{0.0803}} = \boxed{12.02\ pF/m}$$

$$= 4.63$$

For $R_{+/-}$, we have two conductors per phase,

so $R_{+/-} = \frac{0.0728\,\Omega/km}{2} = 0.0364\ \Omega/km$

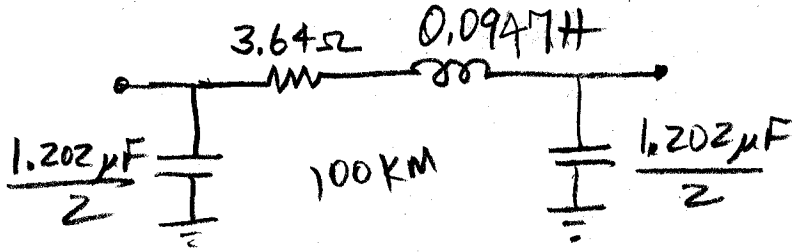
$$= \boxed{3.64\times10^{-5}\ \Omega/m}$$

For $L_{+/-}$, use $L_{+/-} = \frac{\mu_0}{2\pi}\ln\frac{GMD}{GMR_{L+/-}}$

$$L_{+/-} = \frac{4\pi\times10^{-7}}{2\pi}\ln\frac{8.21}{\sqrt[2]{(2)(r_{gmr})(A)}} = 0.2\ln\frac{8.21}{\sqrt[2]{(2)(0.01137)(\cancel{0.2788})(0.229)}}\ \mu H/m$$

$$\boxed{L_{+/-} = 0.947\ \mu H/m}$$

So, the +/- lumped parameter for a $\underbrace{100\ km}_{10^5\ m}$ line would be



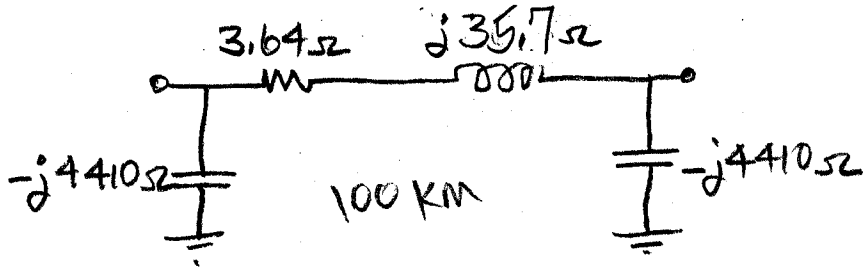
Note –

$C_{+/-} \simeq 10\ pF/m$

$L_{+/-} \simeq 1\ \mu H/m$

$\frac{X_{L60}}{R} \simeq 10$

At 60 Hz, we have



$$\frac{1}{\sqrt{L_{+/-}C_{+/-}}} = 2.96\times10^8\ m/s$$

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OK

Symmetrical Components

Transformation matrices and the decoupling that occurs in balanced three-phase systems. Physical significance of zero sequence.

1. Transformation Matrix

Fortescue's Theorem: An unbalanced set of *N* related phasors can be resolved into *N* systems of phasors called the symmetrical components of the original phasors. For a three-phase system (i.e. *N* = 3), the three sets are:

1. Positive Sequence - three phasors, equal in magnitude, 120° apart, with the same sequence (a-b-c) as the original phasors.

2. Negative Sequence - three phasors, equal in magnitude, 120° apart, with the opposite sequence (a-c-b) of the original phasors.

3. Zero Sequence - three identical phasors (i.e. equal in magnitude, with no relative phase displacement).

The original set of phasors is written in terms of the symmetrical components as follows:

$$\tilde{V}_a = \tilde{V}_{a0} + \tilde{V}_{a1} + \tilde{V}_{a2} \ ,$$

$$\tilde{V}_b = \tilde{V}_{b0} + \tilde{V}_{b1} + \tilde{V}_{b2} \ ,$$

$$\tilde{V}_c = \tilde{V}_{c0} + \tilde{V}_{c1} + \tilde{V}_{c2} \ ,$$

where 0 indicates zero sequence, 1 indicates positive sequence, and 2 indicates negative sequence.

The relationship among the sequence components for a-b-c are

Positive Sequence	Negative Sequence	Zero Sequence
$\tilde{V}_{b1} = \tilde{V}_{a1} \bullet 1\angle -120^{o}$	$\tilde{V}_{b2} = \tilde{V}_{a2} \bullet 1\angle +120^{o}$	$\tilde{V}_{a0} = \tilde{V}_{b0} = \tilde{V}_{c0}$
$\tilde{V}_{c1} = \tilde{V}_{a1} \bullet 1\angle +120^{o}$	$\tilde{V}_{c2} = \tilde{V}_{a2} \bullet 1\angle -120^{o}$	

The symmetrical components of all a-b-c voltages are usually written in terms of the symmetrical components of phase a by defining

$$a = 1\angle +120^{o} \text{ , so that } a^2 = 1\angle +240^{o} = 1\angle -120^{o} \text{ , and } a^3 = 1\angle +360^{o} = 1\angle 0^{o} \ .$$

Substituting into the previous equations for $\tilde{V}_a, \tilde{V}_b, \tilde{V}_c$ yields

$$\tilde{V}_a = \tilde{V}_{a0} + \tilde{V}_{a1} + \tilde{V}_{a2} \ ,$$

$$\tilde{V}_b = \tilde{V}_{a0} + a^2\tilde{V}_{a1} + a\tilde{V}_{a2} \ ,$$

$$\tilde{V}_c = \tilde{V}_{a0} + a\tilde{V}_{a1} + a^2\tilde{V}_{a2} \ .$$

In matrix form, the above equations become

$$\begin{bmatrix} \tilde{V}_a \\ \tilde{V}_b \\ \tilde{V}_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} \tilde{V}_{a0} \\ \tilde{V}_{a1} \\ \tilde{V}_{a2} \end{bmatrix} ,$$

or in matrix form

$$\tilde{V}_{abc} = T \bullet \tilde{V}_{012} \text{ , and } \tilde{V}_{012} = T^{-1} \bullet \tilde{V}_{abc} \ ,$$

where transformation matrix T is

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \text{, and } T^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} .$$

If $\tilde{V}_{abc}$ represents a balanced set (i.e. $\tilde{V}_b = \tilde{V}_a \bullet 1\angle -120^{\circ} = a^2\tilde{V}_a$, $\tilde{V}_c = \tilde{V}_a \bullet 1\angle +120^{\circ} = a\tilde{V}_a$), then substituting into $\tilde{V}_{012} = T^{-1} \bullet \tilde{V}_{abc}$ yields

$$\begin{bmatrix} \tilde{V}_{a0} \\ \tilde{V}_{a1} \\ \tilde{V}_{a2} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} \tilde{V}_a \\ a^2\tilde{V}_a \\ a\tilde{V}_a \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{V}_a \\ 0 \end{bmatrix} .$$

Hence, balanced voltages or currents have only positive sequence components, and the positive sequence components equal the corresponding phase a voltages or currents.

If $\tilde{V}_{abc}$ is an identical set (i.e. $\tilde{V}_a = \tilde{V}_b = \tilde{V}_c$), substituting into $\tilde{V}_{012} = T^{-1} \bullet \tilde{V}_{abc}$ yields

$$\begin{bmatrix} \tilde{V}_{a0} \\ \tilde{V}_{a1} \\ \tilde{V}_{a2} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} \tilde{V}_a \\ \tilde{V}_a \\ \tilde{V}_a \end{bmatrix} = \begin{bmatrix} \tilde{V}_a \\ 0 \\ 0 \end{bmatrix} ,$$

which means that V_a, V_b, V_c have only zero sequence components, and that these components are identical and equal to V_a .

Notice from the top row of $\tilde{V}_{012} = T^{-1} \bullet \tilde{V}_{abc}$ that V_0 is one-third of the sum of the three phase voltages or currents. Therefore, since the sum of three line-to-line voltages is identically zero due to Kirchhoff's voltage law, line-to-line voltages can have no zero sequence components.

1. Relationship Between Zero Sequence Currents and Neutral Current

Consider the relationship between zero sequence current and neutral current The zero sequence current is

$$\tilde{I}_{a0} = \frac{1}{3}\left(\tilde{I}_a + \tilde{I}_b + \tilde{I}_c\right),$$

and, from Kirchhoff's current law, the neutral current is

$$\tilde{I}_n = -\left(\tilde{I}_a + \tilde{I}_b + \tilde{I}_c\right).$$

Because the positive and negative sequence components of the a-b-c currents sum to zero, while the zero sequence components are additive, then $\tilde{I}_n = -3\tilde{I}_{a0}$. Therefore, in a four-wire, three-phase system, the neutral current is three-times the zero sequence current. In a three-wire, three-phase system, there is no zero sequence current.

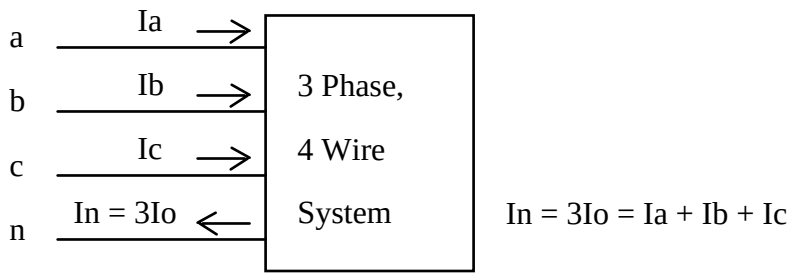


Figure 1. Relationship Between Zero Sequence Currents and Neutral Current

(note – the neutral current is shown above as flowing out)

3. Decoupling in Systems with Balanced Impedances

In a three-phase system with balanced impedances, the relationship among voltage, current, and impedance has the form

$$\begin{bmatrix} \tilde{V}_a \\ \tilde{V}_b \\ \tilde{V}_c \end{bmatrix} = \begin{bmatrix} S & M & M \\ M & S & M \\ M & M & S \end{bmatrix} \begin{bmatrix} \tilde{I}_a \\ \tilde{I}_b \\ \tilde{I}_c \end{bmatrix}, \text{ or } \tilde{V}_{abc} = Z_{abc} \bullet \tilde{I}_{abc},$$

where S represents the self impedances of the phases, and M represents the mutual impedances. This equation can be expressed in terms of sequence components by substituting $\tilde{V}_{abc} = T \bullet \tilde{V}_{012}$ and $\tilde{I}_{abc} = T \bullet \tilde{I}_{012}$, yielding

$$T \bullet \tilde{V}_{012} = Z_{abc} \bullet T \bullet \tilde{I}_{012} \ .$$

Premultiplying by T^{-1} yields

$$\tilde{V}_{012} = T^{-1} Z_{abc} \bullet T \bullet \tilde{I}_{012} = Z_{012} \bullet \tilde{I}_{012} \ ,$$

where $Z_{012} = T^{-1} Z_{abc} \bullet T$. The symmetric form of Z_{abc} given above yields

$$Z_{012} = \begin{bmatrix} S+2M & 0 & 0 \\ 0 & S-M & 0 \\ 0 & 0 & S-M \end{bmatrix},$$

which means that when working in sequence components, a circuit with symmetric impedances is decoupled into three separate impedance networks with $Z_0 = S + 2M$, and $Z_1 = Z_2 = S - M$. Furthermore, if the voltages and currents are balanced, then only the positive sequence circuit must be studied.

In summary, symmetrical components are useful when studying either of the following two situations:

1. Symmetric networks with balanced voltages and currents. In that case, only the positive sequence network must be studied, and that network is the "one-line" network.

2. Symmetric networks with unbalanced voltages and currents. In that case, decoupling applies, and three separate networks must be studied (i.e. positive, negative, and zero sequences). The sequence components of the voltages and currents can be transformed back to a-b-c by using the *T* transformation matrix.

4. Power

For a three-phase circuit, with voltages referenced to neutral,

$$P_{abc} = \tilde{V}_{an}\tilde{I}_a^* + \tilde{V}_{bn}\tilde{I}_b^* + \tilde{V}_{cn}\tilde{I}_c^* = \tilde{V}_{abc}^T \bullet \tilde{I}_{abc}^* \ .$$

Substituting in $\tilde{V}_{abc} = T \bullet \tilde{V}_{012}$ and $\tilde{I}_{abc} = T \bullet \tilde{I}_{012}$ yields

$$P_{abc} = \tilde{V}_{012}^T \bullet T^T \bullet T^* \bullet \tilde{I}_{012}^* \ .$$

Since $T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$, then $T^T = T$. Also, $T^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}$. Therefore,

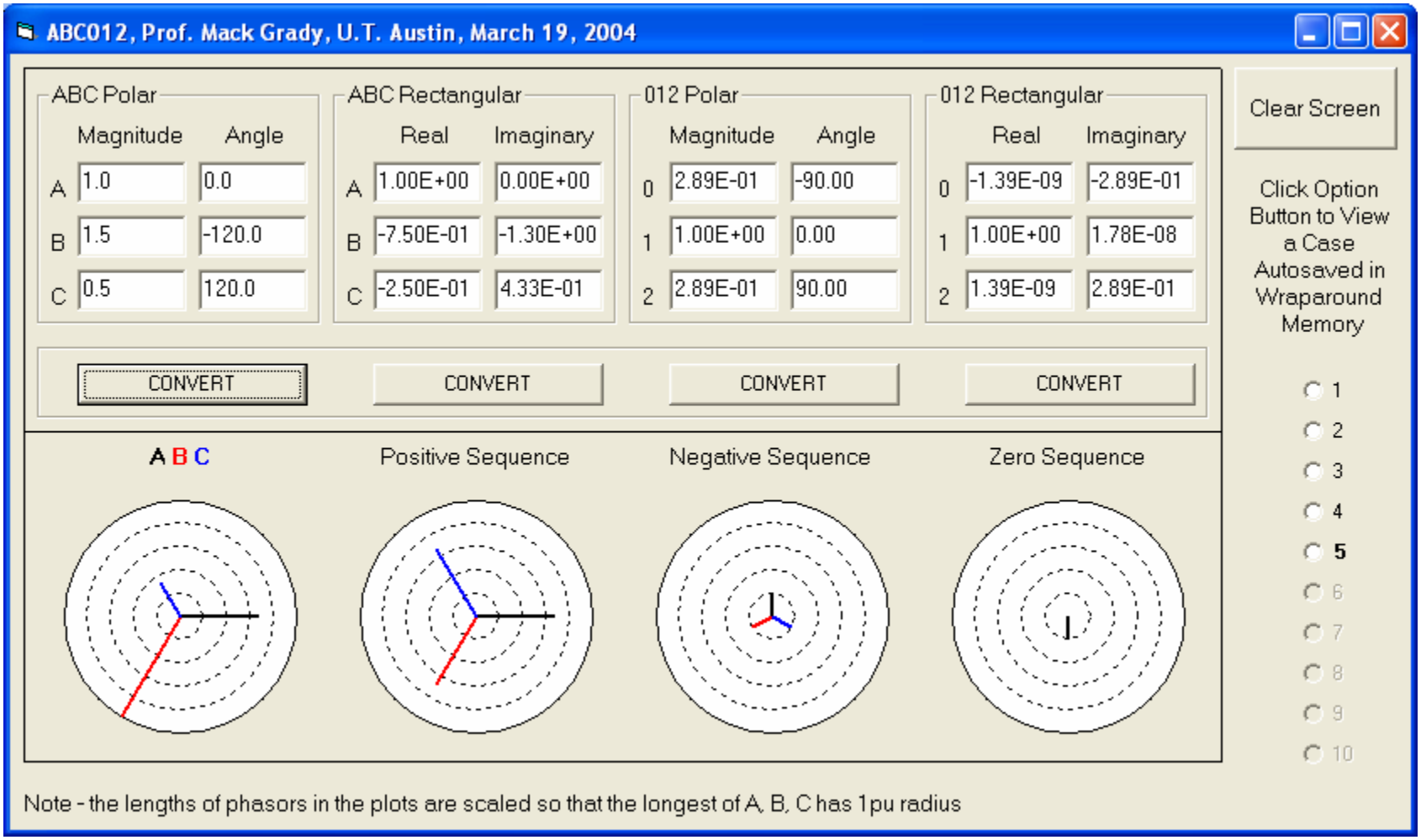
$$T^T \bullet T^* = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \text{ so that } P_{abc} = \tilde{V}_{012}^T \bullet \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \bullet \tilde{I}_{012}^* \text{ , or}$$

$$P_{abc} = 3\left(\tilde{V}_{a0}\tilde{I}_{a0}^* + \tilde{V}_{a1}\tilde{I}_{a1}^* + \tilde{V}_{a2}\tilde{I}_{a2}^*\right).$$

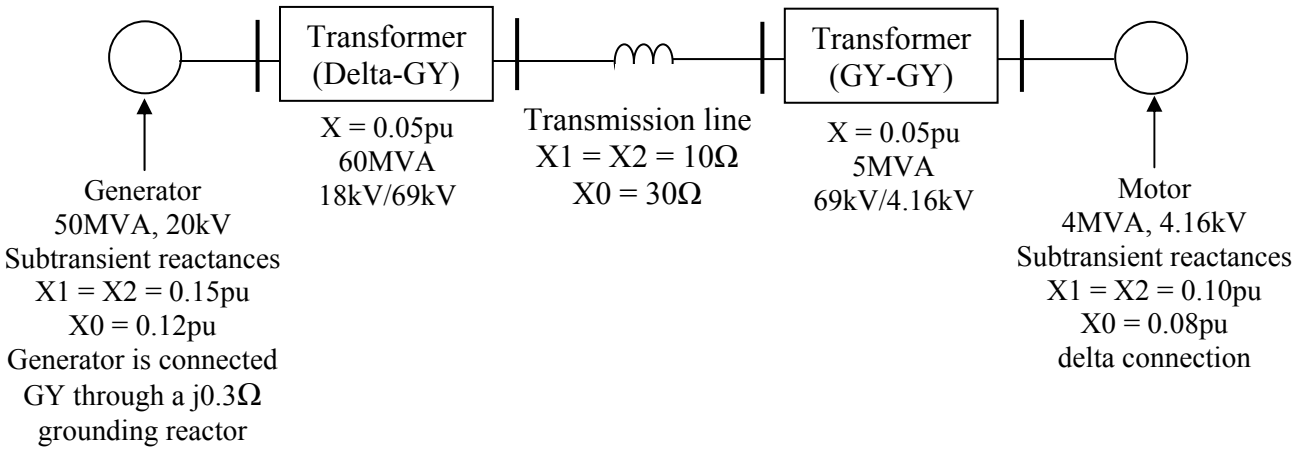
Note the factor of three. If desired, the following power invariant transformation can be used to avoid the factor of three:

$$T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}, \; T^{-1} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix},$$

so that $P_{abc} = \tilde{V}_{a0}\tilde{I}_{a0}^* + \tilde{V}_{a1}\tilde{I}_{a1}^* + \tilde{V}_{a2}\tilde{I}_{a2}^*$.


ABC012, Prof. Mack Grady, U.T. Austin, March 19, 2004
ABC Polar
Magnitude
Angle
A 1.0 0.0
B 1.5 -120.0
C 0.5 120.0
ABC Rectangular
Real
Imaginary
A 1.00E+00 0.00E+00
B -7.50E-01 -1.30E+00
C -2.50E-01 4.33E-01
012 Polar
Magnitude
Angle
0 2.89E-01 -90.00
1 1.00E+00 0.00
2 2.89E-01 90.00
012 Rectangular
Real
Imaginary
0 -1.39E-09 -2.89E-01
1 1.00E+00 1.78E-08
2 1.39E-09 2.89E-01
Clear Screen
Click Option Button to View a Case Autosaved in Wraparound Memory
CONVERT
CONVERT
CONVERT
CONVERT
A B C
Positive Sequence
Negative Sequence
Zero Sequence
1
2
3
4
5
6
7
8
9
10
Note - the lengths of phasors in the plots are scaled so that the longest of A, B, C has 1pu radius

Problem 1. Draw the positive, negative, and zero-sequence one-line diagrams for the system shown. Use 69kV, 100MVA as the base in the transmission line. Generator, transformer, and motor impedances are given in per unit on their own bases.

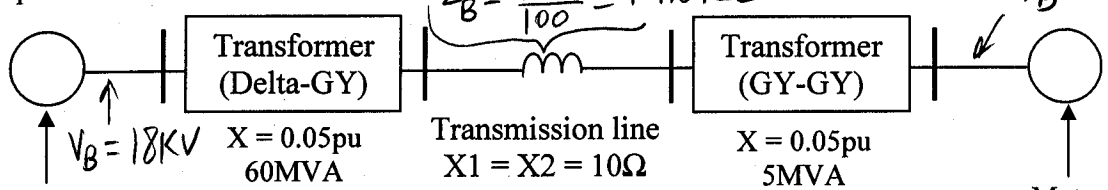


EE369, Final Exam, 2-5pm, Dec. 9, WRW102. Name: KEY

Six equally-weighted problems. Three sheets of notes permitted. Show all work.

Problem 1. Draw the positive, negative, and zero-sequence one-line diagrams for the system shown. Use 69kV, 100MVA as the base in the transmission line. Generator, transformer, and motor impedances are given in per unit on their own bases.

$Z_B = \frac{(69)^2}{100} = 47.61\,\Omega$ $\quad V_B = 4.16\,kV$



$V_B = 18\,kV$

Generator
50MVA, 20kV
Subtransient reactances
X1 = X2 = 0.15pu
X0 = 0.12pu
Generator is connected GY through a j0.3Ω (Actual ohms) grounding reactor

X = 0.05pu
60MVA
18kV/69kV
$0.05\left(\frac{100}{60}\right) = 0.0833$

Transmission line
X1 = X2 = 10Ω
X0 = 30Ω
$j10\,\Omega \rightarrow j0.21\,pu$
$j30\,\Omega \rightarrow j0.63\,pu$

X = 0.05pu
5MVA
69kV/4.16kV
$0.05\left(\frac{100}{5}\right) = 1.000$

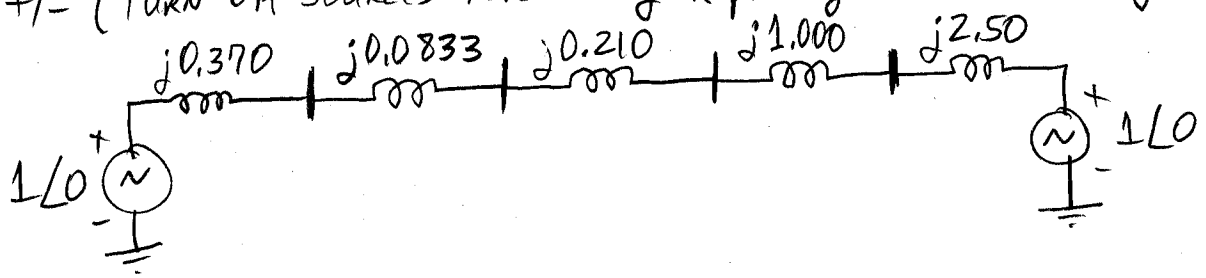
Motor
4MVA, 4.16kV
Subtransient reactances
X1 = X2 = 0.10pu
X0 = 0.08pu
delta connection
$X_1 = X_2 = 0.10\left(\frac{100}{4}\right)\left(\frac{4.16}{4.16}\right)^2 = 2.50$
$X_0 = 0.08\left(\frac{100}{4}\right) = 2.00$

For gen, $X_1 = X_2 = 0.15\left(\frac{100}{50}\right)\left(\frac{20}{18}\right)^2 = 0.370\,pu$

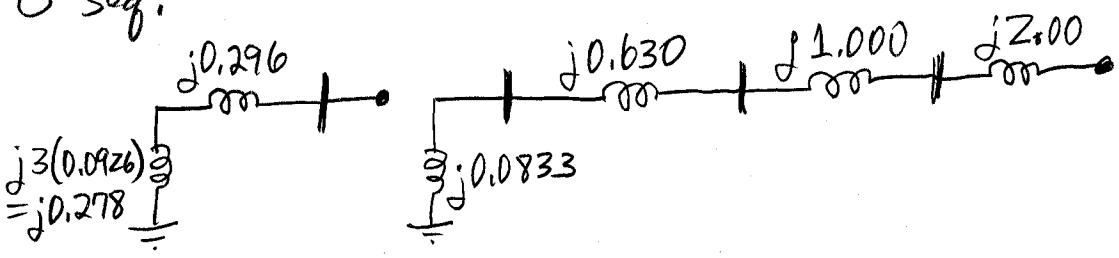
$X_0 = 0.12\left(\frac{100}{50}\right)\left(\frac{20}{18}\right)^2 = 0.296\,pu$

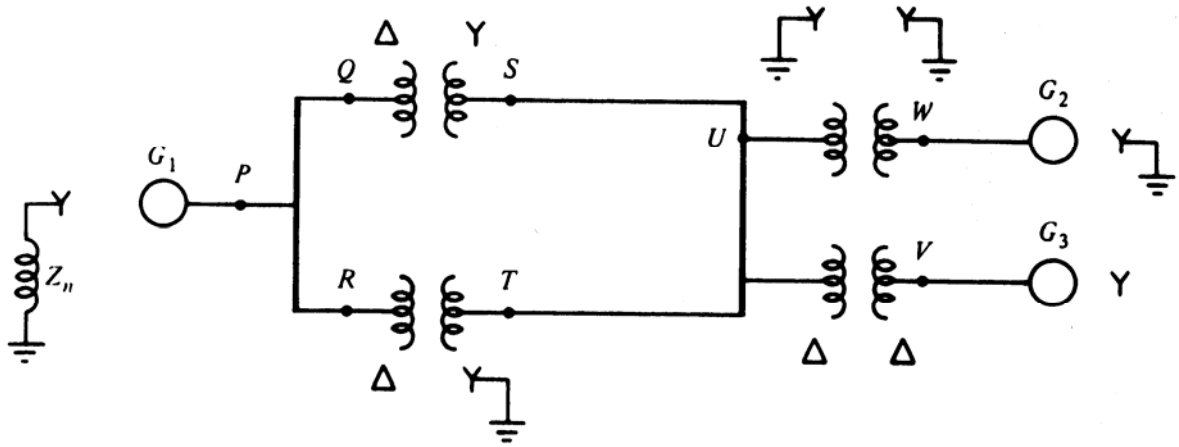
Reactor, $Z_B = \frac{(18)^2}{100} = 3.24$, so $j0.3\,\Omega$ becomes $j0.0926\,pu$

+/- (Turn off sources for − by replacing the $1\angle 0$ voltage with a short)



0 seq.

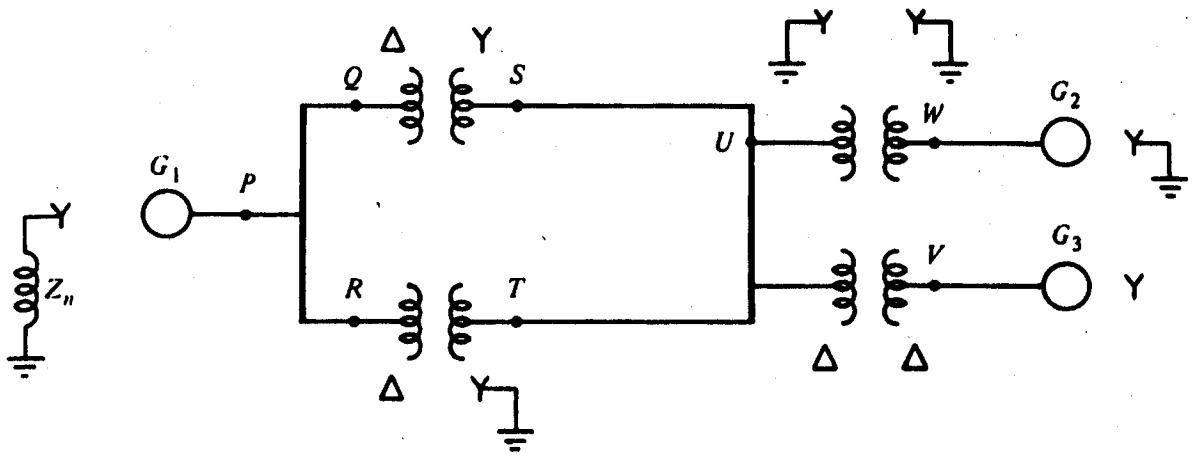




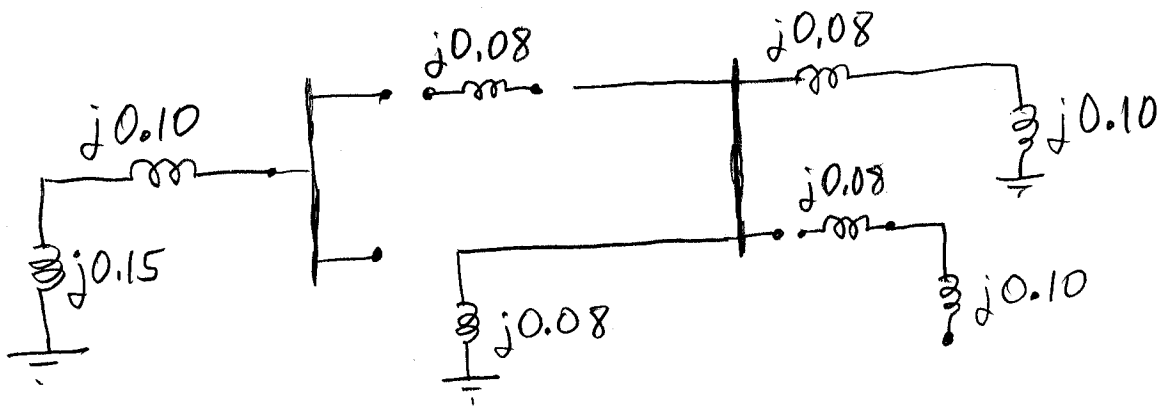
All impedances are given in per unit on system base. The generators have impedances j0.10pu, and the transformers have impedances j0.08pu. The grounding reactor has impedance j0.05pu. Draw the zero-sequence network, showing impedance values. Identify all nodes (i.e., P,Q,R,S,T,U,V,W, and ground).

EE369, Test 8, October 29, 2004. Name: KEY

One sheet of notes permitted. Show all work to receive credit.



All impedances are given in per unit on system base. The generators have impedances j0.10pu, and the transformers have impedances j0.08pu. The grounding reactor has impedance j0.05pu. Draw the zero-sequence network, showing impedance values. Identify all nodes (i.e., P,Q,R,S,T,U,V,W, and ground).



Power System Matrices and Matrix Operations

Nodal equations using Kirchhoff's current law. Admittance matrix and building algorithm. Gaussian elimination. Kron reduction. LU decomposition. Formation of impedance matrix by inversion, Gaussian elimination, and direct building algorithm.

1. Admittance Matrix

Most power system networks are analyzed by first forming the admittance matrix. The admittance matrix is based upon Kirchhoff's current law (KCL), and it is easily formed and very sparse.

Consider the three-bus network shown in Figure that has five branch impedances and one current source.

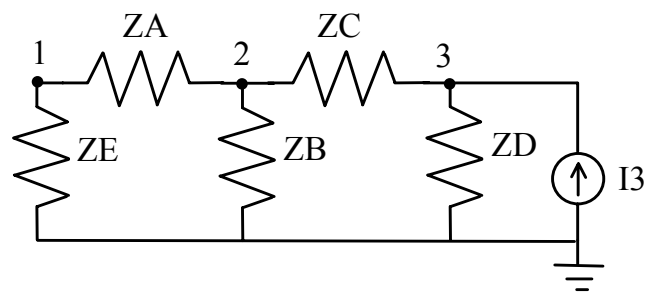


Figure 1. Three-Bus Network

Applying KCL at the three independent nodes yields the following equations for the bus voltages (with respect to ground):

At bus 1, $\dfrac{V_1}{Z_E} + \dfrac{V_1 - V_2}{Z_A} = 0$,

At bus 2, $\dfrac{V_2}{Z_B} + \dfrac{V_2 - V_1}{Z_A} + \dfrac{V_2 - V_3}{Z_C} = 0$,

At bus 3, $\dfrac{V_3}{Z_D} + \dfrac{V_3 - V_2}{Z_C} = I_3$.

Collecting terms and writing the equations in matrix form yields

$$\begin{bmatrix} \frac{1}{Z_E}+\frac{1}{Z_A} & -\frac{1}{Z_A} & 0 \\ -\frac{1}{Z_A} & \frac{1}{Z_A}+\frac{1}{Z_B}+\frac{1}{Z_C} & -\frac{1}{Z_C} \\ 0 & -\frac{1}{Z_C} & \frac{1}{Z_C}+\frac{1}{Z_D} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_3 \end{bmatrix},$$

or in matrix form,

$$YV = I \ ,$$

where Y is the admittance matrix, V is a vector of bus voltages (with respect to ground), and I is a vector of current injections.

Voltage sources, if present, can be converted to current sources using the usual network rules. If a bus has a zero-impedance voltage source attached to it, then the bus voltage is already known, and the dimension of the problem is reduced by one.

A simple observation of the structure of the above admittance matrix leads to the following rule for building Y:

1. The diagonal terms of Y contain the sum of all branch admittances connected directly to the corresponding bus.

2. The off-diagonal elements of Y contain the negative sum of all branch admittances connected directly between the corresponding busses.

These rules make Y very simple to build using a computer program. For example, assume that the impedance data for the above network has the following form, one data input line per branch:

From Bus	To Bus	Branch Impedance (Entered as Complex Numbers)
1	0	ZE
1	2	ZA
2	0	ZB
2	3	ZC
3	0	ZD

The following FORTRAN instructions would automatically build Y, without the need of manually writing the KCL equations beforehand:

```
      COMPLEX Y(3,3),ZB,YB
      DATA Y/9 * 0.0/
1     READ(1,*,END=2) NF,NT,ZB
```

```
      YB = 1.0 / ZB
C  MODIFY THE DIAGONAL TERMS
      IF(NF .NE. 0) Y(NF,NF) = Y(NF,NF) + YB
      IF(NT .NE. 0) Y(NT,NT) = Y(NT,NT) + YB
      IF(NF .NE. 0 .AND. NT .NE. 0) THEN
C  MODIFY THE OFF-DIAGONAL TERMS
        Y(NF,NT) = Y(NF,NT) - YB
        Y(NT,NF) = Y(NT,NF) - YB
      ENDIF
      GO TO 1
   2  STOP
      END
```

Of course, error checking is needed in an actual computer program to detect data errors and dimension overruns. Also, if bus numbers are not compressed (i.e. bus 1 through bus *N*), then additional logic is needed to internally compress the busses, maintaining separate internal and external (i.e. user) bus numbers.

Note that the Y matrix is symmetric unless there are branches whose admittance is direction-dependent. In AC power system applications, only phase-shifting transformers have this asymmetric property. The normal 30° phase shift in wye-delta transformers creates asymmetry.

2. Gaussian Elimination and Backward Substitution

Gaussian elimination is the most common method for solving bus voltages in a circuit for which KCL equations have been written in the form

$$I = YV \ .$$

Of course, direct inversion can be used, where

$$V = Y^{-1} I \ ,$$

but direct inversion for large matrices is computationally prohibitive or, at best, inefficient.

The objective of Gaussian elimination is to reduce the *Y* matrix to upper-right-triangular-plus-diagonal form (URT+D), then solve for *V* via backward substitution. A series of row operations (i.e. subtractions and additions) are used to change equation

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,N} \\ y_{2,1} & y_{2,2} & y_{2,3} & \cdots & y_{2,N} \\ y_{3,1} & y_{3,2} & y_{3,3} & \cdots & y_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{N,1} & y_{N,2} & y_{N,3} & \cdots & y_{N,N} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{bmatrix}$$

into

$$\begin{bmatrix} I_1 \\ I_2^{'} \\ I_3^{'} \\ \vdots \\ I_N^{'} \end{bmatrix} = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,N} \\ 0 & y_{2,2}^{'} & y_{2,3}^{'} & \cdots & y_{2,N}^{'} \\ 0 & 0 & y_{3,3}^{'} & \cdots & y_{3,N}^{'} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y_{N,N}^{'} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{bmatrix},$$

in which the transformed *Y* matrix has zeros under the diagonal.

For illustrative purposes, consider the two equations represented by Rows 1 and 2, which are

$$\begin{array}{lllllllllll} I_1 & = & y_{1,1}V_1 & + & y_{1,2}V_2 & + & y_{1,3}V_3 & + & \cdots & + & y_{1,N}V_N \\ I_2 & = & y_{2,1}V_1 & + & y_{2,2}V_2 & + & y_{2,3}V_3 & + & \cdots & + & y_{2,N}V_N \end{array}.$$

Subtracting ($\frac{y_{2,1}}{y_{1,1}}$ • Row 1) from Row 2 yields

$$\begin{array}{lllllllllll} I_1 & = & y_{1,1}V_1 & + & y_{1,2}V_2 & + & y_{1,3}V_3 & + & \cdots & + & y_{1,N}V_N \\ I_2 - \frac{y_{2,1}}{y_{1,1}}I_1 & = & \left(y_{2,1} - \frac{y_{2,1}}{y_{1,1}}y_{1,1}\right)V_1 & + & \left(y_{2,2} - \frac{y_{2,1}}{y_{1,1}}y_{1,2}\right)V_2 & + & \left(y_{2,3} - \frac{y_{2,1}}{y_{1,1}}y_{1,3}\right)V_3 & + & \cdots & + & \left(y_{2,N} - \frac{y_{2,1}}{y_{1,1}}y_{1,N}\right)V_N \end{array}.$$

The coefficient of V_1 in Row 2 is forced to zero, leaving Row 2 with the desired "reduced" form of

$$I_2^{'} = 0 + y_{2,2}^{'}V_2 + y_{2,3}^{'}V_3 + \cdots + y_{2,N}^{'}V_N .$$

Continuing, Row 1 is then used to "zero" the V_1 coefficients in Rows 3 through N, one row at a time. Next, Row 2 is used to zero the V_2 coefficients in Rows 3 through N, and so forth.

After the Gaussian elimination is completed, and the *Y* matrix is reduced to (URT+D) form, the bus voltages are solved by backward substitution as follows:

For Row *N*,

$$I_N^{'} = y_{N,N}^{'}V_N \text{ , so } V_N = \frac{1}{y_{N,N}^{'}}\left(I_N^{'}\right).$$

Next, for Row *N-1*,

$$I_{N-1}^{'} = y_{N-1,N-1}^{'}V_{N-1} + y_{N-1,N}^{'}V_N \text{ , so } V_{N-1} = \frac{1}{y_{N-1,N-1}^{'}}\left(I_{N-1}^{'} - y_{N-1,N}^{'}V_N\right).$$

Continuing for Row j, where $j = N-2, N-3, \cdots, 2$,

$$I'_j = y'_{j,j}V_j + y'_{j,j+1}V_{j+1} + \cdots + y'_{j,N}V_N \text{ , so}$$

$$V_j = \frac{1}{y'_{j,j}}\left(I'_j - y'_{j,j+1}V_{j+1} - \cdots - y'_{j,N}V_N\right),$$

which, in general form, is described by

$$V_j = \frac{1}{y'_{j,j}}\left(I'_j - \sum_{k=j+1}^{N} y'_{j,k}V_k\right).$$

A simple FORTRAN computer program for solving V in an N-dimension problem using Gaussian elimination and backward substitution is given below.

```
      COMPLEX Y(N,N),V(N),I(N),YMM
C  GAUSSIAN ELIMINATE Y AND I
      NM1    = N - 1
C  PIVOT ON ROW M, M = 1,2,3, ... ,N-1
      DO 1 M = 1,NM1
      MP1    = M + 1
      YMM    = 1.0 / Y(M,M)
C  OPERATE ON THE ROWS BELOW THE PIVOT ROW
      DO 1 J = MP1,N
C  THE JTH ROW OF I
      I(J)   = I(J) - Y(J,M) *YMM * I(M)
C  THE JTH ROW OF Y, BELOW AND TO THE RIGHT OF THE PIVOT
C  DIAGONAL
      DO 1 K = M,N
      Y(J,K) = Y(J,K) - Y(J,M) * YMM * Y(M,K)
   1  CONTINUE
C  BACKWARD SUBSTITUTE TO SOLVE FOR V
      V(N)   = I(N) / Y(N,N)
      DO 2 M = 1,NM1
      J      = N - M
C  BACKWARD SUBSTITUTE TO SOLVE FOR V, FOR
C  ROW J = N-1,N-2,N-3, ... ,1
      V(J)   = I(J)
      JP1    = J + 1
      DO 3 K = JP1,N
      V(J)   = V(J) - Y(J,K) * V(K)
```

```
3   CONTINUE
    V(J)   = V(J) / Y(J,J)
2   CONTINUE
    STOP
    END
```

One disadvantage of Gaussian elimination is that if *I* changes, even though *Y* is fixed, the entire problem must be re-solved since the elimination of *Y* determines the row operations that must be repeated on *I*. Inversion and LU decomposition to not have this disadvantage.

3. Kron Reduction

Gaussian elimination can be made more computationally efficient by simply *not* performing operations whose results are already known. For example, instead of arithmetically forcing elements below the diagonal to zero, simply set them to zero at the appropriate times. Similarly, instead of dividing all elements below and to the right of a diagonal element by the diagonal element, divide only the elements in the diagonal row by the diagonal element, make the diagonal element unity, and the same effect will be achieved. This technique, which is actually a form of Gaussian elimination, is known as Kron reduction.

Kron reduction "pivots" on each diagonal element $y'_{m,m}$, beginning with $y_{1,1}$, and continuing through $y_{N-1,N-1}$. Starting with Row *m* = 1, and continuing through Row *m* = *N* - *1*, the algorithm for Kron reducing $I = YV$ is

1. Divide the elements in Row *m*, that are to the right of the diagonal, by the diagonal element $y'_{m,m}$. (Note - the elements to the left of the diagonal are already zero).

2. Replace element I'_m with $\dfrac{I'_m}{y'_{m,m}}$.

3. Replace diagonal element $y'_{m,m}$ with unity.

4. Modify the Y' elements in rows greater than *m* and columns greater than *m* (i.e. below and to the right of the diagonal element) using

$$y'_{j,k} = y'_{j,k} - y'_{j,m} y'_{m,k} \text{, for } j > m,\ k > m.$$

5. Modify the I' elements below the *mth* row according to

$$I'_j = I'_j - y'_{j,m} I'_m \text{, for } j > m.$$

6. Zero the elements in Column *m* of Y' that are below the diagonal element.

A FORTRAN code for Kron reduction is given below.

```
      COMPLEX Y(N,N),V(N),I(N),YMM
C  KRON REDUCE Y, WHILE ALSO PERFORMING ROW OPERATIONS ON I
     NM1    = N - 1
C  PIVOT ON ROW M, M = 1,2,3, ... ,N-1
      DO 1 M = 1,NM1
      MP1    = M + 1
      YMM    = 1.0 / YMM
C  DIVIDE THE PIVOT ROW BY THE PIVOT
      DO 2 K = MP1,N
      Y(M,K) = Y(M,K) * YMM
   2  CONTINUE
C  OPERATE ON THE I VECTOR
      I(M)   = I(M) * YMM
C  SET THE PIVOT TO UNITY
      Y(M,M) = 1.0
C  REDUCE THOSE ELEMENTS BELOW AND TO THE RIGHT OF THE PIVOT
      DO 3 J = MP1,N
      DO 4 K = MP1,N
      Y(J,K) = Y(J,K) - Y(J,M) * Y(M,K)
   4  CONTINUE
C  OPERATE ON THE I VECTOR
      I(J)   = I(J) - Y(J,M) * I(M)
C  SET THE Y ELEMENTS DIRECTLY BELOW THE PIVOT TO ZERO
      Y(J,M) = 0.0
   3  CONTINUE
   1  CONTINUE
```

4. LU Decomposition

An efficient method for solving V in matrix equation $YV = I$ is to decompose Y into the product of a lower-left-triangle-plus-diagonal (LLT+D) matrix L, and an (URT+D) matrix U, so that $YV = I$ can be written as

$$LUV = I \, .$$

The benefits of decomposing Y will become obvious after observing the systematic procedure for finding V.

It is customary to set the diagonal terms of U equal to unity, so that there are a total of N^2 unknown terms in L and U. $LU = Y$ in expanded form is then

$$\begin{bmatrix} l_{1,1} & 0 & 0 & \cdots & 0 \\ l_{2,1} & l_{2,2} & 0 & \cdots & 0 \\ l_{3,1} & l_{3,2} & l_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{N,1} & l_{N,2} & l_{N,3} & \cdots & l_{N,N} \end{bmatrix} \begin{bmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,N} \\ 0 & 1 & u_{2,3} & \cdots & u_{2,N} \\ 0 & 0 & 1 & \cdots & u_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,N} \\ y_{2,1} & y_{2,2} & y_{2,3} & \cdots & y_{2,N} \\ y_{3,1} & y_{3,2} & y_{3,3} & \cdots & y_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{N,1} & y_{N,2} & y_{N,3} & \cdots & y_{N,N} \end{bmatrix}.$$

Individual l and u terms are easily found by calculating them in the following order:

1. Starting from the top, work down Column 1 of L, finding $l_{1,1}$, then $l_{2,1}$, then $l_{3,1}$, … , $l_{N,1}$. For the special case of Column 1, these elements are $l_{j,1} = y_{j,1}$, j = 1,2,3, …,N.

2. Starting from the left, work across Row 1 of U, finding $u_{1,2}$, then $u_{1,3}$, then $u_{1,4}$, … , $u_{1,N}$. For the special case of Row 1, these elements are $u_{1,k} = \dfrac{y_{1,k}}{l_{1,1}}$, k = 2,3,4, …,N.

3. Work down Column (k = 2) of L, finding $l_{2,2}$, then $l_{3,2}$, then $l_{4,2}$, … , $l_{N,2}$, using

$$l_{j,k} = y_{j,k} - \sum_{m=1}^{k-1} l_{j,m} u_{m,k},\ j = k, k+1, k+2, \cdots, N \text{ , Column } k, 2 \le k \le N \text{ .}$$

4. Work across Row (k = 2) of U, finding $u_{2,3}$, then $u_{2,4}$, then $u_{2,5}$, … , $u_{2,N}$, using

$$u_{k,j} = \frac{y_{k,j} - \sum_{m=1}^{k-1} l_{k,m} u_{m,j}}{l_{k,k}},\ j = k+1, k+2, \cdots, N \text{ , Row } k, 2 \le k \le (N-1) \text{ .}$$

5. Repeat Steps 3 and 4, first for Column k of L, then for Row k of U. Continue for all k = 3,4,5, …,(N−1) for L and U, then for $k = N$ for L.

The procedure given above in Steps 1 - 5 is often referred to as Crout's method. Note that elements of L and U never look "backward" for previously used elements of Y. Therefore, in order to conserve computer memory, L and U elements can be stored as they are calculated in the same locations at the corresponding Y elements. Thus, Crout's method is a memory-efficient "in situ" procedure.

An intermediate column vector is needed to find V. The intermediate vector D is defined as

$$D = UV,$$

so that

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_N \end{bmatrix} = \begin{bmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,N} \\ 0 & 1 & u_{2,3} & \cdots & u_{2,N} \\ 0 & 0 & 1 & \cdots & u_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{bmatrix}.$$

Since $LUV = I$, then $LD = I$. Vector D is found using forward-substitution from

$$\begin{bmatrix} l_{1,1} & 0 & 0 & \cdots & 0 \\ l_{2,1} & l_{2,2} & 0 & \cdots & 0 \\ l_{3,1} & l_{3,2} & l_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{N,1} & l_{N,2} & l_{N,3} & \cdots & l_{N,N} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_N \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_N \end{bmatrix},$$

which proceeds as follows:

From Row 1, $l_{1,1}d_1 = I_1, d_1 = \frac{1}{l_{1,1}}(I_1)$,

From Row 2, $l_{2,1}d_1 + l_{2,2}d_2 = I_2, d_2 = \frac{1}{l_{2,2}}\left(I_2 - l_{2,1}d_1\right)$,

From Row k, $l_{k,1}d_1 + l_{k,2}d_2 + \cdots + l_{k,k}d_k = I_k, d_k = \frac{1}{l_{k,k}}\left(I_k - \sum_{j=1}^{k-1} l_{k,j}d_j\right)$.

Now, since $D = UV$, or

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_N \end{bmatrix} = \begin{bmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,N} \\ 0 & 1 & u_{2,3} & \cdots & u_{2,N} \\ 0 & 0 & 1 & \cdots & u_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{bmatrix},$$

where D and U are known, then V is found using backward substitution.

An important advantage of LU decomposition over Gaussian elimination or Kron reduction is that the I vector is not modified during decomposition. Therefore, once Y has been decomposed into L and U, I can be modified, and V recalculated, with minimal work, using the forward and backward substitution steps shown above.

A special form of L is helpful when Y is symmetric. In that case, let both L and U have unity diagonal terms, and define a diagonal matrix D so that

$$Y = LDU ,$$

or

$$Y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{2,1} & 1 & 0 & \cdots & 0 \\ l_{3,1} & l_{3,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_{N,1} & l_{N,2} & l_{N,3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} d_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & d_{2,2} & 0 & \cdots & 0 \\ 0 & 0 & d_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{N,N} \end{bmatrix} \begin{bmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,N} \\ 0 & 1 & u_{2,3} & \cdots & u_{2,N} \\ 0 & 0 & 1 & \cdots & u_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} .$$

Since Y is symmetric, then $Y = Y^T$, and $LDU = [LDU]^T = U^T D^T L^T = U^T D L^T$. Therefore, an acceptable solution is to allow $L = U^T$. Incorporating this into the above equation yields

$$Y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{2,1} & 1 & 0 & \cdots & 0 \\ l_{3,1} & l_{3,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_{N,1} & l_{N,2} & l_{N,3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} d_{1,1} & d_{1,1}l_{2,1} & d_{1,1}l_{3,1} & \cdots & d_{1,1}l_{N,1} \\ 0 & d_{2,2} & d_{2,2}l_{3,2} & \cdots & d_{2,2}l_{N,2} \\ 0 & 0 & d_{3,3} & \cdots & d_{3,3}l_{N,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{N,N} \end{bmatrix} ,$$

which can be solved by working from top-to-bottom, beginning with Column 1 of Y, as follows:

Working down Column 1 of Y,

$$y_{1,1} = d_{1,1} ,$$

$$y_{2,1} = l_{2,1}d_{1,1}, \text{so } l_{2,1} = y_{2,1} / d_{1,1} ,$$

$$y_{j,1} = l_{j,1}d_{1,1}, \text{so } l_{j,1} = y_{j,1} / d_{1,1} .$$

Working down Column 2 of Y,

$$y_{2,2} = l_{2,1}d_{1,1}l_{2,1} + d_{2,2}, \text{so } d_{2,2} = y_{2,2} - l_{2,1}d_{1,1}l_{2,1} ,$$

$$y_{j,2} = l_{j,1}d_{1,1}l_{2,1} + l_{j,2}d_{2,2}, \text{so } l_{j,2} = \left(y_{j,2} - l_{j,1}d_{1,1}l_{2,1}\right) / d_{2,2}, 2 < j \le N .$$

Working down Column k of Y,

$$y_{k,k} = \sum_{m=1}^{k-1} l_{k,m} d_{m,m} l_{k,m} + d_{k,k}, \text{so } d_{k,k} = y_{k,k} - \sum_{m=1}^{k-1} l_{k,m} d_{m,m} l_{k,m} \ ,$$

$$y_{j,k} = \sum_{m=1}^{k-1} l_{j,m} d_{m,m} l_{k,m}, \text{so } l_{j,k} = \left(y_{j,k} - \sum_{m=1}^{k-1} l_{j,m} d_{m,m} l_{k,m} \right) / d_{k,k}, k < j \le N \ .$$

This simplification reduces memory and computation requirements for LU decomposition by approximately one-half.

5. Bifactorization

Bifactorization recognizes the simple pattern that occurs when doing "in situ" LU decomposition. Consider the first four rows and columns of a matrix that has been LU decomposed according to Crout's method:

$$\begin{bmatrix} l_{1,1} = y_{1,1} & u_{1,2} = y_{1,2} / l_{1,1} & u_{1,3} = y_{1,3} / l_{1,1} & u_{1,4} = y_{1,4} / l_{1,1} \\ l_{2,1} = y_{2,1} & l_{2,2} = y_{2,2} - l_{2,1} u_{1,2} & u_{2,3} = \left(y_{2,3} - l_{2,1} u_{1,3}\right) / l_{2,2} & u_{2,4} = \left(y_{2,4} - l_{2,1} u_{1,4}\right) / l_{2,2} \\ l_{3,1} = y_{3,1} & l_{3,2} = y_{3,2} - l_{3,1} u_{1,2} & l_{3,3} = y_{3,3} - l_{3,1} u_{1,3} - l_{3,2} u_{2,3} & u_{3,4} = \left(y_{3,4} - l_{3,1} u_{1,4} - l_{3,2} u_{2,4}\right) / l_{3,3} \\ l_{4,1} = y_{4,1} & l_{4,2} = y_{4,2} - l_{4,1} u_{1,2} & l_{4,3} = y_{4,3} - l_{4,1} u_{1,3} - l_{4,2} u_{2,3} & l_{4,4} = y_{4,4} - l_{4,1} u_{1,4} - l_{4,2} u_{2,4} - l_{4,3} u_{3,4} \end{bmatrix} .$$

The pattern developed is very similar to Kron reduction, and it can be expressed in the following steps:

1. Beginning with Row 1, divide the elements to the right of the pivot element, and in the pivot row, by $l_{1,1}$, so that

$$y'_{1,k} = \frac{y_{1,k}}{y_{1,1}} \text{ , for } k = 2,3,4, \ldots , N.$$

2. Operate on the elements below and to the right of the pivot element using

$$y'_{j,k} = y_{j,k} - y_{j,1} y'_{1,k} \text{ , for } j = 2,3,4, \ldots , N, \ k = 2,3,4, \ldots , N.$$

3. Continue for pivots $m = 2,3,4, \ldots , (N - 1)$ using

$$y'_{m,k} = \frac{y'_{m,k}}{y'_{m,m}}, k = m+1, m+2, \cdots, N \ ,$$

followed by

$$y'_{j,k} = y'_{j,k} - y'_{j,m} y'_{m,k}, j = m+1, m+2, \cdots, N; k = m+1, m+2, \cdots, N \ ,$$

for each pivot m.

When completed, matrix Y' has been replaced by matrices L and U as follows:

$$Y' = \begin{bmatrix} l_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,5} \\ l_{2,1} & l_{2,2} & u_{2,3} & \cdots & u_{2,5} \\ l_{3,1} & l_{3,2} & l_{3,3} & \cdots & u_{3,5} \\ \vdots & \vdots & \vdots & \ddots & \\ l_{N,1} & l_{N,2} & l_{N,3} & \cdots & l_{N,N} \end{bmatrix},$$

and where the diagonal u elements are unity (i.e. $u_{1,1} = u_{2,2} = u_{3,3} = \cdots = u_{N,N} = 1$).

The corresponding FORTRAN code for bifactorization is

```
      COMPLEX Y(N,N),YMM
C  DO FOR EACH PIVOT M = 1,2,3, ... ,N - 1
      NM1     = N - 1
      DO 1 M  = 1,NM1
C  FOR THE PIVOT ROW
      MP1     = M + 1
      YMM     = 1.0 / Y(M,M)
      DO 2 K  = MP1,N
      Y(M,K)  = Y(M,K) * YMM
   2  CONTINUE
C  BELOW AND TO THE RIGHT OF THE PIVOT ELEMENT
      DO 3 J   = MP1,N
      DO 3 K   = MP1,N
      Y(J,K)   = Y(J,K) - Y(J,M) * Y(M,K)
   3  CONTINUE
   1  CONTINUE
      STOP
      END
```

6. Shipley-Coleman Inversion

For relatively small matrices, it is possible to obtain the inverse directly. The Shipley-Coleman inversion method for inversion is popular because it is easily programmed. The algorithm is

1. For each pivot (i.e. diagonal term) m, $m = 1,2,3, \ldots ,N$, perform the following Steps 2 - 4.

2. Kron reduce all elements in Y, above and below, except those in Column m and Row m using

$$y'_{j,k} = y'_{j,k} - \frac{y'_{j,m} y'_{m,k}}{y_{m,m}},\ j = 1,2,3,\cdots,N,\ j \neq m; k = 1,2,3,\cdots,N, k \neq m\ .$$

3. Replace pivot element $y'_{m,m}$ with its negative inverse, i.e. $\dfrac{-1}{y'_{m,m}}$.

4. Multiply all elements in Row *m* and Column *m*, except the pivot, by $y'_{m,m}$.

The result of this procedure is actually the negative inverse, so that when completed, all terms must be negated. A FORTRAN code for Shipley-Coleman is shown below.

```
      COMPLEX Y(N,N),YPIV
C  DO FOR EACH PIVOT M = 1,2,3, ... ,N
      DO 1 M  = 1,N
      YPIV    = 1.0 / Y(M,M)
C  KRON REDUCE ALL ROWS AND COLUMNS, EXCEPT THE PIVOT ROW
C  AND PIVOT COLUMN
      DO 2 J  = 1,N
      IF(J .EQ. M) GO TO 2
      DO 2 K  = 1,N
      IF(K. NE. M) Y(J,K) = Y(J,K) - Y(J,M) * Y(M,K) * YPIV
   2  CONTINUE
C  INVERT THE PIVOT ELEMENT AND NEGATE IT
      YPIV     = -YPIV
      Y(M,M)   = YPIV
C  WORK ACROSS THE PIVOT ROW AND DOWN THE PIVOT COLUMN,
C  MULTIPLYING BY THE NEW PIVOT VALUE
      DO 3 K   = 1,N
      IF(K .EQ. M) GO TO 3
      Y(M,K) = Y(M,K) * YPIV
      Y(K,M) = Y(K,M) * YPIV
   3  CONTINUE
   1  CONTINUE
C  NEGATE THE RESULTS
      DO 4 J   = 1,N
      DO 4 K   = 1,N
      Y(J,K)   = -Y(J,K)
   4  CONTINUE
      STOP
      END
```

The order of the number of calculations for Shipley-Coleman is N^3 .

7. Impedance Matrix

The impedance matrix is the inverse of the admittance matrix, or

$$Z = Y^{-1} \ ,$$

so that

$$V = ZI \ .$$

The reference bus both Y and Z is ground. Although the impedance matrix can be found via inversion, complete inversion is not common for matrices with more than a few hundred rows and columns because of the matrix storage requirements. In those instances, Z elements are usually found via Gaussian elimination, Kron reduction, or, less commonly, by a direct building algorithm. If only a few of the Z elements are needed, then Gaussian elimination or Kron reduction are best. Both methods are described in following sections.

8. Physical Significance of Admittance and Impedance Matrices

The physical significance of the admittance and impedance matrices can be seen by examining the basic matrix equations $I = YV$ and $V = Y^{-1}I = ZI$. Expanding the *jth* row of $I = YV$ yields $I_j = \sum_{k=1}^{N} y_{j,k} V_k$. Therefore,

$$y_{j,k} = \left. \frac{I_j}{V_k} \right|_{V_m = 0, m = 1,2,\cdots,N, m \neq k} \ ,$$

where, as shown in Figure 2, all busses except k are grounded, V_k is a voltage source attached to bus k, and I_j is the resulting current injection at (i.e. flowing into) bus j. Since all busses that neighbor bus k are grounded, the currents induced by V_k will not get past these neighbors, and the only non-zero injection currents I_j will occur at the neighboring busses. In a large power system, most busses do not neighbor any arbitrary bus k. Therefore, Y consists mainly of zeros (i.e. is sparse) in most power systems.

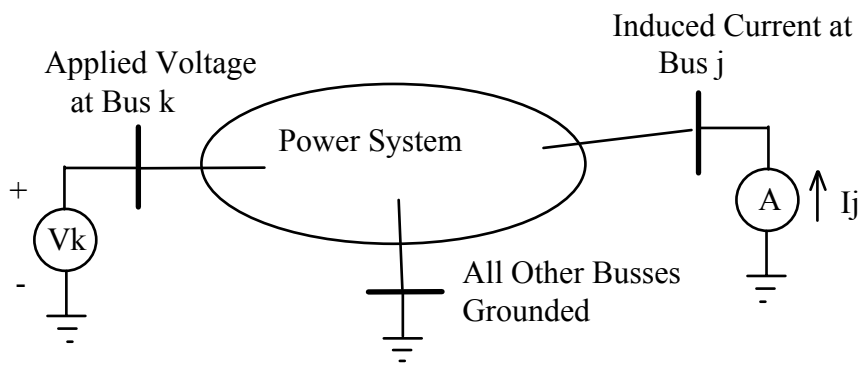


Figure 2. Measurement of Admittance Matrix Term $y_{j,k}$

Concerning *Z*, the *kth* row of $V = Y^{-1}I = ZI$ yields $V_k = \sum_{j=1}^{N} z_{k,j} I_j$. Hence,

$$z_{j,k} = \left.\frac{V_j}{I_k}\right|_{I_m = 0, m=1,2,\cdots,N, m \neq k},$$

where, as shown in Figure 3, I_k is a current source attached to bus *k*, V_j is the resulting voltage at bus *j*, and all busses except *k* are open-circuited. Unless the network is disjoint, then current injection at one bus, when all other busses open-circuited, will raise the potential everywhere in the network. For that reason, *Z* tends to be full.

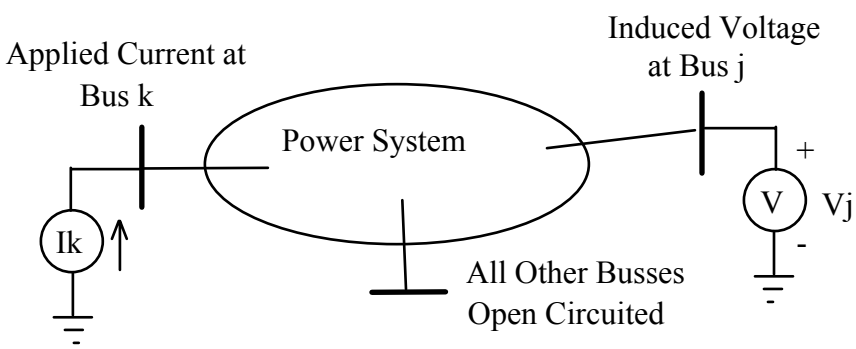


Figure 3. Measurement of Impedance Matrix Term $z_{j,k}$

9. Formation of the Impedance Matrix via Gaussian Elimination, Kron Reduction, and LU Decomposition

Gaussian Elimination

An efficient method of fully or partially inverting a matrix is to formulate the problem using Gaussian elimination. For example, given

$$YZ = I \ ,$$

where Y is a matrix of numbers, Z is a matrix of unknowns, and I is the identity matrix, the objective is to Gaussian eliminate Y, while performing the same row operations on I, to obtain the form

$$Y'Z = I' \ ,$$

where Y' is in (URT+D) form. Then, individual columns of Z can then be solved using backward substitution, one at a time. This procedure is also known as the augmentation method, and it is illustrated as follows:

Y is first Gaussian eliminated, as shown in a previous section, while at the same time performing identical row operations on I. Afterward, the form of $Y'Z = I'$ is

$$\begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,N} \\ 0 & y'_{2,2} & y'_{2,3} & \cdots & y'_{2,N} \\ 0 & 0 & y'_{3,3} & \cdots & y'_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y'_{N,N} \end{bmatrix} \begin{bmatrix} z_{1,1} & z_{1,2} & z_{1,3} & \cdots & z_{1,N} \\ z_{2,1} & z_{2,2} & z_{2,3} & \cdots & z_{2,N} \\ z_{3,1} & z_{3,2} & z_{3,3} & \cdots & z_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{N,1} & z_{N,2} & z_{N,3} & \cdots & z_{N,N} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ I'_{2,1} & I'_{2,2} & 0 & \cdots & 0 \\ I'_{3,1} & I'_{3,2} & I'_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I'_{N,1} & I'_{N,2} & I'_{N,3} & \cdots & I'_{N,N} \end{bmatrix}$$

where Rows 1 of Y' and I' are the same as in Y and I. The above equation can be written in abbreviated column form as

$$\begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,N} \\ 0 & y'_{2,2} & y'_{2,3} & \cdots & y'_{2,N} \\ 0 & 0 & y'_{3,3} & \cdots & y'_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y'_{N,N} \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 & Z_3 & \cdots & Z_N \end{bmatrix} = \begin{bmatrix} I'_1 & I'_2 & I'_3 & \cdots & I'_N \end{bmatrix},$$

where the individual column vectors Z_i and I_i have dimension N x 1. The above equation can be solved as N separate subproblems, where each subproblem computes a column of Z. For example, the kth column of Z can be computed by applying backward substitution to

$$\begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,N} \\ 0 & y'_{2,2} & y'_{2,3} & \cdots & y'_{2,N} \\ 0 & 0 & y'_{3,3} & \cdots & y'_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y'_{N,N} \end{bmatrix} \begin{bmatrix} Z_{1,k} \\ Z_{2,k} \\ Z_{3,k} \\ \vdots \\ Z_{N,k} \end{bmatrix} = \begin{bmatrix} I'_{1,k} \\ I'_{2,k} \\ I'_{3,k} \\ \vdots \\ I'_{N,k} \end{bmatrix} .$$

Each column of Z is solved independently from the others.

Kron Reduction

If Kron reduction, the problem is essentially the same, except that Row 1 of the above equation is divided by $y'_{1,1}$, yielding

$$\begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,N} \\ 0 & y'_{2,2} & y'_{2,3} & \cdots & y'_{2,N} \\ 0 & 0 & y'_{3,3} & \cdots & y'_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y'_{N,N} \end{bmatrix} \begin{bmatrix} Z_{1,k} \\ Z_{2,k} \\ Z_{3,k} \\ \vdots \\ Z_{N,k} \end{bmatrix} = \begin{bmatrix} I'_{1,k} \\ I'_{2,k} \\ I'_{3,k} \\ \vdots \\ I'_{N,k} \end{bmatrix} .$$

Because backward substitution can stop when the last desired *z* element is computed, the process is most efficient if the busses are ordered so that the highest bus numbers (i.e. *N*, *N-1*, *N-2*, etc.) correspond to the desired *z* elements. Busses can be ordered accordingly when forming *Y* to take advantage of this efficiency.

LU Decomposition

Concerning LU decomposition, once the *Y* matrix has been decomposed into *L* and *U*, then we have

$$YZ = LUZ = I ,$$

where *I* is the identity matrix. Expanding the above equation as $L[UZ] = I$ yields

$$\begin{bmatrix} l_{1,1} & 0 & 0 & \cdots & 0 \\ l_{2,1} & l_{2,2} & 0 & \cdots & 0 \\ l_{3,1} & l_{2,3} & l_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{N,1} & l_{N,2} & l_{N,3} & \cdots & l_{N,N} \end{bmatrix} \begin{bmatrix} uz_{1,1} & uz_{1,2} & uz_{1,3} & \cdots & uz_{1,N} \\ uz_{2,1} & uz_{2,2} & uz_{2,3} & \cdots & uz_{2,N} \\ uz_{3,1} & uz_{3,2} & uz_{3,3} & \cdots & uz_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ uz_{N,1} & uz_{N,2} & uz_{N,3} & \cdots & uz_{N,N} \end{bmatrix} = .$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

The special structure of the above equation shows that, in general, *UZ* must be (LLT+D) in form. *UZ* can be found by using forward substitution on the above equation, and then *Z* can be found, one column at a time, by using backward substitution on $UZ = [U][Z]$, which in expanded form is

$$\begin{bmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,N} \\ 0 & 1 & u_{2,3} & \cdots & u_{2,N} \\ 0 & 0 & 1 & \cdots & u_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_{1,1} & z_{1,2} & z_{1,3} & \cdots & z_{1,N} \\ z_{2,1} & z_{2,2} & z_{2,3} & \cdots & z_{2,N} \\ z_{3,1} & z_{3,2} & z_{3,3} & \cdots & z_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{N,1} & z_{N,2} & z_{N,3} & \cdots & z_{N,N} \end{bmatrix} =$$

$$\begin{bmatrix} uz_{1,1} & 0 & 0 & \cdots & 0 \\ uz_{2,1} & uz_{2,2} & 0 & \cdots & 0 \\ uz_{3,1} & uz_{3,2} & uz_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ uz_{N,1} & uz_{N,2} & uz_{N,3} & \cdots & uz_{N,N} \end{bmatrix}.$$

10. Direct Impedance Matrix Building and Modification Algorithm

The impedance matrix can be built directly without need of the admittance matrix, if the physical properties of *Z* are exploited. Furthermore, once *Z* has been built, no matter which method was used, it can be easily modified when network changes occur. This after-the-fact modification capability is the most important feature of the direct impedance matrix building and modification algorithm.

The four cases to be considered in the algorithm are

Case 1. Add a branch impedance between a new bus and the reference bus.

Case 2. Add a branch impedance between a new bus and an existing non-reference bus.

Case 3. Add a branch impedance between an existing bus and the reference bus.

Case 4. Add a branch impedance between two existing non-reference busses.

Direct formation of an impedance matrix, plus all network modifications, can be achieved through these four cases. Each of the four is now considered separately.

Case 1, Add a Branch Impedance Between a New Bus and the Reference Bus

Case 1 is applicable in two situations. First, it is the starting point when building a new impedance matrix. Second, it provides (as does Case 2) a method for adding new busses to an existing impedance matrix.

Both situations applicable to Case 1 are shown in Figure 4. The starting-point situation is on the left, and the new-bus situation is on the right.

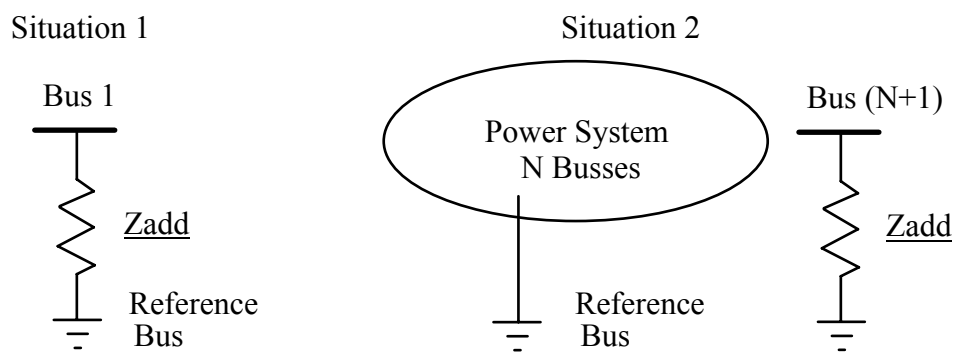


Figure 4. Case 1, Add a Branch Impedance Between a New Bus and the Reference Bus

The impedance matrices for the two situations are

Situation 1 $Z_{1,1} = \underline{Zadd}$,

Situation 2: $$Z_{N+1,N+1} = \begin{bmatrix} \left[Z_{N,N}\right]_{NxN} & \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{Nx1} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}_{1xN} & \left[\underline{Zadd}\right]_{1x1} \end{bmatrix}.$$

The effect of situation 2 is simply to augment the existing $Z_{N,N}$ by a column of zeros, a row of zeros, and a new diagonal element <u>*Zadd*</u>. New bus (*N+1*) is isolated from the rest of the system.

Case 2, Add a Branch Impedance Between a New Bus and an Existing Non-Reference Bus

Consider Case 2, shown in Figure 5, where a branch impedance <u>*Zadd*</u> is added from existing non-reference bus *j* to new bus (*N+1*). Before the addition, the power system has *N* busses, plus a reference bus (which is normally ground), and an impedance matrix $Z_{N,N}$.

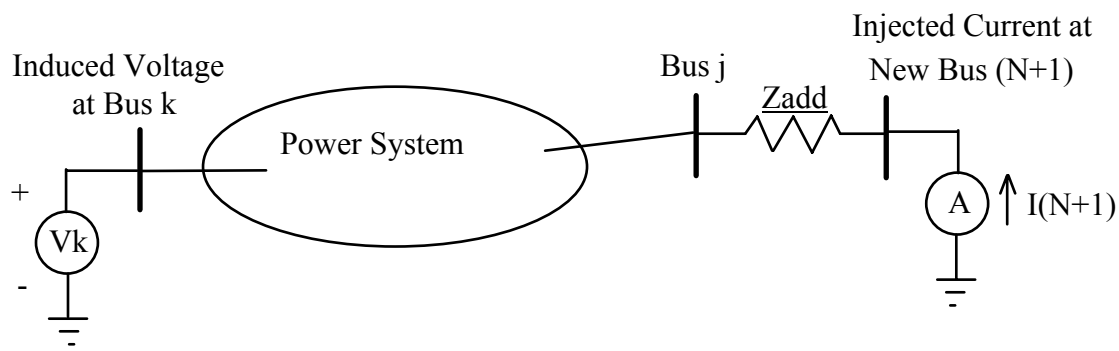


Figure 5. Case 2, Add a Branch Impedance Between New Bus (*N+1*) and Existing Non-Reference Bus j

Since all of the current I_{N+1} injected at new bus (*N+1*) flows through <u>*Zadd*</u> and into Bus *j*, the original *N* power system busses cannot distinguish between current injections I_j and I_{N+1}. Therefore,

$$\frac{\partial V_k}{\partial I_{N+1}} = \frac{\partial V_k}{\partial I_j}, k \neq (N+1) \ ,$$

meaning that impedance matrix elements $z_{k,j}$ and $z_{k,N+1}$ are identical for busses $k \neq (N+1)$, and that the effect on the impedance matrix is to augment it with an additional Row (*N+1*) that is identical to Row *j*.

Likewise, since bus (*N+1*) is an open-circuited radial bus stemming from bus *j*, a current injected at another bus *k* creates identical voltage changes on busses *j* and (*N+1*). This means that

$$\frac{\partial V_{N+1}}{\partial I_k} = \frac{\partial V_j}{\partial I_k}, k \neq (N+1) \ ,$$

so that the effect on the impedance matrix is to augment it with an additional Column (*N+1*) that is identical to Column *j*.

Next, due to the fact that all of injection current $I_{N+1,N+1}$ passes through new branch impedance <u>*Zadd*</u>, the relationship between V_j and V_{N+1} is

$$\frac{\partial V_{N+1}}{\partial I_{N+1}} = \frac{\partial V_j}{\partial I_{N+1}} + \underline{Zadd} \ ,$$

so that the (*N+1*) diagonal term of the impedance matrix becomes

$$z_{N+1,N+1} = z_{j,j} + \underline{Zadd} \ ,$$

where $z_{j,j}$ is the *jth* diagonal term of Z_{NxN} .

Summarizing, the total effect of Case 2 is to increase the dimension of the impedance matrix by one row and column according to

$$Z_{N+1,N+1} = \begin{bmatrix} \lfloor Z_{N,N} \rfloor_{NxN} & \lfloor \text{Column } j \text{ of } Z_{N,N} \rfloor_{Nx1} \\ \lfloor \text{Row } j \text{ of } Z_{N,N} \rfloor_{1xN} & \lfloor z_{j,j} + \underline{Zadd} \rfloor_{1x1} \end{bmatrix} .$$

Case 3, Add a Branch Impedance Between an Existing Bus and the Reference Bus

Now, consider Case 3, where a new impedance-to-reference tie <u>*Zadd*</u> is added to existing Bus *j*. The case is handled as a two-step extension of Case 2, as shown in Figure 4.6. First, extend <u>*Zadd*</u> from Bus *j* to fictitious Bus (*N+1*), as was done in Case 2. Then, tie fictitious Bus (*N+1*) to the reference bus, and reduce the size of the augmented impedance matrix back to (*N* x *N*).

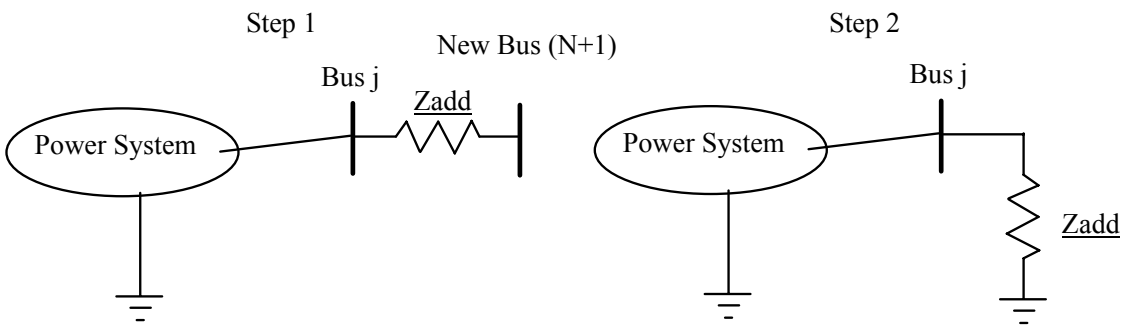


Figure 6. Case 3. Two-Step Procedure for Adding Branch Impedance <u>*Zadd*</u> from Existing Bus *j* to the Reference Bus

Step 1 creates the augmented $Z_{N+1,N+1}$ matrix shown in Case 2. The form of equation $V_{N+1} = Z_{N+1,N+1} I_N$ is

$$\begin{bmatrix} V_{Nx1} \\ V_{REF} = 0 \end{bmatrix} = \begin{bmatrix} \lfloor Z_{N,N} \rfloor_{NxN} & \lfloor \text{Column } j \text{ of } Z_{N,N} \rfloor_{Nx1} \\ \lfloor \text{Row } j \text{ of } Z_{N,N} \rfloor_{1xN} & \lfloor z_{j,j} + \underline{Zadd} \rfloor_{1x1} \end{bmatrix} \begin{bmatrix} I_{Nx1} \\ I_{REF} \end{bmatrix} ,$$

where V_{REF}, $z_{j,j} + \underline{Zadd}$, and I_{REF} are scalars. Defining the row and column vectors as R_j and C_j , respectively, yields

$$\begin{bmatrix} V_{Nx1} \\ V_{REF} = 0 \end{bmatrix} = \begin{bmatrix} \lfloor Z_{N,N} \rfloor_{NxN} & \lfloor C_j \rfloor_{Nx1} \\ \lfloor R_j \rfloor_{1xN} & \lfloor z_{j,j} + \underline{Zadd} \rfloor_{1x1} \end{bmatrix} \begin{bmatrix} I_{Nx1} \\ I_{REF} \end{bmatrix} .$$

At this point, scalar I_{REF} can be eliminated by expanding the bottom row to obtain

$$I_{REF} = -\frac{\lfloor R_j I_{Nx1} \rfloor_{1x1}}{z_{j,j} + \underline{Zadd}} \ .$$

Substituting into the top N rows yields

$$V_{Nx1} = Z_{NxN} I_{Nx1} - \frac{1}{z_{j,j} + \underline{Zadd}} \left(C_j R_j \right) I_{Nx1} = \left(Z_{NxN} - \frac{1}{z_{j,j} + \underline{Zadd}} \left(C_j R_j \right) \right) I_{Nx1} \ ,$$

or

$$V_{Nx1} = Z'_{NxN} I_{Nx1} \ .$$

Expanding $C_j R_j$ gives

$$\begin{bmatrix} z_{1,j} \\ z_{2,j} \\ z_{3,j} \\ \vdots \\ z_{N,j} \end{bmatrix} \begin{bmatrix} z_{j,1} & z_{j,2} & z_{j,3} & \cdots & z_{j,N} \end{bmatrix} = \begin{bmatrix} z_{1,j} z_{j,1} & z_{1,j} z_{j,2} & z_{1,j} z_{j,3} & \cdots & z_{1,j} z_{j,N} \\ z_{2,j} z_{j,1} & z_{2,j} z_{j,2} & z_{2,j} z_{j,3} & \cdots & z_{2,j} z_{j,N} \\ z_{3,j} z_{j,1} & z_{3,j} z_{j,2} & z_{3,j} z_{j,3} & \cdots & z_{3,j} z_{j,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{N,j} z_{j,1} & z_{N,j} z_{j,2} & z_{N,j} z_{j,3} & \cdots & z_{N,j} z_{j,N} \end{bmatrix} ,$$

so that the individual elements of the modified impedance matrix Z'_{Nx1} can be written as

$$z'_{i,k} = z_{i,k} - \frac{z_{i,j} z_{j,k}}{z_{j,j} + \underline{Zadd}}, i = 1,2,3,\cdots,N; k = 1,2,3,\cdots,N \ .$$

Note that the above equation corresponds to Kron reducing the augmented impedance matrix, using element $z_{N+1,N+1}$ as the pivot.

Case 4, Add a Branch Impedance Between Two Existing Non-Reference Busses

The final case to be considered is that of adding a branch between existing busses j and k in an N-bus power system that has impedance matrix $Z_{N,N}$. The system and branch are shown in Figure 4.7.

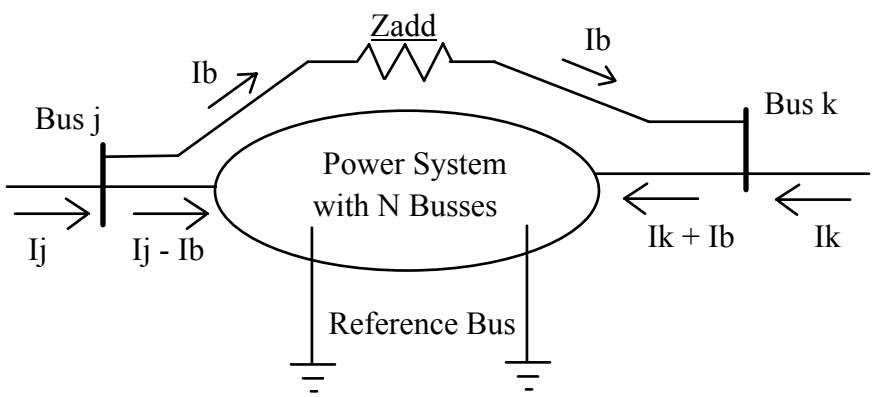


Figure 7. Case 4, Add Branch Impedance <u>*Zadd*</u> between Existing Busses *j* and *k*

As seen in the figure, the actual current injected into the system via Bus *j* is $I_j - I_b$, and the actual injection via Bus *k* is $I_k + I_b$, where

$$I_b = \frac{V_j - V_k}{\underline{Zadd}} \text{ , or } V_j - V_k = (\underline{Zadd})I_b \text{ .}$$

The effect of the branch addition on the voltage at arbitrary bus *m* can be found by substituting the true injection currents at busses *j* and *k* into $V = Z_{N,N}I$, yielding

$$V_m = z_{m,1}I_1 + z_{m,2}I_2 + z_{m,3}I_3 + \cdots z_{m,j}(I_j - I_b) + \cdots + z_{m,k}(I_k + I_b) + \cdots + z_{m,N}I_N \text{ ,}$$

or

$$V_m = z_{m,1}I_1 + z_{m,2}I_2 + z_{m,3}I_3 + \cdots z_{m,j}I_j + \cdots + z_{m,k}I_k + \cdots + \cdots + z_{m,N}I_N - (z_{m,j} - z_{m,k})I_b \text{ .}$$

For busses *j* and *k* specifically,

$$\begin{aligned} V_j &= z_{j,1}I_1 + z_{j,2}I_2 + z_{j,3}I_3 + \cdots + z_{j,j}(I_j - I_b) + \cdots + z_{j,k}(I_k + I_b) + \cdots + z_{j,N}I_N \\ V_k &= z_{k,1}I_1 + z_{k,2}I_2 + z_{k,3}I_3 + \cdots + z_{k,j}(I_j - I_b) + \cdots + z_{k,k}(I_k + I_b) + \cdots + z_{k,N}I_N \text{ .} \end{aligned}$$

Then, $V_j - V_k$ gives

$$V_j - V_k = (z_{j,1} - z_{k,1})I_1 + (z_{j,2} - z_{k,2})I_2 + (z_{j,3} - z_{k,3})I_3 + \cdots + (z_{j,j} - z_{k,j})(I_j - I_b) + \cdots$$
$$(z_{j,k} - z_{k,k})(I_k + I_b) + \cdots + (z_{j,N} - z_{k,N})I_N \ .$$

Substituting in $V_j - V_k = (\underline{Zadd})I_b$ and combining terms then yields

$$0 = (z_{j,1} - z_{k,1})I_1 + (z_{j,2} - z_{k,2})I_2 + (z_{j,3} - z_{k,3})I_3 + \cdots + (z_{j,j} - z_{k,j})I_j + \cdots$$
$$(z_{j,k} - z_{k,k})I_k + \cdots + (z_{j,N} - z_{k,N})I_N - (z_{j,j} + z_{k,k} - z_{j,k} - z_{k,j} + \underline{Zadd})I_b \ .$$

All of the above effects can be included as an additional row and column in equation $V = Z_{N,N}I$ as follows:

$$\begin{bmatrix} [V_N]_{Nx1} \\ [0]_{Nx1} \end{bmatrix} = \begin{bmatrix} [Z_{N,N}]_{NxN} & [\text{Col } j \text{ of } Z_{N,N} - \text{Col } k \text{ of } Z_{N,N}]_{Nx1} \\ [\text{Row } j \text{ of } Z_{N,N} - \text{Row } k \text{ of } Z_{N,N}]_{1xN} & [z_{j,j} + z_{k,k} - z_{j,k} - z_{k,j} + \underline{Zadd}]_{Nx1} \end{bmatrix} \cdot$$
$$\begin{bmatrix} [I_N]_{Nx1} \\ [-I_b]_{1x1} \end{bmatrix}.$$

As was done in Case 3, the effect of the augmented row and column of the above equation can be incorporated into a modified impedance matrix by using Kron reduction, where element $z_{N+1,N+1}$ is the pivot, yielding

$$z'_{i,m} = z_{i,m} - \frac{(z_{i,j} - z_{i,k})(z_{j,m} - z_{k,m})}{z_{j,j} + z_{k,k} - z_{j,k} - z_{k,j} + \underline{Zadd}} \ .$$

Application Notes

Once an impedance matrix is built, no matter which method is used, the modification algorithm can very easily adjust Z for network changes. For example, a branch outage can be effectively achieved by placing an impedance, of the same but negative value, in parallel with the actual impedance, so that the impedance of the parallel combination is infinite.

11. Example Code YZBUILD for Building Admittance Matrix and Impedance Matrix

```
c
c  program yzbuild.   10/10/04.
c
      implicit none
      complex ybus,yline,ycap,ctap,yii,ykk,yik,yki
      complex lmat,umat,unity,czero,ytest,ysave,diff,uz,
     1 uzelem,zbus,elem,alpha
      integer nb,nbus,nfrom,nto,j,k,ipiv,jdum,kdum,jm1,kp1,l
      integer nlu,ipiv1,nm1,ny,nbext,nbint,iout,mxext,mxint
      integer jp1,ipp1,ngy,ngu
      real eps4,eps6,eps9,pi,dr,qshunt,r,x,charge,fill,ymag,umag
```

```
      dimension ybus(150,150),nbext(150),nbint(9999)
      dimension lmat(150,150),umat(150,150),ysave(150,150),
     1 uz(150,150),zbus(150,150),unity(150,150)
      data mxint/150/,mxext/9999/,nbus/0/, nbint/9999 * 0/, iout/6/,
     1 nbext/150 * 0/,czero /(0.0,0.0)/,eps4/1.0e-04/,
     2 eps6/1.0e-06/,eps9/1.0e-09/
      data ybus  /22500 * (0.0,0.0)/
      data zbus  /22500 * (0.0,0.0)/
      data uz    /22500 * (0.0,0.0)/
      data unity /22500 * (0.0,0.0)/
c
      pi = 4.0 * atan(1.0)
      dr = 180.0 / pi

      open(unit=1,file='uz.txt')
      write(1,*) 'errors'
      close(unit=1,status='keep')
      open(unit=1,file='zbus_from_lu.txt')
      write(1,*) 'errors'
      close(unit=1,status='keep')
      open(unit=1,file='ybus_gaussian_eliminated.txt')
      write(1,*) 'errors'
      close(unit=1,status='keep')
      open(unit=1,file='unity_mat.txt')
      write(1,*) 'errors'
      close(unit=1,status='keep')
      open(unit=1,file='zbus_from_gaussian_elim.txt')
      write(1,*) 'errors'
      close(unit=1,status='keep')

      open(unit=6,file='yzbuild_screen_output.txt')
      open(unit=2,file='ybus.txt')
      open(unit=1,file='bdat.csv')
      write(6,*) 'program yzbuild, 150 bus version, 10/10/04'
      write(2,*) 'program yzbuild, 150 bus version, 10/10/04'
      write(6,*) 'read bus data from file bdat.csv'
      write(6,*) 'number              b'
      write(2,*) 'read bus data from file bdat.csv'
      write(2,*) 'number              b'
c
c  read the bdat.csv file
c
    1 read(1,*,end=5,err=30) nb,qshunt
      if(nb .eq. 0) go to 5
      write(6,1002) nb,qshunt
      write(2,1002) nb,qshunt
 1002 format(i7,5x,f10.4)
      if(nb .le. 0 .or. nb .gt. mxext) then
        write(6,*) 'illegal bus number - stop'
        write(2,*) 'illegal bus number - stop'
        write(*,*) 'illegal bus number - stop'
        stop
      endif
      nbus = nbus + 1
      if(nbus .gt. mxint) then
        write(6,*) 'too many busses - stop'
        write(2,*) 'too many busses - stop'
        write(*,*) 'too many busses - stop'
        stop
```

```
      endif
      nbext(nbus) = nb
      nbint(nb)   = nbus
      ycap        = cmplx(0.0,qshunt)
      ybus(nbus,nbus) = ybus(nbus,nbus) + ycap
      go to 1
    5 close(unit=1,status='keep')
c
      open(unit=1,file='ldat.csv')
      write(6,*)
      write(6,*) 'read line/transformer data from file ldat.csv'
      write(6,*) ' from     to           r           x',
     1 '           b'
      write(2,*) 'read line/transformer data from file ldat.csv'
      write(2,*) ' from     to           r           x',
     1 '           b'
c
c  read the ldat.csv file
c
   10 read(1,*,end=15,err=31) nfrom,nto,r,x,charge
      if(nfrom .eq. 0 .and. nto .eq. 0) go to 15
      write(6,1004) nfrom,nto,r,x,charge
      write(2,1004) nfrom,nto,r,x,charge
 1004 format(1x,i5,2x,i5,3(2x,f10.4))
      if(nfrom .lt. 0 .or. nfrom .gt. mxext .or. nto .lt. 0
     1 .or. nto .gt. mxext) then
        write(6,*) 'illegal bus number - stop'
        write(2,*) 'illegal bus number - stop'
        write(*,*) 'illegal bus number - stop'
        stop
      endif
      if(nfrom .eq. nto) then
        write(6,*) 'same bus number given on both ends - stop'
        write(2,*) 'same bus number given on both ends - stop'
        write(*,*) 'same bus number given on both ends - stop'
        stop
      endif
      if(r .lt. -eps6) then
        write(6,*) 'illegal resistance - stop'
        write(2,*) 'illegal resistance - stop'
        write(*,*) 'illegal resistance - stop'
        stop
      endif
      if(charge .lt. -eps6) then
        write(6,*) 'line charging should be positive'
        write(2,*) 'line charging should be positive'
        write(*,*) 'line charging should be positive'
        stop
      endif
      if(nfrom .ne. 0) then
        nfrom = nbint(nfrom)
        if(nfrom .eq. 0) then
          write(6,*) 'bus not included in file bdat.csv - stop'
          write(2,*) 'bus not included in file bdat.csv - stop'
          write(*,*) 'bus not included in file bdat.csv - stop'
          stop
        endif
      endif
      if(nto   .ne. 0) then
        nto   = nbint(nto)
```

```
        if(nto .eq. 0) then
          write(6,*) 'bus not included in file bdat.csv - stop'
          write(2,*) 'bus not included in file bdat.csv - stop'
          write(*,*) 'bus not included in file bdat.csv - stop'
          stop
        endif
      endif
      yline = cmplx(r,x)
      if(cabs(yline) .lt. eps6) go to 10
      yline = 1.0 / yline
      ycap  = cmplx(0.0,charge / 2.0)

c
c  the line charging terms
c
      if(nfrom .ne. 0) ybus(nfrom,nfrom) = ybus(nfrom,nfrom) + ycap
      if(nto   .ne. 0) ybus(nto  ,nto  ) = ybus(nto  ,nto  ) + ycap
c
c  shunt elements
c
      if(nfrom .ne. 0 .and. nto   .eq. 0) ybus(nfrom,nfrom) =
     1 ybus(nfrom,nfrom) + yline
      if(nfrom .eq. 0 .and. nto   .ne. 0) ybus(nto  ,nto  ) =
     1 ybus(nto  ,nto  ) + yline
c
c  transmission lines
c
      if(nfrom .ne. 0 .and. nto .ne. 0) then
        ybus(nfrom,nto  ) = ybus(nfrom,nto  ) - yline
        ybus(nto  ,nfrom) = ybus(nto  ,nfrom) - yline
        ybus(nfrom,nfrom) = ybus(nfrom,nfrom) + yline
        ybus(nto  ,nto  ) = ybus(nto  ,nto  ) + yline
      endif
      go to 10
c
c  write the nonzero ybus elements to file ybus
c
   15 close(unit=1,status='keep')
         write(6,*)
      write(6,*) 'nonzero elements of ybus (in rectangular form)'
      write(2,*) 'nonzero elements of ybus (in rectangular form)'
      write(6,*) '-internal-   -external-'
      write(2,*) '-internal-   -external-'
      ny = 0

      do j = 1,nbus
        do k = 1,nbus
          ysave(j,k) = ybus(j,k)
          if(cabs(ybus(j,k)) .ge. eps9) then
            ny = ny + 1
            write(6,1005) j,k,nbext(j),nbext(k),ybus(j,k)
            write(2,1005) j,k,nbext(j),nbext(k),ybus(j,k)
 1005 format(2i5,3x,2i5,2e20.8)
          endif
        end do
      end do

      close(unit=2,status='keep')
      fill = ny / float(nbus * nbus)
      write(iout,525) ny,fill
```

```
  525 format(/1x,'number of nonzero elements in ybus = ',i5/1x,
     1'percent fill = ',2pf8.2/)
c
c  bifactorization - replace original ybus with lu
c
      nm1 = nbus - 1

      do ipiv = 1,nm1
        write(iout,530) ipiv
  530 format(1x,'lu pivot element = ',i5)
        ipiv1 = ipiv + 1
        alpha = 1.0 / ybus(ipiv,ipiv)
        do k = ipiv1,nbus
          ybus(ipiv,k) = alpha * ybus(ipiv,k)
        end do
        do j = ipiv1,nbus
          alpha   = ybus(j,ipiv)
          do k = ipiv1,nbus
            ybus(j,k) = ybus(j,k) - alpha * ybus(ipiv,k)
          end do
        end do
      end do

      write(iout,530) nbus
      write(iout,532)
  532 format(/1x,'nonzero lu follows'/)
      open(unit=4,file='lu.txt')
      nlu = 0

      do j = 1,nbus
        do k = 1,nbus
          ymag = cabs(ybus(j,k))
          if(ymag .gt. eps9) then
            nlu = nlu + 1
            write(iout,1005) j,k,nbext(j),nbext(k),ybus(j,k)
            write(4,1005) j,k,nbext(j),nbext(k),ybus(j,k)
          endif
        end do
      end do

      fill = nlu / float(nbus * nbus)
      write(iout,535) nlu,fill
  535 format(/1x,'number of nonzero elements in lu = ',i5/1x,
     1'percent fill = ',2pf8.2/)
      close(unit=4,status='keep')
c
c  check l times u
c
      write(iout,560)
  560 format(1x,'lu .ne. ybus follows'/)

      do j = 1,nbus
        do k = 1,nbus
          lmat(j,k) = czero
          umat(j,k) = czero
          if(j .ge. k) lmat(j,k) = ybus(j,k)
          if(j .lt. k) umat(j,k) = ybus(j,k)
        end do
        umat(j,j) = 1.0
      end do
```

```
      do j = 1,nbus
        do k = 1,nbus
          ytest   = czero
          do l = 1,k
            ytest = ytest + lmat(j,l) * umat(l,k)
          end do
          diff = ysave(j,k) - ytest
          if(cabs(diff) .gt. eps4) write(iout,1005) j,k,nbext(j),
     1     nbext(k),diff
        end do
      end do
c
c  form uz (urt + diag)
c
      write(iout,536) nbus
  536 format(1x,'uz pivot = ',i5)
      uz(nbus,nbus) = 1.0 / ybus(nbus,nbus)
      nm1 = nbus - 1

      do kdum = 1,nm1
        k   = nbus - kdum
        write(iout,536) k
        uz(k,k) = 1.0 / ybus(k,k)
        kp1     = k + 1
        do j = kp1,nbus
          uzelem  = czero
          jm1     = j - 1
          alpha   = 1.0 / ybus(j,j)
          do l = k,jm1
            uzelem  = uzelem - ybus(j,l) * uz(l,k)
          end do
          uz(j,k) = uzelem * alpha
        end do
      end do

      write(iout,537)
  537 format(/1x,'nonzero uz follows'/)
      open(unit=8,file='uz.txt')
      nlu     = 0

      do j = 1,nbus
        do k = 1,nbus
          ymag    = cabs(uz(j,k))
          if(ymag .gt. eps9) then
            nlu     = nlu + 1
            write(iout,1005) j,k,nbext(j),nbext(k),uz(j,k)
            write(8,1005) j,k,nbext(j),nbext(k),uz(j,k)
          endif
        end do
      end do

      fill = nlu / float(nbus * nbus)
      write(iout,545) nlu,fill
  545 format(/1x,'number of nonzero elements in uz = ',i5/1x,
     1'percent fill = ',2pf8.2/)
      close(unit=8,status='keep')
c
c  form z
c
```

```
      open(unit=10,file='zbus_from_lu.txt')

      do kdum = 1,nbus
        k   = nbus - kdum + 1
        write(iout,550) k
  550 format(1x,'zbus column = ',i5)
        do jdum = 1,nbus
          j   = nbus - jdum + 1
          zbus(j,k) = czero
          if(j .ge. k) zbus(j,k) = uz(j,k)
          if(j .ne. nbus) then
            jp1 = j + 1
            do l    = jp1,nbus
              zbus(j,k) = zbus(j,k) - ybus(j,l) * zbus(l,k)
            end do
          end if
        end do
      end do

      write(iout,565)
  565 format(/1x,'writing zbus_from_lu.txt file')

      do j = 1,nbus
        do k = 1,nbus
          write(10,1005) j,k,nbext(j),nbext(k),zbus(j,k)
          write( 6,1005) j,k,nbext(j),nbext(k),zbus(j,k)
        end do
      end do

      close(unit=10,status='keep')
c
c  check ybus * zbus
c
      write(iout,5010)
 5010 format(/1x,'nonzero ybus * zbus follows')

      do j = 1,nbus
        do k = 1,nbus
          elem    = czero
          do l = 1,nbus
            elem    = elem + ysave(j,l) * zbus(l,k)
          end do
          ymag    = cabs(elem)
          if(ymag .gt. eps4) write(iout,1005) j,k,nbext(j),nbext(k),elem
        end do
      end do
c
c  copy ysave back into ybus, and zero zbus
c
      do j = 1,nbus
        unity(j,j) = 1.0
        do k = 1,nbus
          ybus(j,k) = ysave(j,k)
          zbus(j,k) = czero
        end do
      end do
c
c  gaussian eliminate ybus, while performing the same operations
c  on the unity matrix
c
```

```
      nm1 = nbus - 1
      do ipiv = 1,nm1
        write(iout,561) ipiv
  561 format(1x,'gaussian elimination ybus pivot = ',i5)
        alpha      = 1.0 / ybus(ipiv,ipiv)
        ipp1 = ipiv + 1
c
c  pivot row operations for ybus and unity
c
        do k = 1,nbus
          if(k .gt. ipiv) then
            ybus(ipiv,k)  = ybus(ipiv,k)  * alpha
            else
            unity(ipiv,k) = unity(ipiv,k) * alpha
          endif
        end do
c
c  pivot element
c
        ybus(ipiv,ipiv) = 1.0
c
c  kron reduction of ybus and unity below the pivot row and to
c  the right of the pivot column
c
        do j = ipp1,nbus
          alpha   = ybus(j,ipiv)
          do k = 1,nbus
            if(k .gt. ipiv)
     1        ybus(j,k)  = ybus(j,k)  - alpha * ybus(ipiv,k)
            if(k .lt. ipiv)
     1        unity(j,k) = unity(j,k) - alpha * unity(ipiv,k)
          end do
        end do
c
c  elements directly below the pivot element
c
        do j = ipp1,nbus
          alpha   = ybus(j,ipiv)
          ybus(j,ipiv)  = ybus(j,ipiv)  - alpha * ybus(ipiv,ipiv)
          unity(j,ipiv) = unity(j,ipiv) - alpha * unity(ipiv,ipiv)
        end do

      end do
c
c  last row
c
      write(iout,561) nbus
      alpha   = 1.0 / ybus(nbus,nbus)

      do k = 1,nbus
        unity(nbus,k) = unity(nbus,k) * alpha
      end do

      ybus(nbus,nbus) = 1.0
      ngy = 0
      ngu = 0
      write(iout,562)
  562 format(/1x,'writing gaussian eliminated ybus and unity to files')
      open(unit=12,file='ybus_gaussian_eliminated.txt')
      open(unit=13,file='unity_mat.txt')
```

```
      do j = 1,nbus
        do k = 1,nbus
          ymag = cabs(ybus(j,k))
          if(ymag .ge. eps9) then
            write(12,1005) j,k,nbext(j),nbext(k),ybus(j,k)
            ngy   = ngy + 1
          endif
          umag = cabs(unity(j,k))
          if(umag .ge. eps9) then
            write(13,1005) j,k,nbext(j),nbext(k),unity(j,k)
            ngu   = ngu + 1
          endif
        end do
      end do

      close(unit=12,status='keep')
      close(unit=13,status='keep')
      fill = ngy / float(nbus * nbus)
      write(iout,555) ngy,fill
  555 format(/1x,'number of nonzero elements in gaussian eliminated',
     1' ybus = ',i5/1x,
     1'percent fill = ',2pf8.2/)
      fill = ngu / float(nbus * nbus)
      write(iout,556) ngu,fill
  556 format(/1x,'number of nonzero elements in gaussian eliminated',
     1' unity matrix = ',i5/1x,
     1'percent fill = ',2pf8.2/)
c
c  back substitute to find z
c
      do k = 1,nbus
        zbus(nbus,k) = unity(nbus,k)
      end do

      do jdum = 2,nbus
        j = nbus - jdum + 1
        write(iout,550) j
        do kdum = 1,nbus
          k = nbus - kdum + 1
          zbus(j,k) = unity(j,k)
          jp1 = j + 1
          do l = jp1,nbus
            zbus(j,k) = zbus(j,k) - ybus(j,l) * zbus(l,k)
          end do
        end do
      end do

      write(iout,566)
  566 format(/1x,'writing zbus_from_gaussian_elim.txt file')
      open(unit=11,file='zbus_from_gaussian_elim.txt')

      do j = 1,nbus
        do k = 1,nbus
          write(11,1005) j,k,nbext(j),nbext(k),zbus(j,k)
          write( 6,1005) j,k,nbext(j),nbext(k),zbus(j,k)
        end do
      end do

      close(unit=11,status='keep')
```

```
c
c  check ybus * zbus
c
      write(iout,5010)

      do j = 1,nbus
        do k = 1,nbus
          elem    = czero
          do l = 1,nbus
            elem    = elem + ysave(j,l) * zbus(l,k)
          end do
          ymag    = cabs(elem)
          if(ymag .gt. eps4) write(iout,1005) j,k,nbext(j),nbext(k),elem
        end do
      end do

      stop
   30 write(iout,*) 'error in reading bdat.csv file - stop'
      write(*,*)    'error in reading bdat.csv file - stop'
      stop
   31 write(iout,*) 'error in reading ldat.csv file - stop'
      write(*,*)    'error in reading ldat.csv file - stop'
      stop

      end
```

Fortran program yzbuild.f (and executable yzbuild.exe) reads bus data file bdat.csv, and line/transformer data file ldat.csv, and then produces the corresponding impedance matrix using LU decomposition, and again using Gaussian elimination. The program is dimensioned for systems as large as 150 busses.

Comma-separated file bdat.csv contains the bus data, using the format: bus number , shunt q (in pu, where capacitance positive, and inductance is negative)

Comma-separated file ldat.csv contains the branch data, using the format: from bus number , to bus number , r(pu) , x(pu) , line charging (in pu, for the total length of line)

It is OK to have ties to ground in ldat.csv. Ground is bus = 0.

Two sample systems are included. One has three busses, and the other has eighty-three busses. To try out yzbuild, simply copy and rename the data files for either system as bdat.csv and ldat.csv, and then click on yzbuild.exe.

The data below are given in per unit. R and X are the series impedances of the transmission lines (in per unit ohms), and B is the line charging of the transmission lines (in per unit mhos).

Build the Y matrix. Show the complex terms individually. Do not perform any complex arithmetic such as inversions, and do not combine the terms.

From Bus	To Bus	R – pu ohms	jX – pu ohms	jB – pu mhos
1	2	0.004	j0.053	j0
2	3	0.02	j0.25	j0.22
3	4	0.02	j0.25	j0.22
2	4	0.01	j0.15	j0.11

EE369, Test 6, Oct. 8, 2004. Name: KEY

One sheet of notes permitted. Show all your work.

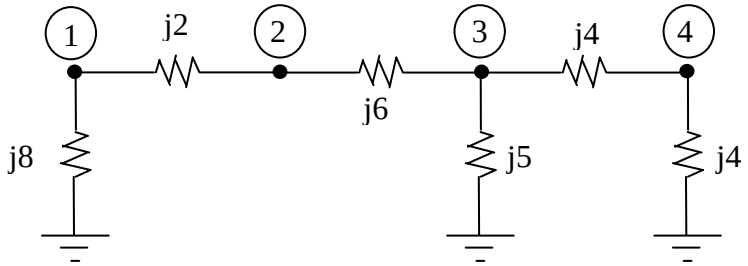
The data below are given in per unit. R and X are the series impedances of the transmission lines (in per unit ohms), and B is the line charging of the transmission lines (in per unit mhos).

Build the Y matrix. Show the complex terms individually. Do not perform any complex arithmetic such as inversions, and do not combine the terms.

From Bus	To Bus	R – pu ohms	jX – pu ohms	jB – pu mhos
1	2	0.004	j0.053	j0
2	3	0.02	j0.25	j0.22
3	4	0.02	j0.25	j0.22
2	4	0.01	j0.15	j0.11

	1	2	3	4
1	$\frac{1}{0.004+j0.053}$	$\frac{-1}{0.004+j0.053}$		
2	$\frac{-1}{0.004+j0.053}$	$\frac{1}{0.004+j0.053}+\frac{1}{0.02+j0.25}+j\frac{0.22}{2}+\frac{1}{0.01+j0.15}+j\frac{0.11}{2}$	$\frac{-1}{0.02+j0.25}$	$\frac{-1}{0.01+j0.15}$
3		$\frac{-1}{0.02+j0.25}$	$\frac{1}{0.02+j0.25}+j\frac{0.22}{2}+\frac{1}{0.02+j0.25}+j\frac{0.22}{2}$	$\frac{-1}{0.02+j0.25}$
4		$\frac{-1}{0.01+j0.15}$	$\frac{-1}{0.02+j0.25}$	$\frac{1}{0.02+j0.25}+j\frac{0.22}{2}+\frac{1}{0.01+j0.15}+j\frac{0.11}{2}$

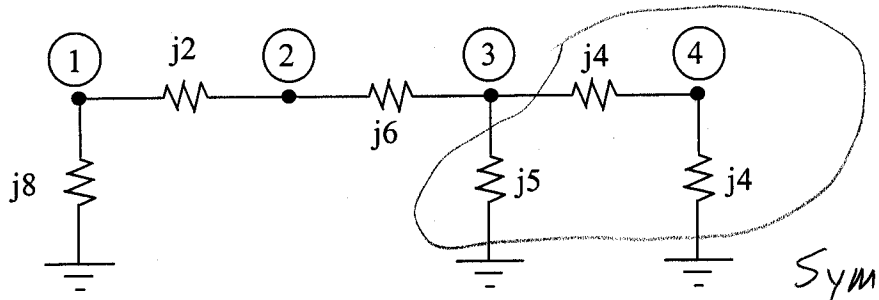
Use the definition of Z-matrix elements to find matrix elements z_{11}, z_{12}, z_{13}, z_{14}.



EE369, Test 7, Oct. 15, 2004. Name (printed): KEY

Clearly show all work needed to obtain your answer.

Use the definition of Z-matrix elements to find matrix elements z_{11}, z_{12}, z_{13}, z_{14}.



$j5 \| j8 = j\left[\frac{40}{13}\right] = j3.08$

Symmetry $z_{ij} = z_{ji}$ (except where phase shifting transformers exist)

$$z_{ij} = \frac{\partial V_i}{\partial I_j} = \frac{V_i}{I_j}\Big|_{I_k=0,\ k\neq j}$$

$$z_{11} = j8 \| (j2 + j6 + j3.08) = j8 \| j11.08 = \underline{j4.65}$$

$$z_{21} = z_{11}\left(\frac{V_2}{V_1}\right) = j4.65\left(\frac{6+3.08}{2+6+3.08}\right) = \underline{j3.81} = z_{12}$$

$$z_{31} = z_{11}\left(\frac{V_3}{V_1}\right) = j4.65\left(\frac{3.08}{2+6+3.08}\right) = \underline{j1.293} = z_{13}$$

$$z_{41} = z_{31}\left(\frac{V_4}{V_3}\right) = j1.293\left(\frac{4}{4+4}\right) = \underline{j0.647} = z_{14}$$

Voltage Divider

Check z_{14},

$$z_{44} = j4 \| (j4 + j5 \| (j6 + j2 + j8)) = j4 \| (j4 + j5 \| j16)$$
$$= j4 \| (j4 + j3.81) = j4 \| j7.81 = j2.65$$

$$z_{34} = z_{44}\left(\frac{V_3}{V_4}\right) = j2.65\left(\frac{5\|16}{5\|16+4}\right) = j2.65\left(\frac{3.81}{3.81+4}\right) = j1.293$$

$$z_{14} = z_{34}\left(\frac{V_1}{V_3}\right) = j1.293\left(\frac{8}{8+2+6}\right) = j0.647$$

checks

Programming Considerations

Methods for efficient programming. Sparsity techniques. Optimal bus ordering.

1. Sparsity Programming

Many power system matrices tend to be very large and very sparse. A typical example is the admittance matrix *Y*. Each diagonal element of *Y* is non-zero, and each branch connected between busses occurs twice in *Y*, so that a branch between busses *j* and *k* appears in $y_{j,k}$ and $y_{k,j}$. Additional parallel branches and shunt elements do not create additional *y* elements.

Consider a power system with 250 busses and 750 branches. Assume that none of the branches are parallel branches. The dimension of *Y* is 250 x 250, or 62,500 elements. However, the number of non-zero terms in *Y* equals 250 + 2(750), or 1750. Therefore, only 2.8% of the *Y* elements are non-zero. The savings are even more impressive for larger systems, since Y grows by N^2, but the number of non-zero terms grows by *N*. For example, a 2500 bus, 7500 branch Y has only 0.28% non-zero terms.

In most cases, the coding of large computer programs involves trade-offs between execution time and memory requirements. Sparsity storage techniques, however, reduce both execution time *and* memory requirements. The only drawback of sparsity programming is that it is more difficult to code than full-matrix programming.

There are several techniques for sparsity programming. The method presented here is known as the sequential trace method, and although the example is given for *Y*, it applies equally well to the loadflow Jacobian matrix.

Four vectors are used for storing and accessing the sparse *Y*:

1. Instead of storing *Y* as a square matrix of dimension *N* x *N*, *Y* is stored as a column vector, whose length *M* is estimated (at the time of program compilation) using typical sparsity ratios of 1% - 5% of *N* x *N*. The location from the top of *Y* is known as the stack location.

2. Vector *NFIRST*, of length *N*, points to the stack location in *Y* of the first non-zero element of a row (in column order). For example, the stack position of the first non-zero element in Row *k* is found in *NFIRST(k)*.

3. Vector *NCOL*, of length *M*, gives the *Y* column number of the associated *Y* stack position.

4. Vector *NEXT*, of length *M*, gives the stack location in *Y* of the next non-zero element in the corresponding row. A zero value indicates the end of the row.

The term "sequential trace" is derived from the fact that all elements of a *Y* row can be found, in a column-order sequence, by starting with the lowest occupied column number of the row, and continuing through the highest occupied column number of the row.

As an example, let Row 23 of Y have three elements: $y_{23,5}$ in stack location 987, $y_{23,23}$ in stack location 560, and $y_{23,138}$ in stack location 1230. These elements can be found as follows:

1. The stack location of the first non-zero element (in column order) of Row 23 is found by looking in *NFIRST*(23). In this example, *NFIRST*(23) equals 987.

2. The corresponding column number is found in *NCOL*(987), which equals 5. Thus, the equivalent *Y*(23,5) element is found in stack position *Y*(987).

3. The stack position of the next non-zero *Y* element in Row 23 is found by looking at *NEXT*(987). The value of *NEXT*(987) is 560.

4. The corresponding column number is found in *NCOL*(560), which equals 23. Thus, the equivalent *Y*(23,23) element is found in stack position *Y*(560).

5. The stack position of the next non-zero *Y* element in Row 23 is found by looking at *NEXT*(560). The value of *NEXT*(560) is 1230.

6. The corresponding column number is found in *NCOL*(1230), which equals 138. Thus, the equivalent *Y*(23,138) element is found in stack position *Y*(1230).

7. A zero in *NEXT*(1230) indicates that the end Row 23 has been reached.

Table 1 gives the corresponding numbers for this example.

Table 1: Example of Sequential Trace Method for Sparsity Storage, where the First Non-Zero Column for Row 23 is found in *NFIRST*(23) = 987

Stack Location	NCOL	NEXT	*Y*
⋮	⋮	⋮	⋮
560	23	1230	$y_{23,23}$ value
⋮	⋮	⋮	⋮
987	5	560	$y_{23,5}$ value
⋮	⋮	⋮	⋮
1230	138	0	$y_{23,138}$ value
⋮	⋮	⋮	⋮

When *Y* is first built, the elements for each row are contiguous. However, as *Y* is Gaussian eliminated or LU decomposed, "fill-ins" are created, and they are normally added to the end of the *Y* stack. The four vectors must be modified appropriately, and the elements of a row are no longer contiguous.

Note - if *Y* is to be Gaussian eliminated, this can be performed simultaneously while *Y* is being formed in row order, thereby never saving the LLT part of the eliminated *Y* matrix. For example, Row 1 is formed and saved. Next, Row 2 is formed, then Gaussian eliminated by Row

1, and then saved. Next, Row 3 is formed, eliminated by Row 1, then by Row 2, and then saved, etc.

2. Optimal Bus Ordering

Gaussian elimination and LU decomposition reduce matrices through a series of row operations. Once the decision has been made to use sparsity programming, then it is very important in power system applications to order the busses efficiently so that the advantages of sparsity programming are not nullified by the creation of too many "fill-ins" (i.e. zero elements that become non-zeros).

Consider the simple five-bus system given figuratively in Figure 1.

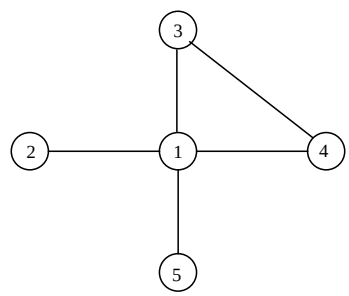


Figure 1. Simple Five-Bus System

The form of Y, for the given bus order, is

$$\begin{bmatrix} x & x & x & x & x \\ x & x & & & \\ x & & x & x & \\ x & & x & x & \\ x & & & & x \end{bmatrix},$$

and the final form of the Gaussian eliminated Y is a full (URT+D) matrix, or

$$\begin{bmatrix} x & x & x & x & x \\ & x & \otimes & \otimes & \otimes \\ & & x & x & \otimes \\ & & & x & \otimes \\ & & & & x \end{bmatrix},$$

where the $\otimes$ terms are "fill-ins."

However, if the bus numbers for busses 1 and 5 are interchanged, then the initial Y has the form

$$\begin{bmatrix} x & & & & x \\ & x & & & x \\ & & x & x & x \\ & & x & x & x \\ x & x & x & x & x \end{bmatrix},$$

and the final form of the Gaussian eliminated Y is

$$\begin{bmatrix} x & & & & x \\ & x & & & x \\ & & x & x & x \\ & & & x & x \\ & & & & x \end{bmatrix},$$

which has no fill-ins.

While this is a simple example, the importance of bus ordering cannot be overstated. Of course, an efficient computer code does not require the user to order the busses ahead of time. Rather, the user simply inputs his/her desired bus numbers, in no particular order, and the program develops an "internal" (to the program) bus order. The following three methods are often used:

Method 1: Before Gaussian elimination begins, order the busses according to the number of connected branches, ignoring parallel branches and shunt elements, so that the busses with the fewest branches are first, and the busses with the most branches are last.

Method 2: Order the busses so that after each step of Gaussian elimination, the next bus in the order is the one that has the fewest branches, including intermediate fill-in branches.

Method 3: Order the busses so that after each step of Gaussian elimination, the next bus in the order is the one that creates the fewest fill-ins. This requires a simulated row-reduction for all candidate busses.

Method 1 is the simplest and provides the highest incremental benefit. Method 2 usually provides the best overall speed and memory benefits. Method 3 is more time-consuming to compute, and the time and storage savings gained by Method 3 usually do not offset its additional ordering computation time.

The impact of optimal ordering on the Gaussian elimination of Y for six sample systems is shown in Table 2.

Table 2: Impact of Optimal Bus Ordering Methods on Gaussian Elimination of *Y*

System	Elements in in *Y*	---------- Elements in Gaussian-Eliminated *Y* ---------- User Bus Order	Method 1	Method 2	Method 3
18 Bus	52	66	41	35	35
83 Bus	303	592	292	264	267
125 Bus	421	1,357	614	419	422
264 Bus	976	5,736	1,204	1,005	1,004
515 Bus	1,761	14,142	1,992	1,763	1,751
2245 Bus	8,257	94,468	25,775	9,473	9,218*

* Estimated using similar results for loadflow Jacobian matrix

The impact of optimal bus ordering on the Jacobian matrix, before and after Gaussian elimination, is shown in Figures 2a, 2b, and 2c, which correspond to optimal ordering methods 1, 2, and 3, respectively.

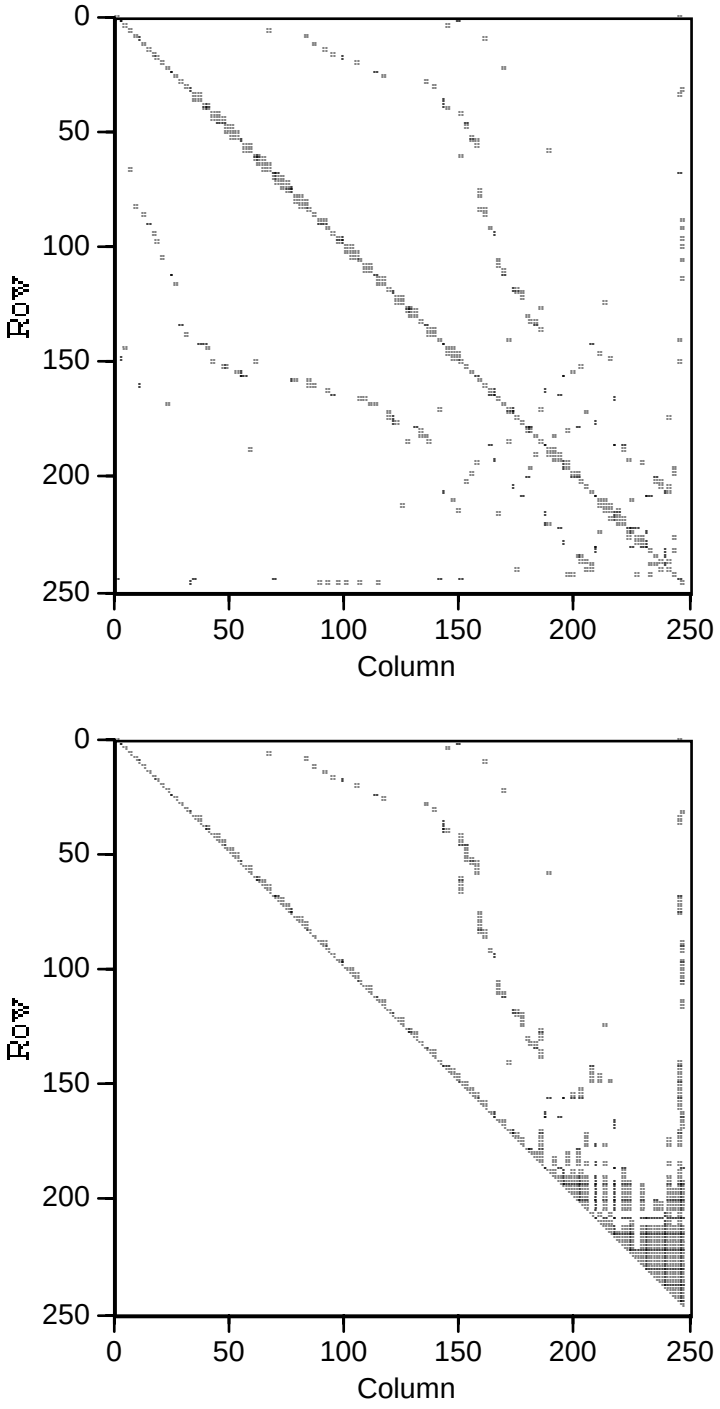


Figure 2a. Jacobian Matrix for 125 Bus System, Before and After Gaussian Elimination, Using Optimal Ordering Method 1

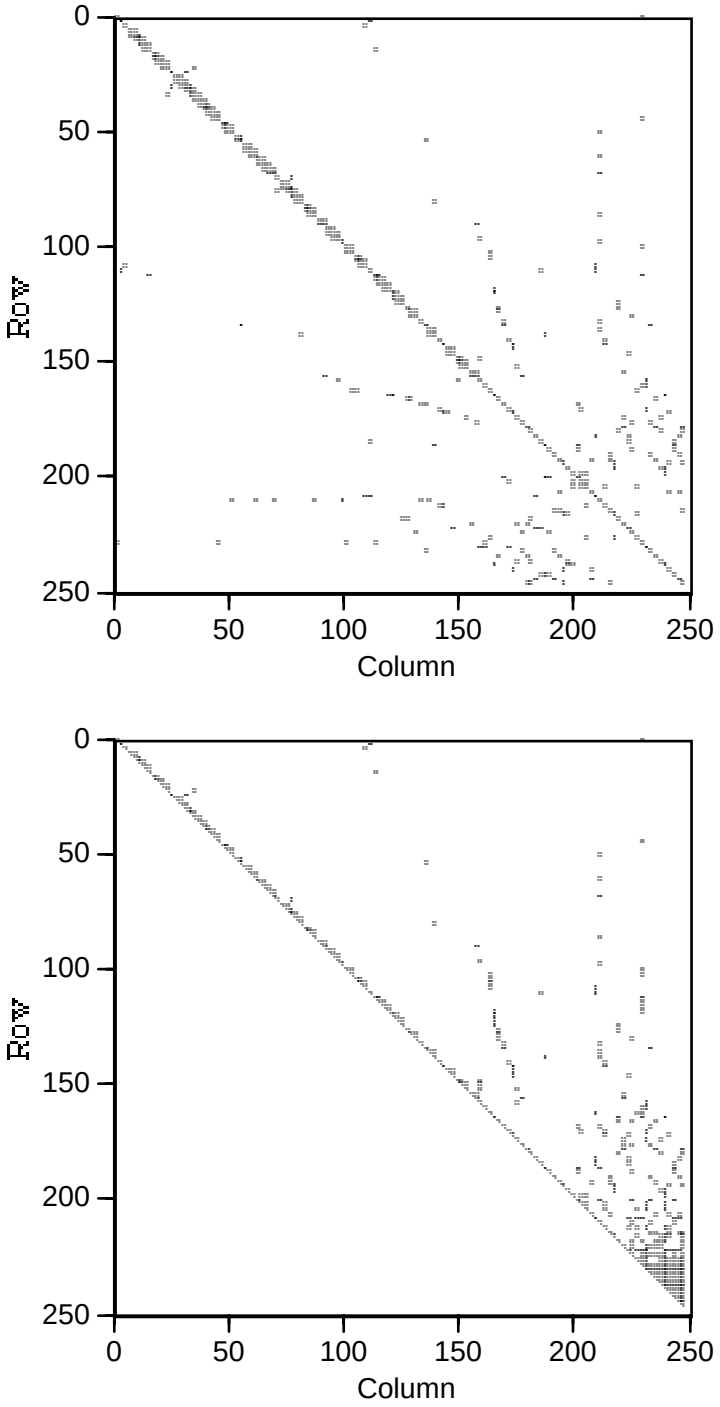


Figure 2b. Jacobian Matrix for 125 Bus System, Before and After Gaussian Elimination, Using Optimal Ordering Method 2

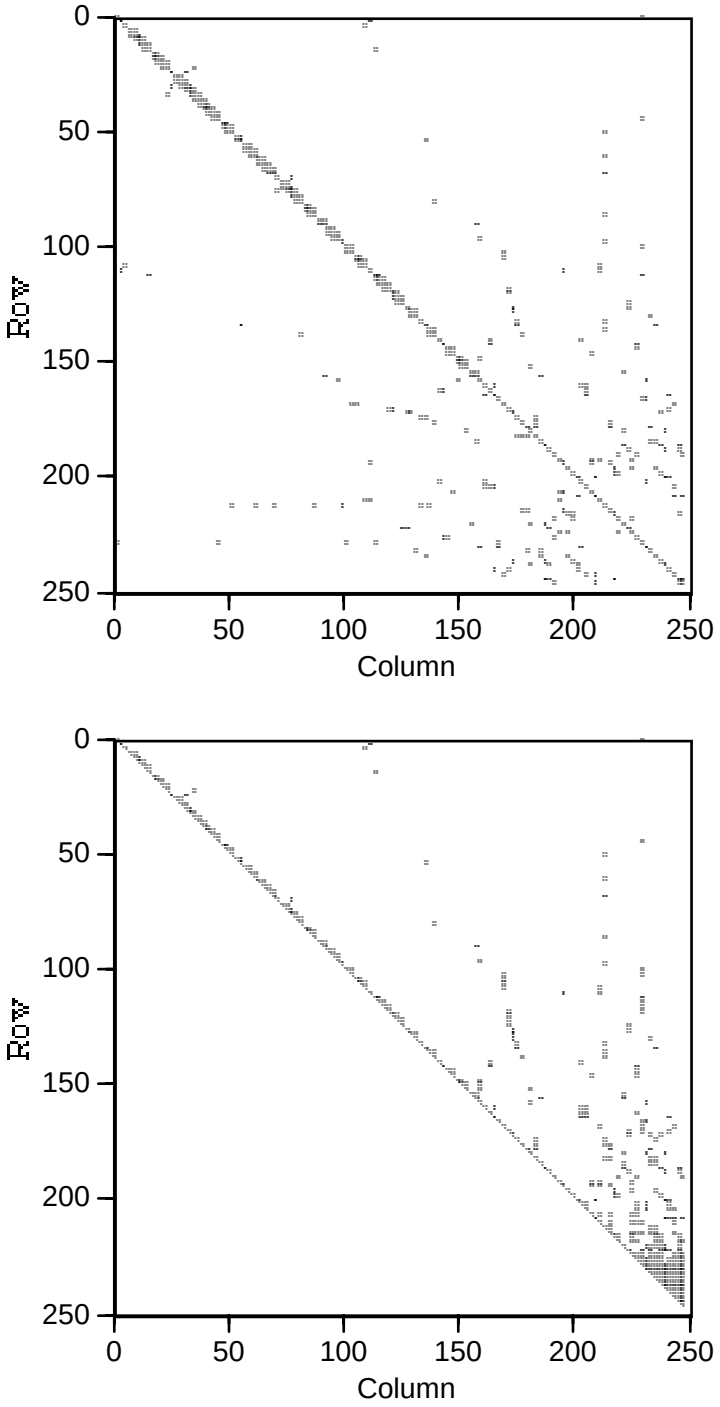


Figure 2c. Jacobian Matrix for 125 Bus System, Before and After Gaussian Elimination, Using Optimal Ordering Method 3

3. Example Code BORDER for Performing Optimal Ordering Methods 1, 2, and 3

```
c
c  Program BORDER optimally orders busses according to methods
c  1, 2, and 3.  Reads formatted BDAT.BOR and LDAT.BOR files in
c  PCFLO fixed column format.  Results placed in files border.1,
c  border.2, and border.3.
c
c  Mack Grady.  See date below in variable versio.
c
      character*6 versio/'981025'/
      logical ybus,taken,ltrue,lfalse,ysave
      dimension ybus(100,100),nbext(100),nbint(9999),nbran(100),
     1 taken(100),ysave(100,100),jbus(100)
      data mxint/100/,mxext/9999/,nbus/0/, nbint/9999 * 0/,
     1 nbext/100 * 0/,eps6/1.0e-06/
      data ybus  /10000 * .false./,ltrue/.true./,lfalse/.false./
      data iout/15/
      pi = 4.0 * atan(1.0)
      dr = 180.0 / pi
      open(unit=1,file='bdat.bor')
      open(unit=iout,file='exlog.bor')
      write(6,*) 'Program BORDER, Version ',versio
      write(6,*) 'Read bus data from PCFLO fixed column format ',
     1 'file BDAT.BOR'
      write(6,*) 'number        q shunt'
      write(iout,*) 'Program BORDER, Version ',versio
      write(iout,*) 'Read bus data from PCFLO fixed column format ',
     1 'file BDAT.BOR'
      write(iout,*) 'number        q shunt'
c
c  read the bdat file
c
    1 read(1,1001,end=5) nb,qshunt
 1001 format(i4,t58,f8.0)
      if(nb .eq. 0) go to 5
      write(6,1002) nb,qshunt
      write(iout,1002) nb,qshunt
 1002 format(i7,5x,f10.2)
      if(nb .le. 0 .or. nb .gt. mxext) then
        write(6,*) 'Illegal bus number - stop'
        write(iout,*) 'Illegal bus number - stop'
        stop
      endif
      nbus = nbus + 1
      if(nbus .gt. mxint) then
        write(6,*) 'Too many busses - stop'
        write(iout,*) 'Too many busses - stop'
        stop
      endif
      nbext(nbus) = nb
      nbint(nb)   = nbus
      ybus(nbus,nbus) = ltrue
      go to 1
    5 close(unit=1,status='keep')
c
      open(unit=1,file='ldat.bor')
      write(6,*) 'Read line/transformer data from PCFLO fixed column ',
     1 'format file LDAT.BOR'
```

```
      write(6,*) ' from     to           r           x',
     1 '      charge         tap  phase shift'
      write(iout,*) 'Read line/transformer data from PCFLO fixed ',
      1 'column format file LDAT.BOR'
      write(iout,*) ' from     to           r           x',
     1 '      charge         tap  phase shift'
c
c  read the ldat file
c
   10 read(1,1003,end=15) nfrom,nto,r,x,charge,tap,phase
 1003 format(2i4,2x,3f12.0,t76,f7.0,f8.0)
      if(nfrom .eq. 0 .and. nto .eq. 0) go to 15
      write(6,1004) nfrom,nto,r,x,charge,tap,phase
      write(iout,1004) nfrom,nto,r,x,charge,tap,phase
 1004 format(1x,i5,2x,i5,4(2x,f10.4),3x,f10.2)
      if(nfrom .lt. 0 .or. nfrom .gt. mxext .or. nto .lt. 0
     1 .or. nto .gt. mxext) then
        write(6,*) 'illegal bus number - stop'
        write(iout,*) 'illegal bus number - stop'
        stop
      endif
      if(nfrom .eq. nto) then
        write(6,*) 'same bus number given on both ends - stop'
        write(iout,*) 'same bus number given on both ends - stop'
        stop
      endif
      if(r .lt. -eps6 .or. tap .lt. -eps6) then
        write(6,*) 'illegal resistance or tap - stop'
        write(iout,*) 'illegal resistance or tap - stop'
        stop
      endif
      if(charge .lt. -eps6) then
        write(6,*) 'line charging should be positive'
        write(iout,*) 'line charging should be positive'
        stop
      endif
      if(nfrom .ne. 0) then
        nfrom = nbint(nfrom)
        if(nfrom .eq. 0) then
          write(6,*) 'bus not included in file bdat - stop'
          write(iout,*) 'bus not included in file bdat - stop'
          stop
        endif
      endif
      if(nto   .ne. 0) then
        nto   = nbint(nto)
        if(nto .eq. 0) then
          write(6,*) 'bus not included in file bdat - stop'
          write(iout,*) 'bus not included in file bdat - stop'
          stop
        endif
      endif
      phase = phase / dr
      if(abs(phase) .gt. eps6 .and. abs(tap) .lt. eps6) tap = 1.0
      if((nfrom .eq. 0 .or. nto .eq. 0) .and. tap .gt. eps6) then
        write(6,*) 'shunt elements should not have taps - stop'
        write(iout,*) 'shunt elements should not have taps - stop'
        stop
      endif
      if(tap .gt. eps6 .and. charge .gt. eps6) then
```

```
        write(6,*) 'transformers should not have line charging'
        write(iout,*) 'transformers should not have line charging'
        stop
      endif
c
c  the line charging terms
c
      if(nfrom .ne. 0) ybus(nfrom,nfrom) = ltrue
      if(nto   .ne. 0) ybus(nto  ,nto  ) = ltrue
c
c  shunt elements
c
      if(nfrom .ne. 0 .and. nto   .eq. 0) ybus(nfrom,nfrom) = ltrue
      if(nfrom .eq. 0 .and. nto   .ne. 0) ybus(nto  ,nto  ) = ltrue
c
c  transmission lines
c
      if(nfrom .ne. 0 .and. nto .ne. 0 .and. tap .lt. eps6) then
        ybus(nfrom,nto  ) = ltrue
        ybus(nto  ,nfrom) = ltrue
        ybus(nfrom,nfrom) = ltrue
        ybus(nto  ,nto  ) = ltrue
      endif
c
c  transformers
c
      if(nfrom .ne. 0 .and. nto .ne. 0 .and. tap .ge. eps6) then
        ybus(nfrom,nfrom) = ltrue
        ybus(nto  ,nto  ) = ltrue
        ybus(nfrom,nto  ) = ltrue
        ybus(nto  ,nfrom) = ltrue
      endif
      go to 10
   15 close(unit=1,status='keep')
c
c  count the number of nonzero terms in each row
c
      nterms    = 0
      do 20 j   = 1,nbus
      nbran(j)  = 0
      taken(j)  = lfalse
      do 20 k   = 1,nbus
      ysave(j,k) = ybus(j,k)
      if(ybus(j,k)) then
        nbran(j) = nbran(j) + 1
        nterms   = nterms   + 1
      endif
   20 continue
      write(6,1111) nterms
      write(iout,1111) nterms
c
c  simulate the gaussian elimination using file=bdat order
c
      ntfill   = nterms
      do 96 j  = 1,nbus
      taken(j) = ltrue
      do 97 k  = 1,nbus
      if(taken(k) .or. .not.ybus(k,j)) go to 97
      do 98 l  = 1,nbus
      if(ybus(j,l) .and. .not.ybus(k,l)) then
```

```
        ybus(k,l) = ltrue
        ntfill    = ntfill + 1
      endif
   98 continue
      ybus(k,j)   = lfalse
      ntfill      = ntfill - 1
   97 continue
   96 continue
      write(6,1113) ntfill
      write(iout,1113) ntfill
 1113 format(1x,'Total reduced YBUS terms for BDAT.BOR bus order = ',
     1 i5/)
c
      do 99 j  = 1,nbus
      taken(j) = lfalse
      do 99 k  = 1,nbus
      ybus(j,k) = ysave(j,k)
   99 continue
c
c  method 1
c
      open(unit=1,file='border.1')
      norder    = 1
      nfound    = 0
      nties     = 2
      write(6,1111) nterms
      write(iout,1111) nterms
 1111 format(/1x,'YBUS terms before reduction = ',i8/)
 1105 format(1x,'Method = ',i5,', nties = ',i5,
     1  ', found = ',i5)
   30 do 25 j   = 1,nbus
      if(taken(j) .or. nbran(j) .ne. nties) go to 25
      taken(j)  = ltrue
      nfound    = nfound + 1
      jbus(nfound) = j
      write(1,1100) norder,nfound,nbext(j)
      write(6,1100) norder,nfound,nbext(j)
      write(iout,1100) norder,nfound,nbext(j)
 1100 format(1x,'Method = ',i5,', int. bus = ',i5,',  ext. bus = ',i5)
   25 continue
      write(6,1105) norder,nties,nfound
      write(iout,1105) norder,nties,nfound
      nties     = nties + 1
      if(nfound .ne. nbus) go to 30
      close(unit=1,status='keep')
c
c  simulate the gaussian elimination for method 1 order
c
      do 35 j  = 1,nbus
      taken(j) = lfalse
   35 continue
      ntfill   = nterms
      do 36 jj = 1,nbus
      j        = jbus(jj)
      taken(j) = ltrue
      do 37 k  = 1,nbus
      if(taken(k) .or. .not.ybus(k,j)) go to 37
      do 38 l  = 1,nbus
      if(ybus(j,l) .and. .not.ybus(k,l)) then
        ybus(k,l) = ltrue
```

```
        ntfill    = ntfill + 1
      endif
   38 continue
      ybus(k,j)   = lfalse
      ntfill      = ntfill - 1
   37 continue
   36 continue
      write(6,1122) norder,ntfill
      write(iout,1122) norder,ntfill
 1122 format(/1x,'Total reduced YBUS terms for method = ',i5,
     1' equals ',i5/)
c
c  method 2
c
      open(unit=1,file='border.2')
      norder    = 2
      do 40 j   = 1,nbus
      nbran(j)  = 0
      taken(j)  = lfalse
      do 40 k   = 1,nbus
      ybus(j,k) = ysave(j,k)
      if(ybus(j,k)) nbran(j) = nbran(j) + 1
   40 continue
      nfound    = 0
      ntfill    = nterms
      nties     = 2
      write(6,1111) nterms
      write(iout,1111) nterms
 1108 format(1x,'Method = ',i5,', nties = ',i5,
     1  ', found = ',i5,', tot. terms = ',i5)
   50 do 45 j   = 1,nbus
      if(taken(j) .or. nbran(j) .ne. nties) go to 45
      taken(j)  = ltrue
      nfound    = nfound + 1
      write(1,1100) norder,nfound,nbext(j)
      write(6,1100) norder,nfound,nbext(j)
      write(iout,1100) norder,nfound,nbext(j)
c
c  gaussian eliminate all the remaining rows with row j
c
      do 60 k   = 1,nbus
      if(k .eq. j .or. taken(k)) go to 60
      if(ybus(k,j)) then
        do 65 l   = 1,nbus
        if(l .eq. j) go to 65
        if(ybus(j,l) .and. .not.ybus(k,l)) then
          ybus(k,l) = ltrue
          nbran(k)  = nbran(k) + 1
          ntfill    = ntfill   + 1
        endif
   65   continue
c
c  for the term under the pivot
c
        ybus(k,j) = lfalse
        nbran(k)  = nbran(k) - 1
        ntfill    = ntfill   - 1
      endif
   60 continue
      write(6,1108) norder,nties,nfound,ntfill
```

```
      write(iout,1108) norder,nties,nfound,ntfill
      if(nfound .eq. nbus) go to 59
      nties     = 2
      if(nfound .eq. (nbus - 1)) nties = 1
      go to 50
   45 continue
      nties = nties + 1
      go to 50
   59 close(unit=1,status='keep')
      write(6,1122) norder,ntfill
      write(iout,1122) norder,ntfill
c
c  method 3
c
      open(unit=1,file='border.3')
      norder    = 3
      do 160 j  = 1,nbus
      taken(j)  = lfalse
      do 160 k  = 1,nbus
      ybus(j,k) = ysave(j,k)
  160 continue
      nfound    = 0
      nfills    = 0
      ntfill    = nterms
      write(6,1111) nterms
      write(iout,1111) nterms
  170 write(6,1107) norder,nfills,nfound,ntfill
      write(iout,1107) norder,nfills,nfound,ntfill
 1107 format(1x,'Method = ',i5,', nfills = ',i5,
     1  ', found = ',i5,', tot. terms = ',i5)
c
c  simulated gaussian elimination using the remaining pivot candidates
c
      mfirst    = 0
      nxtbus    = 0
      do 165 m  = 1,nbus
      if(taken(m)) go to 165
      lstbus    = m
      nfills    = 0
      do 180 k  = 1,nbus
      if(k .eq. m .or. taken(k)) go to 180
      if(ybus(k,m)) then
        do 185 l  = 1,nbus
        if(l .eq. m) go to 185
        if(ybus(m,l) .and. .not.ybus(k,l)) nfills = nfills + 1
  185   continue
c
c  for the term under the pivot
c
        nfills  = nfills - 1
      endif
  180 continue
        if(mfirst .eq. 0) then
          mfirst = 1
          nxtbus = m
          minfil = nfills
        endif
        if(nfills .lt. minfil) then
          nxtbus = m
          minfil = nfills
```

```
        endif
  165 continue
      if(nxtbus .eq. 0) then
        if(nfound .ne. (nbus - 1)) then
          write(6,1120)
          write(iout,1120)
 1120     format(/1x,'Error in method 3 ordering - stop')
          stop
        endif
        nxtbus = lstbus
        minfil = 0
      endif
      j      = nxtbus
      nfills = minfil
      taken(j)  = ltrue
      nfound    = nfound + 1
      write(1,1100) norder,nfound,nbext(j)
      write(6,1100) norder,nfound,nbext(j)
      write(iout,1100) norder,nfound,nbext(j)
      if(nfound .eq. nbus) then
        write(6,1107) norder,nfills,nfound,ntfill
        write(iout,1107) norder,nfills,nfound,ntfill
        go to 175
      endif
c
c  gaussian eliminate all the remaining rows with row j
c
      do 290 k   = 1,nbus
      if(k .eq. j .or. taken(k)) go to 290
      if(ybus(k,j)) then
        do 295 l   = 1,nbus
        if(l .eq. j) go to 295
        if(ybus(j,l) .and. .not.ybus(k,l)) then
          ybus(k,l) = ltrue
          nbran(k)  = nbran(k) + 1
          ntfill    = ntfill   + 1
        endif
  295   continue
c
c  for the term under the pivot
c
        ybus(k,j) = lfalse
        nbran(k)  = nbran(k) - 1
        ntfill    = ntfill   - 1
      endif
  290 continue
      go to 170
  175 close(unit=1,status='keep')
      write(6,1122) norder,ntfill
      write(iout,1122) norder,ntfill
      stop
      end
```

Motor Starting

Steady-state analysis of the motor starting problem.

1. Characteristics of Motors During Start-Up

During start-up, motors behave like fixed impedances that, at rated voltage, draw approximately five-times their full-load running current, and at a very low power factor. If the motor terminal voltage drops to less than 80% during start-up, most motors will not start.

2. Determination of Motor Start-Up Voltage

The equivalent circuit for calculating motor starting voltages is shown in Figure 1.

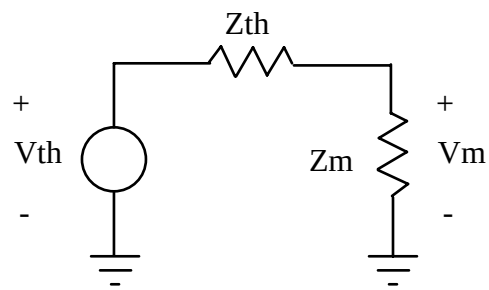


Figure 1. Motor Starting Equivalent Circuit

This Thevenin equivalent circuit has open-circuit voltage V_{TH} = 1.0 pu, and Thevenin impedance Z_{TH} , which is the "short circuit," positive-sequence impedance of the system as "seen" at the motor terminals. Z_{TH} is the motor-bus diagonal element of the positive-sequence impedance matrix, which is computed with all other large machines represented by their respective subtransient impedances.

The motor is represented at start-up as fixed impedance $Z_M = R_M + jX_M$, where R_M and X_M are calculated to give approximately five-times full-load running current @ 1.0 pu voltage, and at a startup-up power factor of approximately 0.20.

Given the circuit parameters, the motor start-up voltage is then computed from the voltage divider equation, or

$$V_M = V_{TH} \left| \frac{Z_M}{Z_{TH} + Z_M} \right| .$$

V_M should not be less than 0.80 pu. if the motor is start properly.

This problem can be worked "in reverse," where the objective is to determine how large a motor will start at a bus. A FORTRAN code for this purpose is shown below.

c

```
c  program motors.f reads zbus1 and vsoln files, and then computes
c  the kw motor that can be started at each bus without dragging the
c  voltage below vmin (in percent).  the results are written to file
c  mstart.
c
      complex zzz,zmotor
      character*12 anames,an
      dimension anames(9999),hpmin(9999)
      data anames /9999 * ' '/, hpmin /9999 * -1.0/
c
c  abase = base current,
c  vbase = base voltage - volts (three-phase),
c  pbase = base power - va (three-phase),
c  arun  = running amps per kw,
c  spf   = motor power factor when starting,
c  rpf   = motor power factor when running,
c  start = multiplier of run current when starting,
c  vmin  = minimum acceptable starting voltage - pu.
c
      data pbase/100000.0/,vbase/450.0/,spf/0.2/,start/5.0/
      data vmin/0.8/,rpf/0.85/eps/1.0e-03/
c
      abase = pbase / 3.0 / vbase * sqrt(3.0)
      arun  = 1000.0 / 3.0 / rpf / vbase * sqrt(3.0)
      write(6,5007) abase,arun
 5007 format(1x,'base current = ',f10.2/1x,'running current ',
     1'for 1 kw = ',f10.2)
c
c  zmotor = motor starting impedance for a 1 kw motor
c
      zmotor = abase / arun / start *
     1 cmplx(spf,sqrt(1.0 - spf * spf))
      write(6,5006) zmotor
 5006 format(1x,'z start for a 1 kw motor = ',2e15.6)
      rm = real(zmotor)
      xm = aimag(zmotor)
      open(unit=1,file='vsoln')
      open(unit=2,file='zbus1')
      open(unit=3,file='mstart')
    1 read(1,5000,end=2)  kbus,an
 5000 format(1x,i4,2x,a12)
      if(kbus .le. 0 .or. kbus .gt. 9999) go to 500
      anames(kbus) = an
      go to 1
    2 read(2,5002,end=3) jbus,kbus,zzz
 5002 format(1x,2i5,2e15.6)
      if(jbus .ne. kbus) go to 2
      rs  = real(zzz)
      xs  = aimag(zzz)
      a   = (vmin * vmin - 1) * (rm * rm + xm * xm)
      b   = 2.0 * vmin * vmin * (rs * rm + xs * xm)
      c   = vmin * vmin * (rs * rs + xs * xs)
      rad = sqrt(b * b - 4.0 * a * c)
      h1  = 1.0 / ((-b + rad) / 2.0 / a)
      h2  = 1.0 / ((-b - rad) / 2.0 / a)
      h   = -1.0
      if(h1 .lt. 0.0 .and. h2 .gt. 0.0) h = h2
      if(h2 .lt. 0.0 .and. h1 .gt. 0.0) h = h1
      if(h1 .gt. 0.0 .and. h2 .gt. 0.0) h = amin1(h1,h2)
      if(h .le. 0.0) go to 501
```

```
      if(hpmin(jbus) .lt. 0.0) hpmin(jbus) = h
      if(h .lt. hpmin(jbus)) hpmin(jbus) = h
      write(6,5003) jbus,anames(jbus),zzz,h
 5003 format(1x,'num = ',i4,', name = ',a12,', zsys = ',2e12.4,
     1 ', kw = ',f7.1)
      vcheck = cabs(zmotor / h / (zzz + zmotor / h))
      if(vcheck .lt. (vmin - eps) .or. vcheck .gt. (vmin + eps))
     1 go to 502
      go to 2
    3 do 5 jbus = 1,9999
      if(hpmin(jbus) .lt. 0.0) go to 5
      write(3,5011) jbus,anames(jbus),hpmin(jbus)
 5011 format(1x,'bus number = ',i5,', name = ',a12,
     1', kw starting cap. = ',f7.1)
    5 continue
      stop
  500 write(6,5001)
 5001 format(1x,'error in vsoln file')
      stop
  501 write(6,5005)
 5005 format(1x,'error in kw calculation')
      stop
  502 write(6,5010)
 5010 format(1x,'error in vcheck')
      stop
      end
```

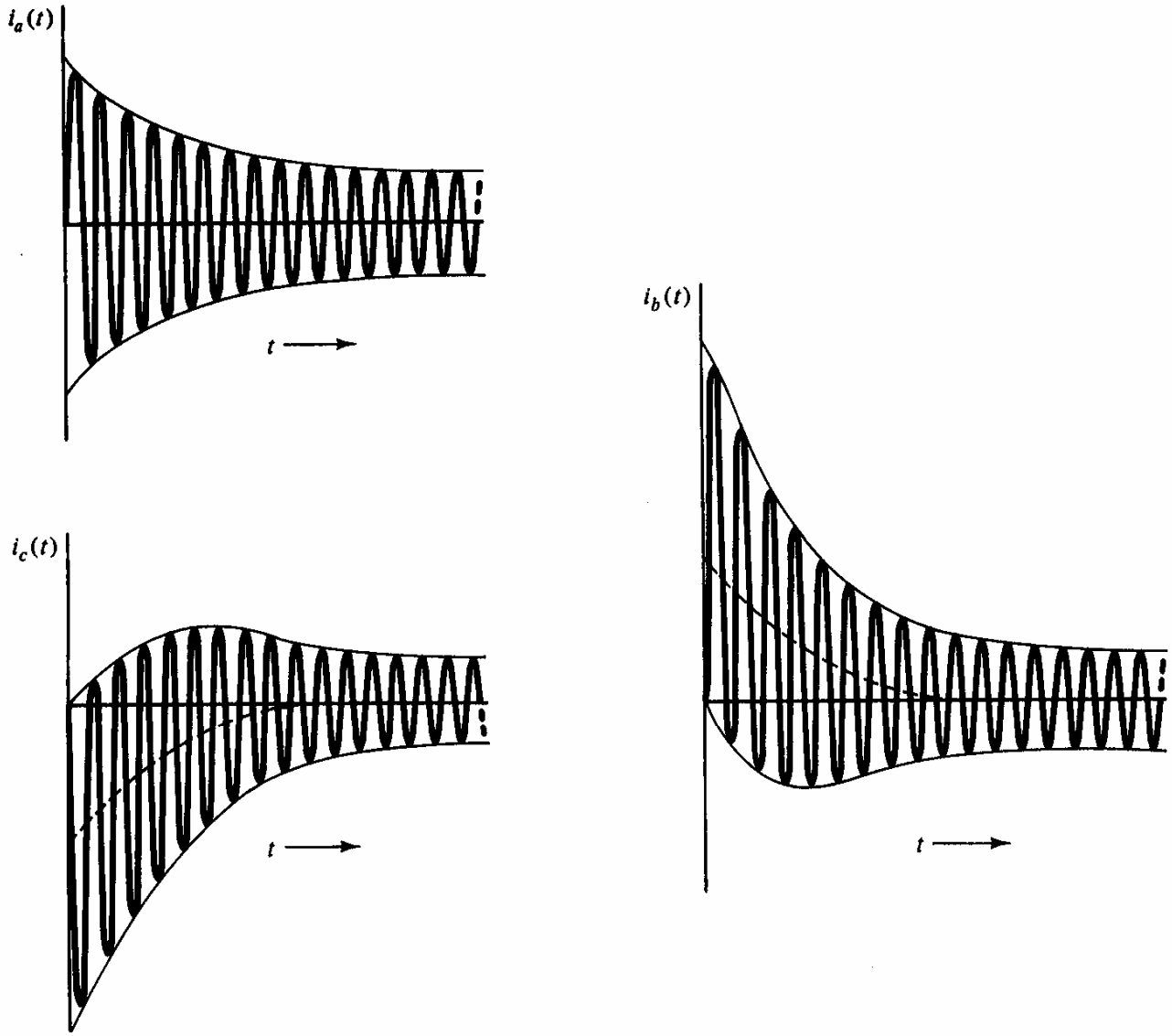


Figure 7.5 Short-circuit currents.

Grady 2007, p. 231

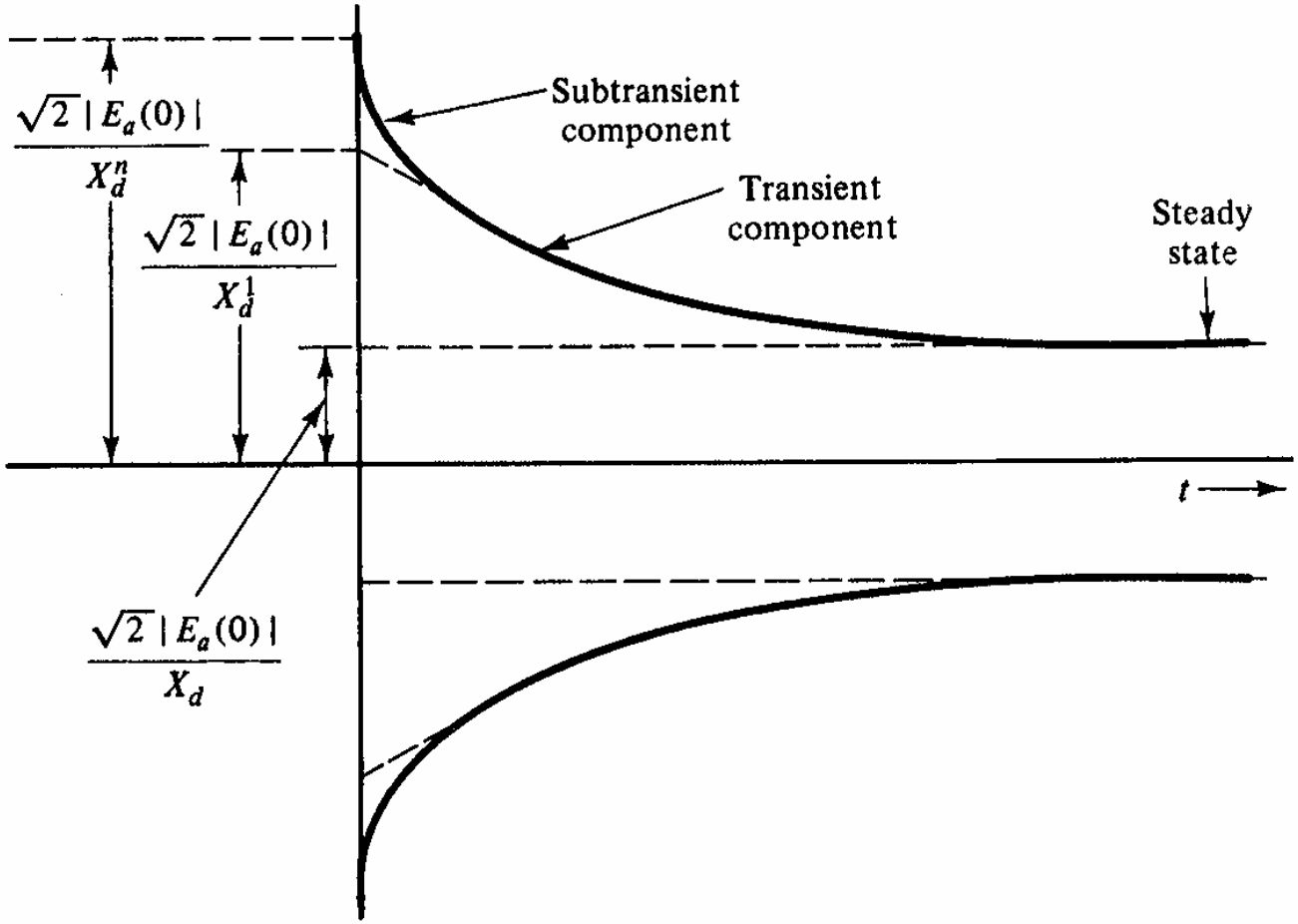


Figure 7.6 Short-circuit current components.

Grady 2007, p. 232

TABLE 12.1 TYPICAL SYNCHRONOUS MACHINE REACTANCES

	Turbine-generators (two-pole)		Salient-pole generators with dampers
	Conventionally cooled*	Conductor cooled	
X_d	1.20	1.80	1.25
X_q	1.16	1.75	0.70
X'_d	0.15	0.30	0.30
X''_d	0.09	0.22	0.20
X^-	0.09	0.22	0.20
X^0	0.03	0.12	0.18

*Reactances are representative of smaller air-cooled and hydrogen-cooled machines.

Bergen, Chapter 12, p.468

Grady 2007, p. 233

Short Circuits

1. Introduction

Voltage sags are due mostly to faults on either transmission systems or distribution feeders. Transmission faults affect customers over a wide area, possibly dozens of miles, but distribution faults usually affect only the customers on the faulted feeder or on adjacent feeders served by the same substation transformer.

Single-phase faults (i.e., line-to-ground) are the most common type of faults, followed by line-to-line, and three-phase. Since single-phase and line-to-line faults are unbalanced, their resulting sag voltages are computed using symmetrical components. Transformer connections affect the propagation of positive, negative, and zero sequence components differently. Thus, the characteristics of a voltage sag changes as it propagates through a network.

Typically, a transmission voltage sag passes through two levels of transformers before reaching a 480V load (e.g., 138kV:12.47kV at the entrance to the facility, and 12.47kV:480V at the load). 120V loads likely have a third transformer (e.g., 480V:120V). It is not intuitively obvious how the sag changes, but the changes can be computed using symmetrical components and are illustrated in this report.

2. Symmetrical Components

An unbalanced set of N related phasors can be resolved into N systems of phasors called the symmetrical components of the original phasors. For a three-phase system (i.e. $N = 3$), the three sets are:

1. Positive Sequence - three phasors, equal in magnitude, 120° apart, with the same sequence (a-b-c) as the original phasors.
2. Negative Sequence - three phasors, equal in magnitude, 120° apart, with the opposite sequence (a-c-b) of the original phasors.
3. Zero Sequence - three identical phasors (i.e. equal in magnitude, with no relative phase displacement).

The original set of phasors is written in terms of the symmetrical components as follows:

$$\tilde{V}_a = \tilde{V}_{a0} + \tilde{V}_{a1} + \tilde{V}_{a2} \ ,$$
$$\tilde{V}_b = \tilde{V}_{b0} + \tilde{V}_{b1} + \tilde{V}_{b2} \ ,$$
$$\tilde{V}_c = \tilde{V}_{c0} + \tilde{V}_{c1} + \tilde{V}_{c2} \ ,$$

where 0 indicates zero sequence, 1 indicates positive sequence, and 2 indicates negative sequence.

The relationships among the sequence components for a-b-c are

Positive Sequence	Negative Sequence	Zero Sequence
$\widetilde{V}_{b1} = \widetilde{V}_{a1} \bullet 1\angle -120^{\circ}$	$\widetilde{V}_{b2} = \widetilde{V}_{a2} \bullet 1\angle +120^{\circ}$	$\widetilde{V}_{a0} = \widetilde{V}_{b0} = \widetilde{V}_{c0}$
$\widetilde{V}_{c1} = \widetilde{V}_{a1} \bullet 1\angle +120^{\circ}$	$\widetilde{V}_{c2} = \widetilde{V}_{a2} \bullet 1\angle -120^{\circ}$	

The symmetrical components of all a-b-c voltages are usually written in terms of the symmetrical components of phase a by defining

$$a = 1\angle +120^{\circ} \text{ , so that } a^2 = 1\angle +240^{\circ} = 1\angle -120^{\circ} \text{ , and } a^3 = 1\angle +360^{\circ} = 1\angle 0^{\circ}.$$

Substituting into the previous equations for $\widetilde{V}_a, \widetilde{V}_b, \widetilde{V}_c$ yields

$$\begin{aligned}
\widetilde{V}_a &= \widetilde{V}_{a0} + \widetilde{V}_{a1} + \widetilde{V}_{a2} \text{ ,} \\
\widetilde{V}_b &= \widetilde{V}_{a0} + a^2\widetilde{V}_{a1} + a\widetilde{V}_{a2} \text{ ,} \\
\widetilde{V}_c &= \widetilde{V}_{a0} + a\widetilde{V}_{a1} + a^2\widetilde{V}_{a2} \text{ .}
\end{aligned}$$

In matrix form, the above equations become

$$\begin{bmatrix} \widetilde{V}_a \\ \widetilde{V}_b \\ \widetilde{V}_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} \widetilde{V}_{a0} \\ \widetilde{V}_{a1} \\ \widetilde{V}_{a2} \end{bmatrix} \text{ , } \begin{bmatrix} \widetilde{V}_{a0} \\ \widetilde{V}_{a1} \\ \widetilde{V}_{a2} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} \widetilde{V}_a \\ \widetilde{V}_b \\ \widetilde{V}_c \end{bmatrix} \tag{1}$$

or in matrix form

$$\widetilde{V}_{abc} = T \bullet \widetilde{V}_{012} \text{ , and } \widetilde{V}_{012} = T^{-1} \bullet \widetilde{V}_{abc} \text{ ,} \tag{2}$$

where transformation matrix T is

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \text{, and } T^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} . \tag{3}$$

If $\widetilde{V}_{abc}$ represents a balanced set (i.e. $\widetilde{V}_b = \widetilde{V}_a \bullet 1\angle -120^{\circ} = a^2\widetilde{V}_a$, $\widetilde{V}_c = \widetilde{V}_a \bullet 1\angle +120^{\circ} = a\widetilde{V}_a$), then substituting into $\widetilde{V}_{012} = T^{-1} \bullet \widetilde{V}_{abc}$ yields

$$\begin{bmatrix} \widetilde{V}_{a0} \\ \widetilde{V}_{a1} \\ \widetilde{V}_{a2} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} \widetilde{V}_a \\ a^2\widetilde{V}_a \\ a\widetilde{V}_a \end{bmatrix} = \begin{bmatrix} 0 \\ \widetilde{V}_a \\ 0 \end{bmatrix} .$$

Hence, balanced voltages or currents have only positive sequence components, and the positive sequence components equal the corresponding phase a voltages or currents.

However, balanced voltages are rare during voltage sags. Most often, one phase is affected significantly, and the other two less significantly. Thus, all three sequence voltages $\tilde{V}_{a0}, \tilde{V}_{a1}, \tilde{V}_{a2}$ exist during most sags, and these sequence voltages are shifted differently by transformers when propagating through a system. When recombined to yield phase voltages $\tilde{V}_a, \tilde{V}_b, \tilde{V}_c$, it is clear that the form of phase voltages must also change as transformers are encountered.

3. Transformer Phase Shift

The conventional positive-sequence and negative-sequence model for a three-phase transformer is shown below. Admittance y is a series equivalent for resistance and leakage reactance, tap t is the tap (in per unit), and angle θ is the phase shift.

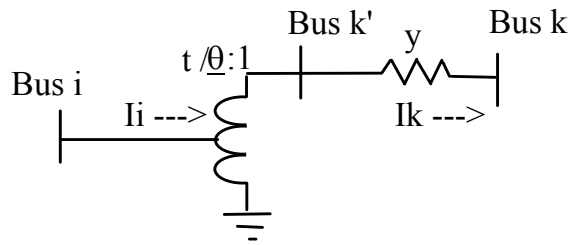


Figure 1. Positive- and Negative-Sequence Model of Three-Phase Transformer

For grounded-wye:grounded-wye and delta:delta transformers, θ is +0˚, and thus positive- and negative-sequence voltages and currents pass through unaltered (in per unit). However, for wye-delta and delta-wye transformers, θ is sequence-dependent and is defined as follows:

- For positive sequence, θ is +30˚ if bus i is the high-voltage side, or –30˚ if bus i is the low-voltage side

  and oppositely

- For negative sequence, θ is –30˚ if bus i is the high-voltage side, or +30˚ if bus i is the low-voltage side

In other words, positive sequence voltages and currents on the high-voltage side *lead* those on the low-voltage side by 30˚. Negative sequence voltages and currents on the high-voltage side *lag* those on the low-voltage side by 30˚.

For zero-sequence voltages and currents, transformers do not introduce a phase shift, but they may block zero-sequence propagation as shown in Figure 2.

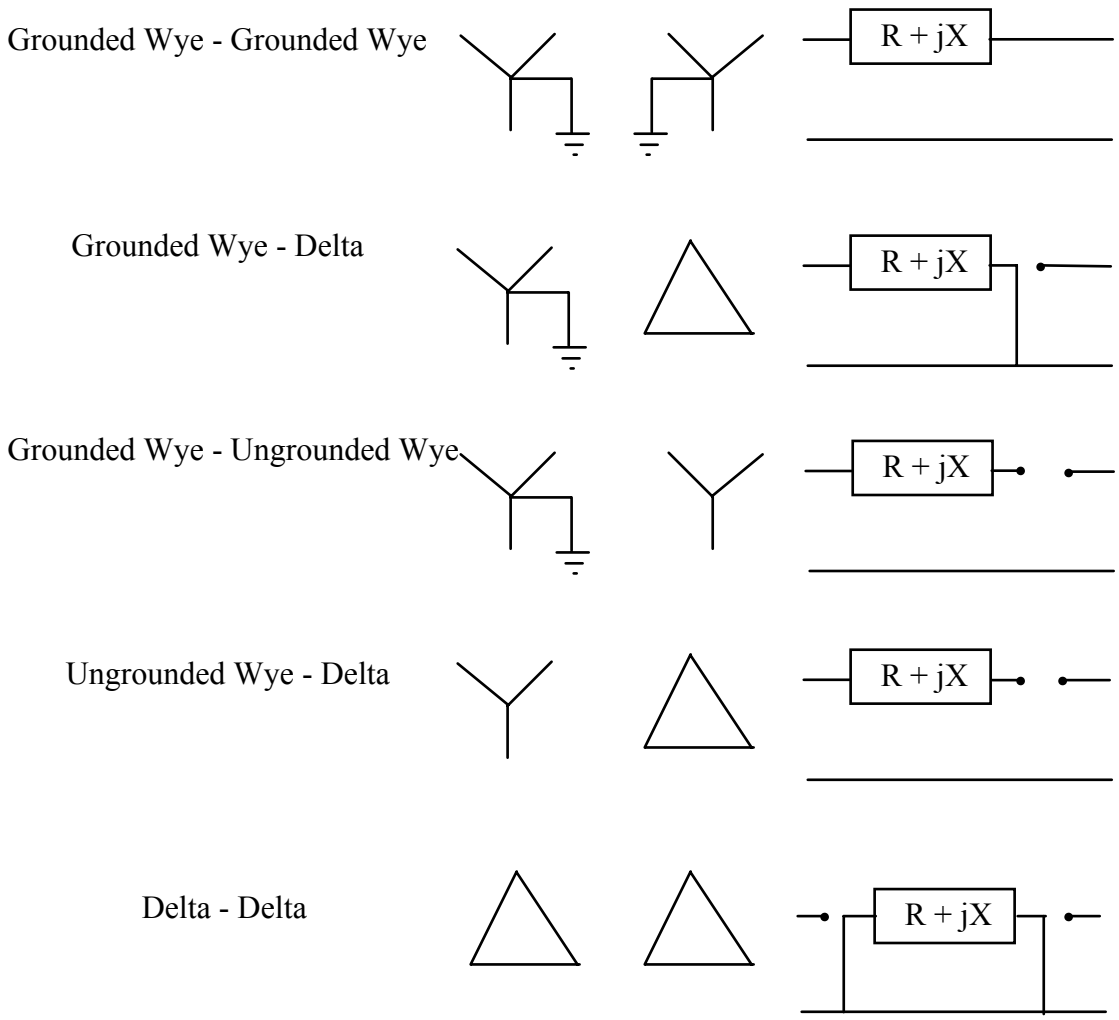


Figure 2. Zero-Sequence Model of Three-Phase Transformer

It can be seen in the above figure that only the grounded-wye:grounded-wye transformer connection permits the flow of zero-sequence from one side of a transformer to the other.

Thus, due to phase shift and the possible blocking of zero-sequence, transformers obviously play an important role in unbalanced voltage sag propagation.

4. System Impedance Matrices

Fault currents and voltage sags computations require elements of the impedance matrix Z for the study system. While each of the three sequences has its own impedance matrix, positive- and negative-sequence matrices are usually identical. Impedance elements are usually found by

- building the system admittance matrix Y, and then inverting it to obtain the entire Z,

or by

- using Gaussian elimination and backward substitution to obtain selected columns of Z.

The admittance matrix Y is easily built according to the following rules:

- The diagonal terms of Y contain the sum of all branch admittances connected directly to the corresponding bus.

- The off-diagonal elements of *Y* contain the negative sum of all branch admittances connected directly between the corresponding busses.

The procedure is illustrated by the three-bus example in Figure 3.

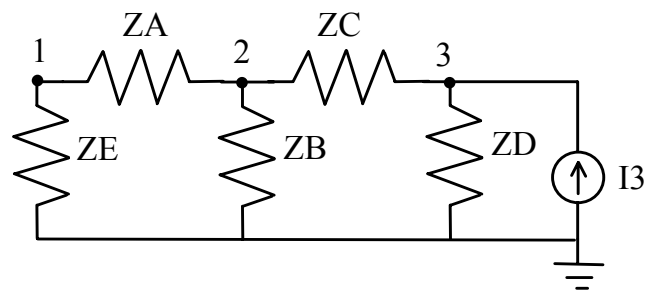


Figure 3. Three-Bus Admittance Matrix Example

Applying KCL at the three independent nodes yields the following equations for the bus voltages (with respect to ground):

At bus 1, $\dfrac{V_1}{Z_E} + \dfrac{V_1 - V_2}{Z_A} = 0$,

At bus 2, $\dfrac{V_2}{Z_B} + \dfrac{V_2 - V_1}{Z_A} + \dfrac{V_2 - V_3}{Z_C} = 0$,

At bus 3, $\dfrac{V_3}{Z_D} + \dfrac{V_3 - V_2}{Z_C} = I_3$.

Collecting terms and writing the equations in matrix form yields

$$\begin{bmatrix} \frac{1}{Z_E}+\frac{1}{Z_A} & -\frac{1}{Z_A} & 0 \\ -\frac{1}{Z_A} & \frac{1}{Z_A}+\frac{1}{Z_B}+\frac{1}{Z_C} & -\frac{1}{Z_C} \\ 0 & -\frac{1}{Z_C} & \frac{1}{Z_C}+\frac{1}{Z_D} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_3 \end{bmatrix},$$

or in matrix form,

$$YV = I \ ,$$

Besides being the key for fault calculations, the impedance matrix, $Z = Y^{-1}$, is also physically significant. Consider Figure 4.

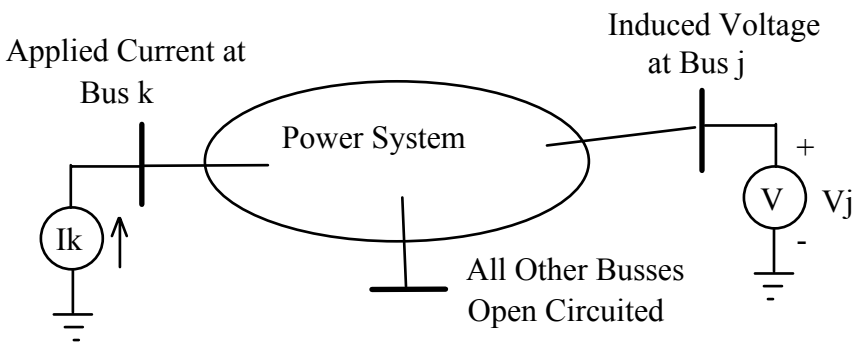


Figure 4. Physical Significance of the Impedance Matrix

Impedance matrix element $z_{j,k}$ is defined as

$$z_{j,k} = \left.\frac{V_j}{I_k}\right|_{I_m=0, m=1,2,\cdots,N, m\neq k} , \tag{4}$$

where I_k is a current source attached to bus *k*, V_j is the resulting voltage at bus *j*, and all busses except *k* are open-circuited. The depth of a voltage sag at bus k is determined directly by multiplying the phase sequence components of the fault current at bus k by the matrix elements $z_{j,k}$ for the corresponding phase sequences.

5. Short Circuit Calculations

Short circuit calculations require positive, negative, and zero sequence impedance information, depending on whether or the fault is balanced or not. For example, the commonly-studied, but relatively rare, three-phase fault is balanced. Therefore, only positive sequence impedances are required for its study.

Consider the three-phase fault represented by the one-line diagram in Figure 5, where V_{TH} and Z_{TH} are the Thevenin equivalent circuit parameters for bus k.

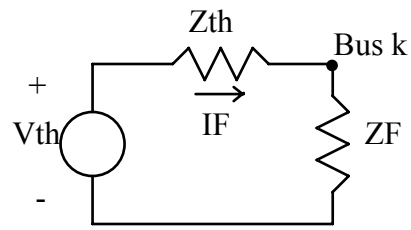


Figure 5. Three-Phase Fault at Bus *k*

The fault current and voltage are clearly

$$I_k^F = \frac{V_{TH}}{Z_{TH} + Z_F} \text{ , and } V_k^F = V_{TH} - Z_{TH} I_k^F = V_{TH}\left[\frac{Z_F}{Z_{TH} + Z_F}\right].$$

In a large power system, the Thevenin equivalent impedance for a bus is the corresponding diagonal impedance matrix element, and the Thevenin equivalent voltage is usually assumed to be 1.0 /0 pu.

The type of machine models used when building impedance matrices affects the Thevenin equivalent impedances and fault calculations. Rotating machines actually have time-varying impedances when subjected to disturbances. However, for simplification purposes, their impedances are usually divided into three zones - subtransient (first few cycles), transient (5 cycles - 60 cycles), and steady-state (longer than 60 cycles). When performing fault studies, the time period of interest is usually a few cycles, so that machines are represented by their subtransient impedances when forming the impedance matrices.

Developing the equations for fault studies requires adept use of both a-b-c and 0-1-2 forms of the circuit equations. The use of sequence components implies that the system impedances (but not the system voltages and currents) are symmetric. In general, there are six equations and six unknowns to be solved, regardless of the type of fault studied.

It is common in fault studies to assume that the power system is initially unloaded and that all voltages are 1.0 per unit. When there are multiple sources, this assumption *requires* that there are no shunt elements connected, such as loads, capacitors, etc., *except* for rotating machines (whose Thevenin equivalent voltages are 1.0 pu.).

Since wye-delta transformers shift positive, negative, and zero sequence components differently, it is important to model transformers according to the rules given earlier. This means that the pre-fault voltages all have magnitude 1.0 pu., but that the pre-fault voltage angles can be 0°,$+30^o$, or -30°, depending upon the net transformer phase shift between them and the chosen reference bus.

Balanced Three-Phase Fault

Consider the three-phase fault at bus *k*, as shown in Figure 6.

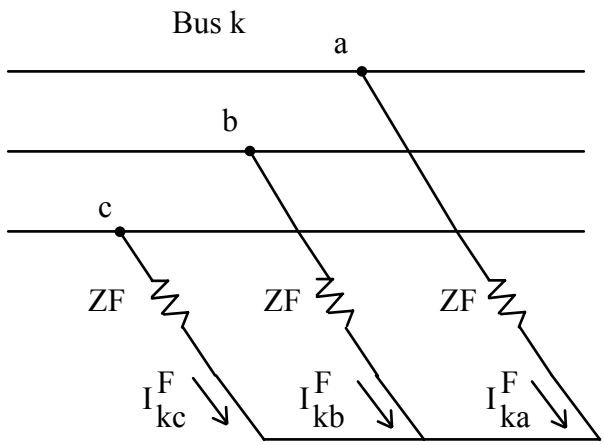


Figure 6: Three-Phase Fault at Bus *k*

The Thevenin equivalent circuit equation, assuming no other current injections in the system, is

$$\begin{bmatrix} V_{ka}^F \\ V_{kb}^F \\ V_{kc}^F \end{bmatrix} = \begin{bmatrix} V_{ka}^{\Pr e} \\ V_{kb}^{\Pr e} \\ V_{kc}^{\Pr e} \end{bmatrix} - \begin{bmatrix} z_{ka,ka} & z_{ka,kb} & z_{ka,kc} \\ z_{kb,ka} & z_{kb,kb} & z_{kb,kc} \\ z_{kc,ka} & z_{kc,kb} & z_{kc,kc} \end{bmatrix} \begin{bmatrix} I_{ka}^F \\ I_{kb}^F \\ I_{kc}^F \end{bmatrix},$$

or in sequence form,

$$\begin{bmatrix} V_{k0}^F \\ V_{k1}^F \\ V_{k2}^F \end{bmatrix} = \begin{bmatrix} V_{k0}^{\Pr e} \\ V_{k1}^{\Pr e} \\ V_{k2}^{\Pr e} \end{bmatrix} - \begin{bmatrix} z_{k0,k0} & 0 & 0 \\ 0 & z_{k1,k1} & 0 \\ 0 & 0 & z_{k2,k2} \end{bmatrix} \begin{bmatrix} I_{k0}^F \\ I_{k1}^F \\ I_{k2}^F \end{bmatrix}.$$

In abbreviated form, the above equations are

$$V_{kabc}^F = V_{kabc}^{\Pr e} - Z_{k-k,abc} I_{kabc}^F \text{ , and } V_{k012}^F = V_{k012}^{\Pr e} - Z_{k-k,012} I_{k012}^F \text{ ,} \tag{5}$$

where V_k^F consists of the voltages at bus *k* during the fault, $V_k^{\Pr e}$ consists of the pre-fault voltages, I_k^F gives the fault currents, and Z_{k-k} contains the individual impedance elements extracted from the impedance matrix.

The above matrix equations represents three equations (repeated in abc and 012 form), but there are six unknowns represented by V_k^F and I_k^F, so that three additional equations are required. The additional equations are found by observing that

$$V_{kabc}^F = Z_F I_{kabc}^F \text{ , or } V_{k012}^F = Z_F I_{k012}^F \text{ .}$$

Substituting into the Thevenin equation, and recognizing that all zero- and negative-sequence voltages and currents are zero for a balanced fault yields

$$\begin{bmatrix} 0 \\ Z_F I_{k1}^F \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ V_{k1}^{\Pr e} \\ 0 \end{bmatrix} - \begin{bmatrix} z_{k0,k0} & 0 & 0 \\ 0 & z_{k1,k1} & 0 \\ 0 & 0 & z_{k2,k2} \end{bmatrix} \begin{bmatrix} 0 \\ I_{k1}^F \\ 0 \end{bmatrix},$$

so that the positive sequence fault current is found to be

$$I_{k1}^F = \frac{V_{k1}^{\Pr e}}{z_{k1,k1} + Z_F}, I_{k0}^F = 0, I_{k2}^F = 0 \,. \tag{6}$$

Substituting into Thevenin equation

$$V_{k012}^F = V_{k012}^{\Pr e} - Z_{k-k,012} I_{k012}^F \tag{7}$$

yields the fault voltage at Bus *k*. Similarly, because the impedance matrix relates the voltages at network busses to current injections at network busses, the voltage at any other bus *j* is found using

$$V_{j012}^F = V_{j012}^{\Pr e} - Z_{j-k,012} I_{k012}^F \,. \tag{8}$$

Note that the minus sign is needed because the fault current has been drawn as positive outward. Once the fault voltages are known at neighboring busses, the contribution currents through the connected branches can be easily found.

Single-Phase to Ground Fault

Consider the single-phase fault at bus *k*, as shown in Figure 7.

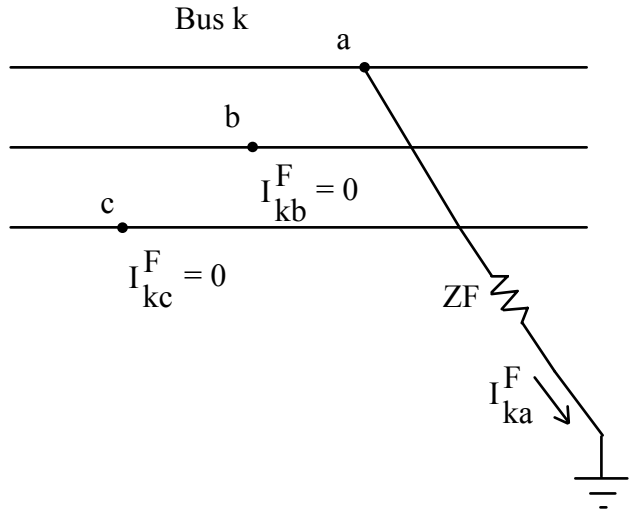


Figure 7: Single-Phase Fault at Bus *k*, Phase a

As before, the Thevenin equivalent circuit equations, assuming no other current injections in the system, is

$$V^F_{k012} = V^{\Pr e}_{k012} - Z_{k-k,012} I^F_{k012}.$$

Examining I^F_{k012} shows that in this case

$$I^F_{k012} = T^{-1} I^F_{kabc} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} I^F_{ka} \\ I^F_{kb} = 0 \\ I^F_{kc} = 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} I^F_{ka} \\ I^F_{ka} \\ I^F_{ka} \end{bmatrix}. \tag{9}$$

Substituting into the Thevenin equation yields

$$\begin{bmatrix} V^F_{k0} \\ V^F_{k1} \\ V^F_{k2} \end{bmatrix} = \begin{bmatrix} V^{\Pr e}_{k0} = 0 \\ V^{\Pr e}_{k1} = V^{\Pr e}_{ka} \\ V^{\Pr e}_{k2} = 0 \end{bmatrix} - \begin{bmatrix} z_{k0,k0} & 0 & 0 \\ 0 & z_{k1,k1} & 0 \\ 0 & 0 & z_{k2,k2} \end{bmatrix}\begin{bmatrix} I^F_{ka}/3 \\ I^F_{ka}/3 \\ I^F_{ka}/3 \end{bmatrix}$$

Add the three rows yields

$$V^F_{k0} + V^F_{k1} + V^F_{k2} = V^F_{ka} = V^{\Pr e}_{ka} - \frac{1}{3} I^F_{ka}\left(z_{k0,k0} + z_{k1,k1} + z_{k2,k2}\right).$$

From the circuit it is obvious that

$$V_{ka}^{F} = I_{ka}^{F} Z_F \ ,$$

so that

$$I_{ka}^{F} Z_F = V_{ka}^{\Pr e} - \frac{1}{3} I_{ka}^{F} \left(z_{k0,k0} + z_{k1,k1} + z_{k2,k2} \right) .$$

Solving for I_{ka}^{F} yields

$$I_{ka}^{F} = \frac{3V_{ka}^{\Pr e}}{z_{k0,k0} + z_{k1,k1} + z_{k2,k2} + 3Z_F} \ . \tag{10}$$

Now, using

$$I_{k0}^{F} = I_{k1}^{F} = I_{k2}^{F} = \frac{I_{ka}^{F}}{3} \ ,$$

all network voltages can be found from

$$V_{j012}^{F} = V_{j012}^{\Pr e} - Z_{j-k,012} I_{k012}^{F} \ .$$

Note that if $z_{k0,k0} < z_{k1,k2}$, a single-phase fault will have a higher value than does a three-phase fault.

Line-to-Line Fault

Consider the line-to-line fault at bus *k*, as shown in Figure 8.

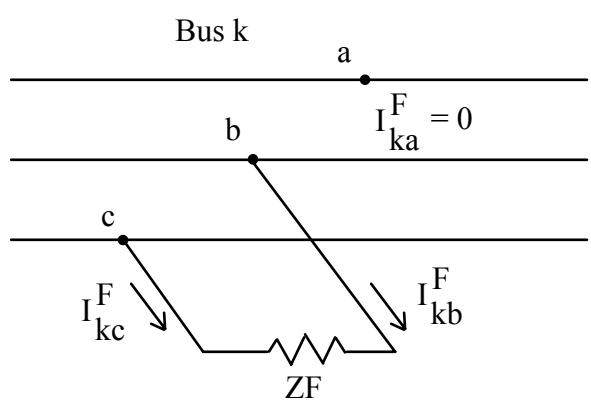


Figure 8. Line-to-Line Fault Between Phases b and c at Bus *k*

Examining I_{k012}^{F} shows that in this case

$$I_{k012}^{F} = T^{-1} I_{kabc}^{F} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} I_{ka}^{F} = 0 \\ I_{kb}^{F} \\ I_{kc}^{F} = -I_{kb}^{F} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ I_{kb}^{F}\left(a - a^2\right) \\ I_{kb}^{F}\left(a^2 - a\right) \end{bmatrix} . \tag{11}$$

Note that there is no zero sequence fault current.

Substituting into the Thevenin equation yields

$$\begin{bmatrix} V_{k0}^{F} \\ V_{k1}^{F} \\ V_{k2}^{F} \end{bmatrix} = \begin{bmatrix} 0 \\ V_{ka}^{\Pr e} \\ 0 \end{bmatrix} - \begin{bmatrix} z_{k0,k0} & 0 & 0 \\ 0 & z_{k1,k1} & 0 \\ 0 & 0 & z_{k2,k2} \end{bmatrix}\begin{bmatrix} 0 \\ I_{kb}^{F}\left(a - a^2\right)/3 \\ I_{kb}^{F}\left(a^2 - a\right)/3 \end{bmatrix} .$$

Subtracting the last two rows of the Thevenin equation yields

$$V_{k1}^{F} - V_{k2}^{F} = V_{ka}^{\Pr e} - z_{k1,k1}\frac{I_{kb}^{F}\left(a - a^2\right)}{3} + z_{k2,k2}\frac{I_{kb}^{F}\left(a^2 - a\right)}{3} ,$$

or

$$I_{kb}^{F}\left[\frac{z_{k2,k2}\left(a^2 - a\right)}{3} - \frac{z_{k1,k1}\left(a - a^2\right)}{3}\right] = V_{k1}^{F} - V_{k2}^{F} - V_{ka}^{\Pr e} .$$

From the circuit, we see that

$$I_{kb}^{F} = \frac{V_{kb}^{F} - V_{kc}^{F}}{Z_F} .$$

Using $V_{abc} = TV_{012}$, we find that

$$V_{kb}^{F} - V_{kc}^{F} = V_{k1}^{F}\left(a^2 - a\right) + V_{k2}^{F}\left(a - a^2\right),$$

so that

$$I_{kb}^{F} = \frac{V_{k1}^{F}\left(a^2 - a\right) + V_{k2}^{F}\left(a - a^2\right)}{Z_F}, \text{ or } I_{kb}^{F} = \frac{\left(V_{k1}^{F} - V_{k2}^{F}\right)\left(a^2 - a\right)}{Z_F} .$$

Combining equations yields

$$I_{kb}^{F}\left[\frac{z_{k2,k2}\left(a^{2}-a\right)}{3}-\frac{z_{k1,k1}\left(a-a^{2}\right)}{3}\right]=\frac{I_{kb}^{F}Z_{F}}{\left(a^{2}-a\right)}-V_{ka}^{\Pr e} \ .$$

Collecting terms yields

$$I_{kb}^{F}\left[\frac{z_{k2,k2}\left(a^{2}-a\right)}{3}-\frac{z_{k1,k1}\left(a-a^{2}\right)}{3}-\frac{Z_{F}}{\left(a^{2}-a\right)}\right]=-V_{ka}^{\Pr e} \ ,$$

or

$$I_{kb}^{F}\left[\frac{Z_{F}}{\left(a^{2}-a\right)}+\frac{\left(a-a^{2}\right)}{3}\left(z_{k1,k1}+z_{k2,k2}\right)\right]=V_{ka}^{\Pr e} \ .$$

Simplifying yields

and where $I_{kc}^{F}=-I_{kb}^{F}, I_{ka}^{F}=0$. All network voltages can now be found from

$$I_{kb}^{F}=\frac{-j\sqrt{3}V_{ka}^{\Pr e}}{z_{k1,k1}+z_{k2,k2}+Z_{F}} \ , \tag{12}$$

$$V_{j012}^{F}=V_{j012}^{\Pr e}-Z_{j-k,012}I_{k012}^{F} \ .$$

Voltage Sag Propagation Along Feeders

1. Introduction

Short circuit equations provide the theoretical framework for determining the voltage sag at a bus due to a fault anywhere in the system. However, the short circuit equations by themselves provide little insight. We now proceed with examples to provide this insight by showing how a sag propagates for various transformer situations.

2. Impact of Transmission System Faults on Customers

Consider the typical situation shown in Figure 1. A fault occurs at bus k of the transmission system, causing a voltage sag that affects a substation (bus j) and the customers connected to its feeders. There can be as many as three transformers between the customer's load and the transmission fault point, and each of these transformers can have a 30° phase shift. Typically, all three of the transformers shown (i.e., T1, T2, and T3) are delta connected on the high side, and grounded-wye connected on the low side.

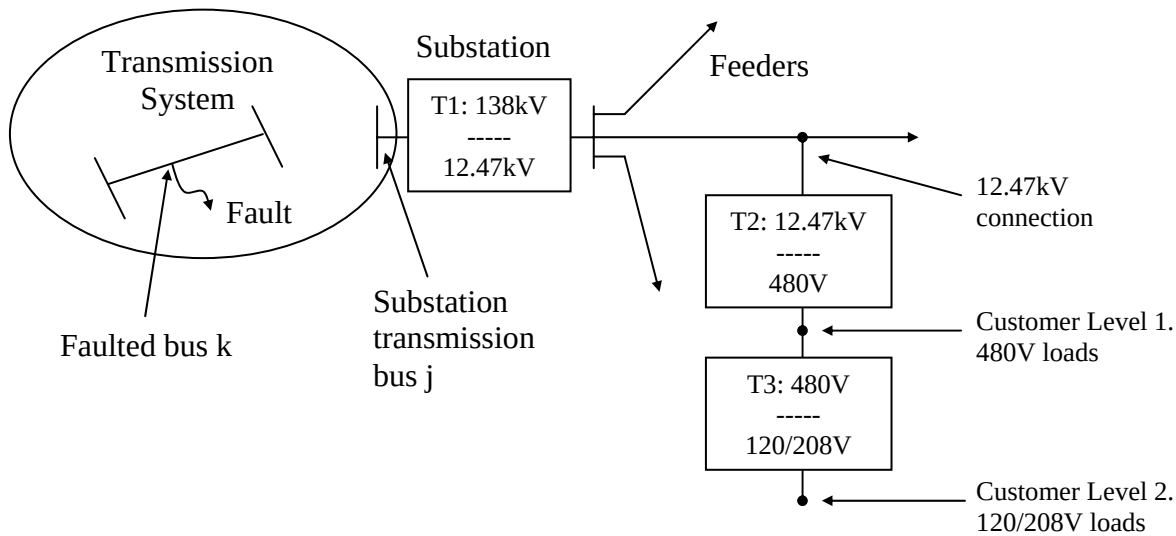


Figure 1. Example System for Analyzing the Propagation of Transmission Voltage Sags into Customer Low-Voltage Busses

The standard assumption for fault calculations is that

- the circuit is initially unloaded, or at least that the voltages are all close to 1.0 per unit.

Using this assumption, *and further assuming that there are no significant contributors of fault current on the feeders*, then the actual location of the customer is not important because all points on the three 12.47kV feeders shown (including the substation 12.47kV bus) will experience the same sag. Furthermore, the sag experienced on the substation 12.47kV bus will

be the same as on substation 138kV bus j, except for possible zero-sequence component blocking and positive/negative-phase shifts.

The significance of the above paragraph is that

> *for transmission faults, one monitor at either the substation 138kV bus or at the substation 12.47kV bus is adequate to predict voltage sag levels anywhere on the substation's feeders, provided there are no significant contributors of fault current on the feeders.*

If the transmission fault is electrically far away, then the sag experienced at the substation and at the customer site will be small. Alternatively, if the fault is immediately at substation 138kV bus j, then the sag will be the most severe possible. Thus, it is reasonable to assume that an electrical "proximity" factor exists, where a proximity factor of zero (i.e., 0%) indicates that the fault is at substation 138kV bus j, and a proximity factor of unity (i.e., 100%) indicates that the transmission fault bus k is very far away. From knowledge of the physical significance of the impedance matrix, and from examining Thevenin equations

$$V_{k012}^{F} = V_{k012}^{\Pr e} - Z_{k-k,012} I_{k012}^{F},$$

$$V_{j012}^{F} = V_{j012}^{\Pr e} - Z_{j-k,012} I_{k012}^{F},$$

this proximity factor P is approximated using the ratio of positive-sequence impedances

$$P = 1 - \left| \frac{z_{j1,k1}}{z_{k1,k1}} \right| . \tag{1}$$

By coding the short circuit equations into a Visual Basic program, and employing (1), voltage sag propagation for the situation described in Figure 1 can now be illustrated. Assuming that the transmission fault is relatively close to the substation (i.e., proximity factor = 25%), and that T1, T2, and T3 are all delta:grounded-wye transformers, the line-to-neutral voltages for single-phase, phase-to-phase, and three-phase transmission faults are shown in Figures 2 – 4, respectively. Both phasor plots and magnitude bar charts are given.

It is important to note that if a transformer is connected grounded-wye:grounded-wye or delta:delta, then the voltage sag on the low-voltage side of the transformer is the same as on the high-voltage side, as illustrated in Figure 5 for the single-phase fault.

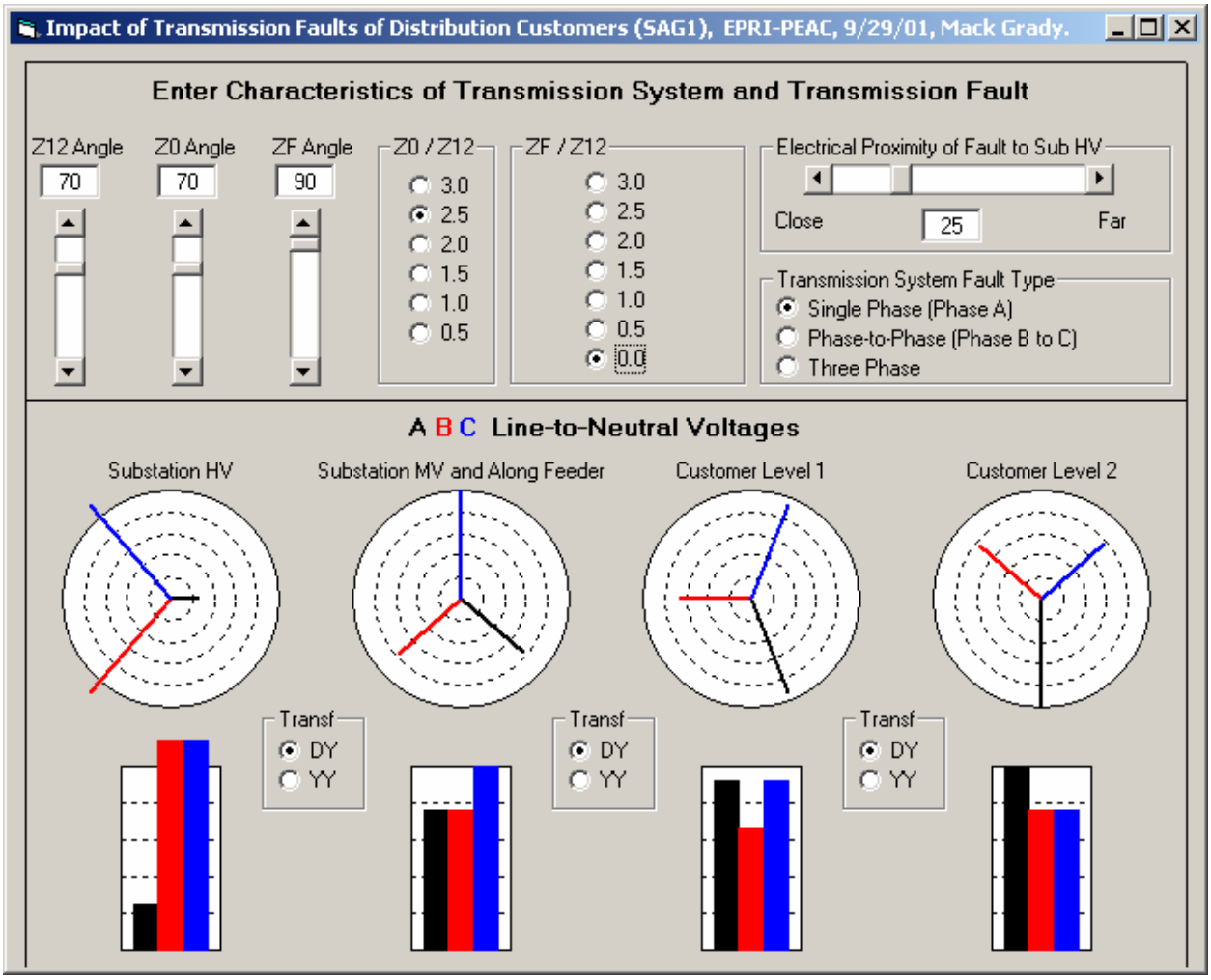


Figure 2. Propagation of Close-In Single Phase Fault on the Transmission System (all three transformers have delta:grounded-wye connections)

Note the voltage swell on phases b and c at the substation 138kV bus. Note also that two phases are affected after the first transformation, then one phase is affected after two transformations, and again two phases are affected after three transformations.

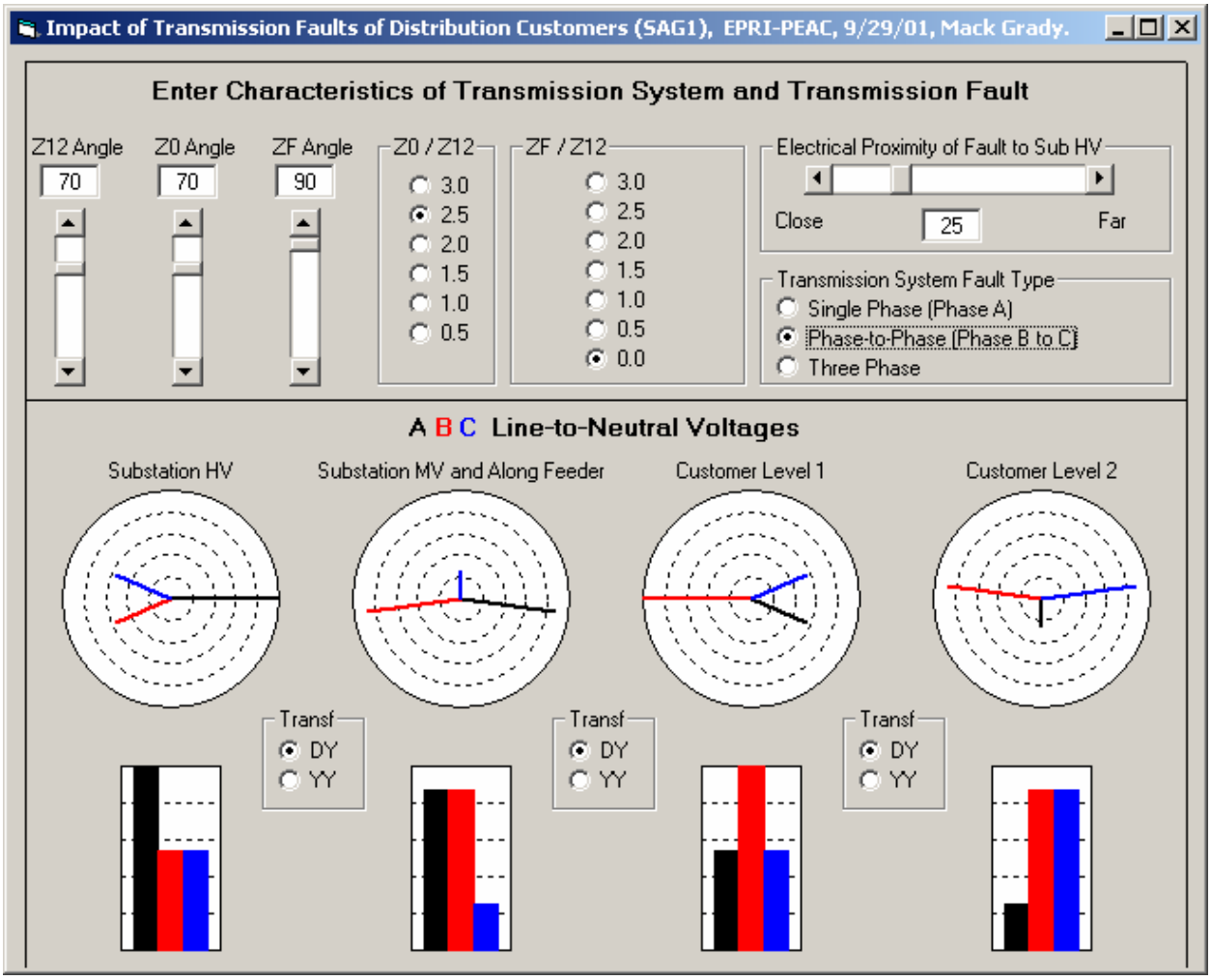


Figure 3. Propagation of Close-In Phase-to-Phase Fault on the Transmission System (all three transformers have delta:grounded-wye connections)

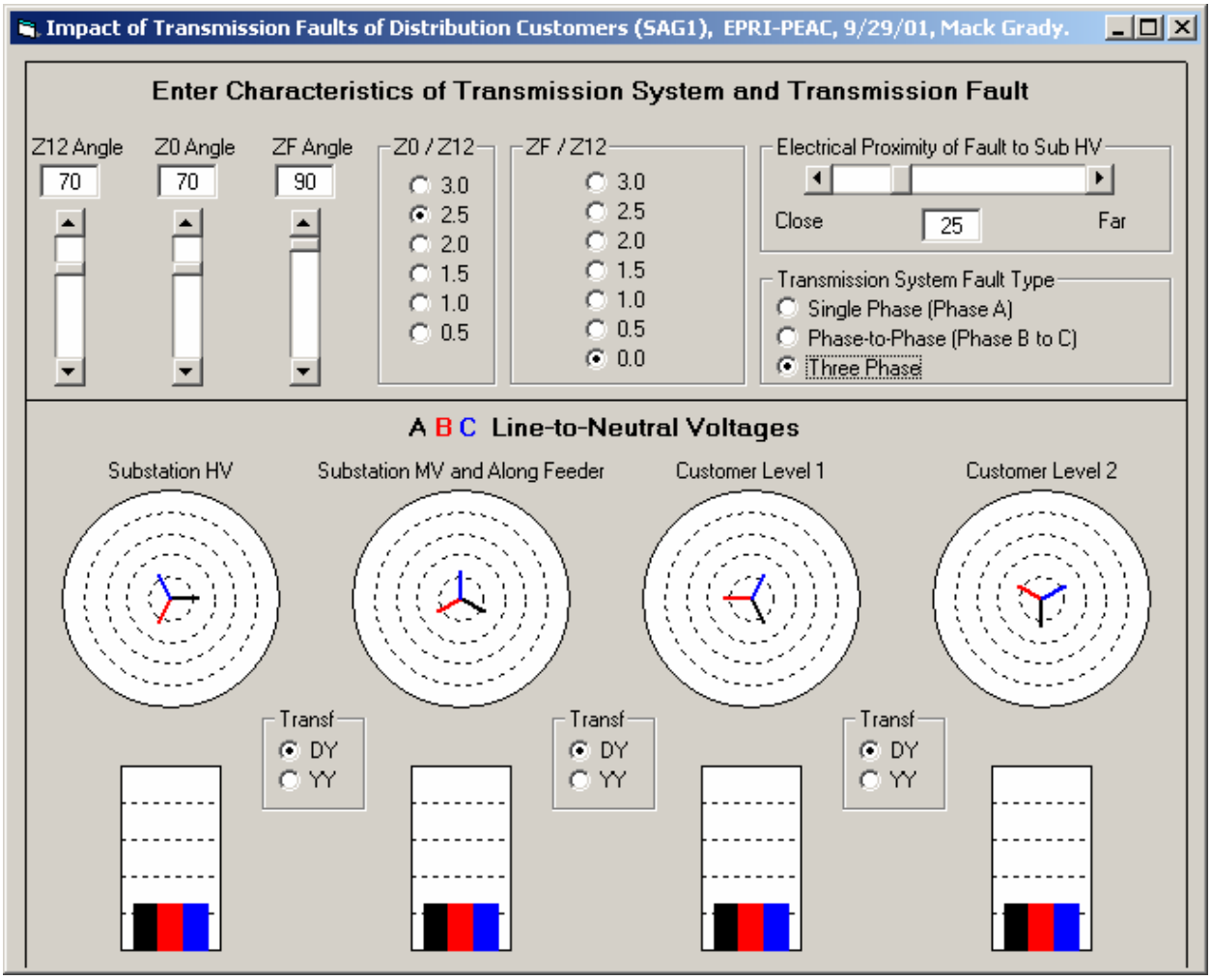


Figure 4. Propagation of Close-In Three-Phase Fault on the Transmission System (all three transformers have delta:grounded-wye connections)

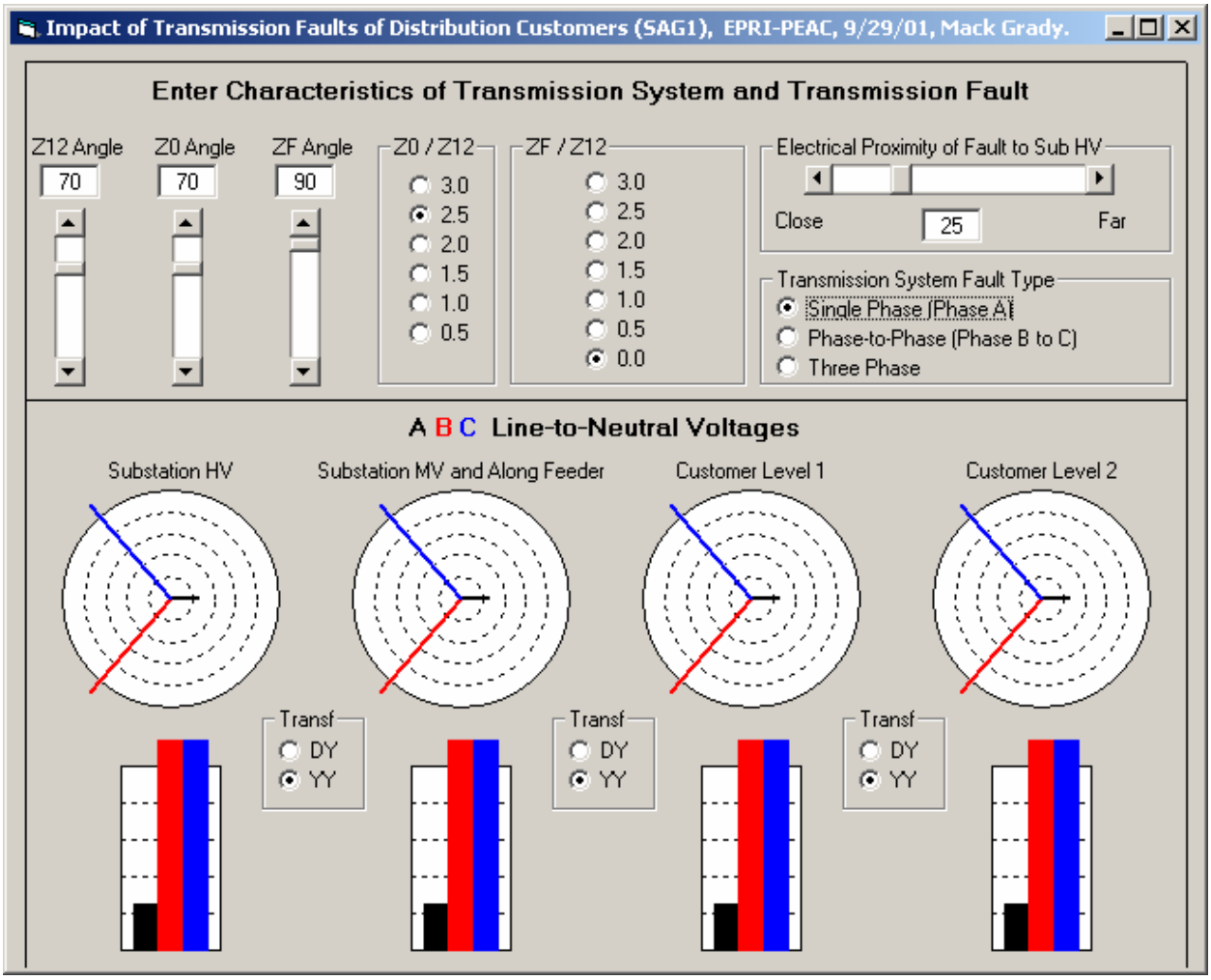


Figure 5. Situation in Figure 2 Repeated, but with all Three Transformers Having Grounded-Wye:Grounded-Wye Connections

3. Impact of Distribution System Faults on Adjacent Feeders

Now, consider the situation in Figure 6 where a fault occurs on an adjacent feeder, and a monitor records the voltage waveform at the substation 12.47kV bus.

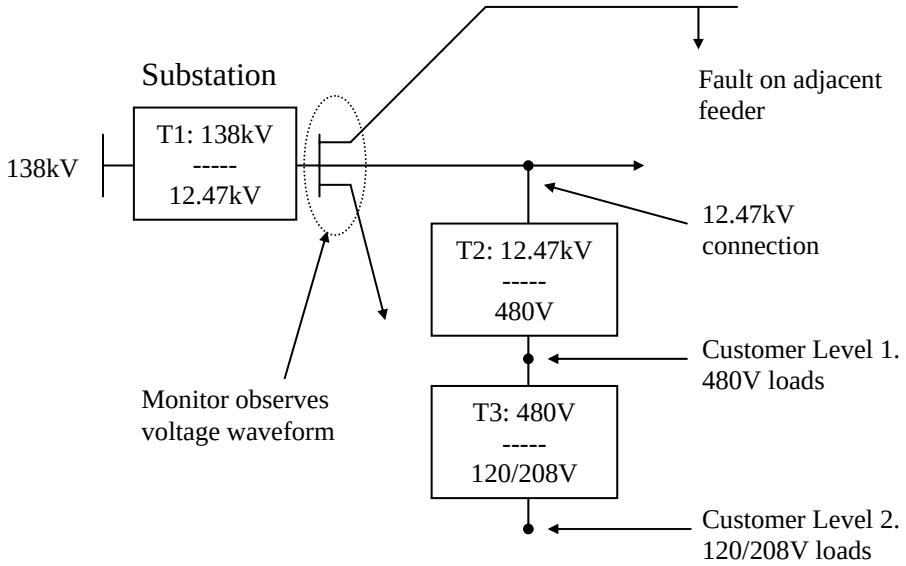


Figure 6. Substation Monitor Records Voltages when a Sag Occurs on an Adjacent Feeder.

As in Section 2, unless the customer's feeder has significant contributors to the fault current, the voltage sag at the substation 12.47kV bus will appear everywhere along the customer's 12.47kV feeder. However, to predict the voltage sag at Customer Levels 1 and 2, the a-b-c line-to-neutral voltages at the substation 12.47kV bus must be

- converted to positive/negative/zero-sequence components,
- shifted with the appropriate transformer phase shifts,
- converted back to a-b-c.

Ideally, the monitor should report phasor voltage magnitudes and angles during the sag condition, rather than time-sampled data points for which phase angles must be estimated.

The conversion/shifting/re-conversion procedure has been coded into a Visual Basic program and is illustrated in Figure 7 to illustrate the propagation of a sag due to a single-phase fault through two levels of delta:grounded-wye transformers inside a customer's facility. The substation 12.47kV voltage entered is approximately the same as predicted in Figure 2.

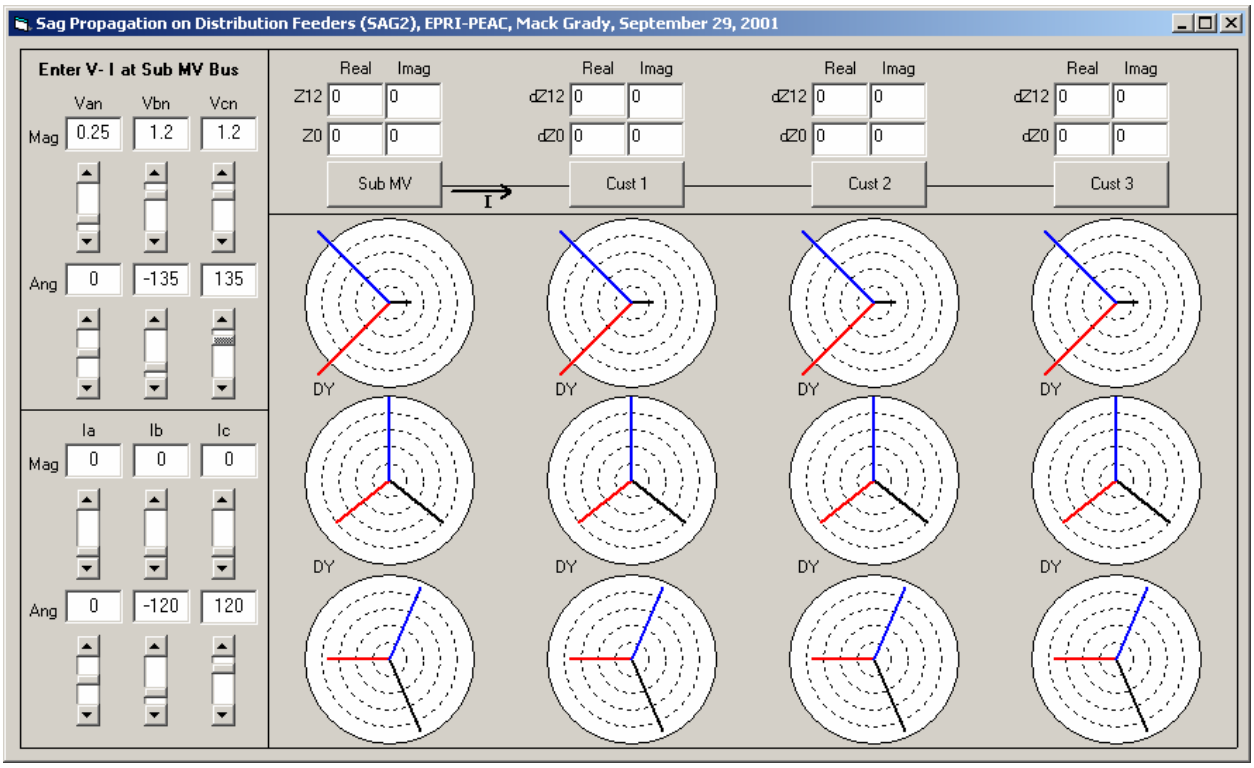


Figure 7. Propagation of Voltage Sag Due to Single-Phase Fault on Adjacent Feeder

Each vertical column in Figure 7 indicates the following voltages: 1. on the feeder, 2. after one level of delta:grounded-wye transformer, and 3. after a second level of delta:grounded-wye transformer. The four vertical columns correspond to various locations down the feeder, provided that sequence impedances of the feeder are entered. For Figure 7, all four columns are identical since no feeder currents and feeder impedances were entered.

4. Impact of Distribution System Faults on the Customer's Feeder

To analyze voltage sags on a faulted feeder, feeder currents must be known. Consider the situation in Figure 8. The Thevenin-equivalent impedance at Customer 1 is Z1, followed by incremental impedances dZ between the other customers that are distributed down the feeder.

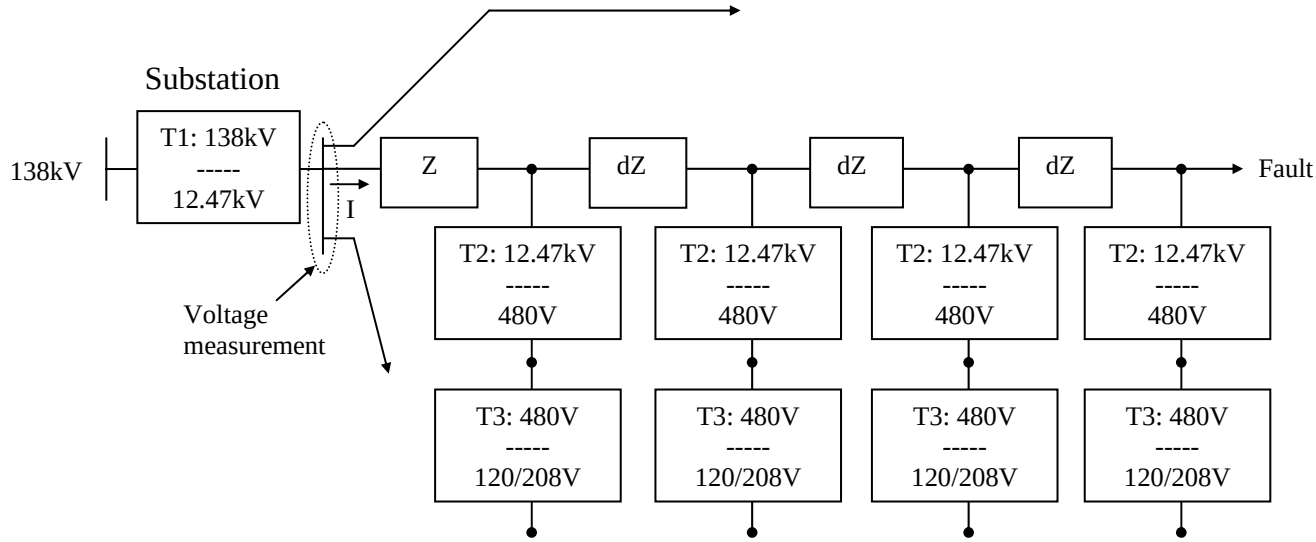


Figure 8. Model of Faulted Feeder with Four Distributed Customers

If the fault is to the right of the last customer (as shown), and if the customer load currents are small in relation to the measured fault I at the substation transformer, then the voltage drops along the feeder can be easily calculated using current times impedance, as illustrated in Figure 9. More study is needed, supplemented with field tests, to determine when and with what certainty the latter assumption can be made.

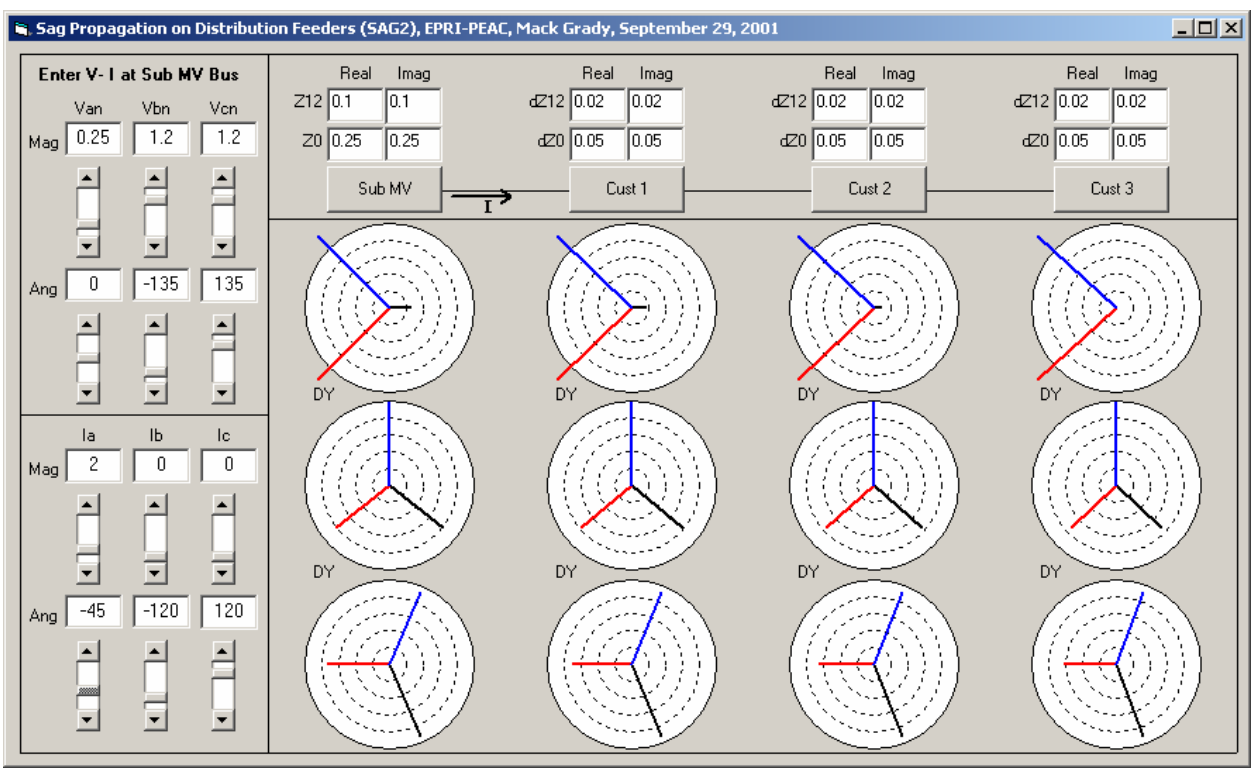


Figure 9. Propagation of Voltage Sag Due to Single-Phase Fault on Customer's Feeder

Homework Using SAG3.EXE

1. File EASTSH_1.TXT. The observations for a transmission fault are recorded on the MV side of a delta-wye substation transformer.

- Determine prefault phasors Van, Vbn, Vcn, Ia, Ib, Ic.
- Compute three-phase P, Q, and power factor.
- Determine phasors Van, Vbn, and Vcn during the steady-state portion of the fault.
- Use program SAG1.EXE to help determine the faulted phase (i.e., a, or b, or c), and possibly make other conclusions about the transmission fault.

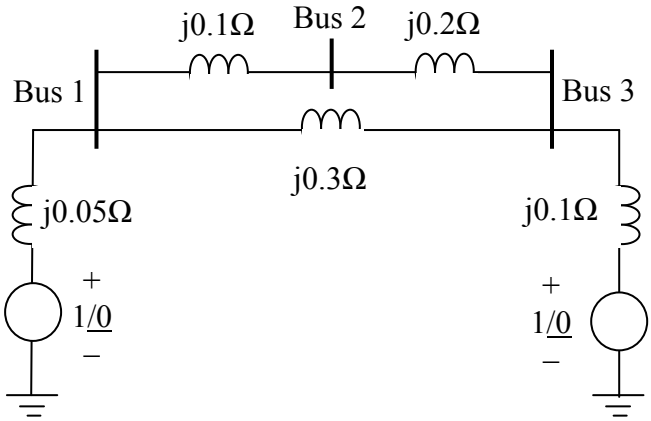
2. Repeat the above, using file AustinData.txt. Also,

- Use ABC012 to convert the pre-fault abc voltage and currents to 012 (polar form). Comment on the degree of balance. Do you notice anything unusual about the sequence components?
- what phenomena accounts for the change in current waveshape from pre-fault, during fault, and postfault?

3. File CAP_DIST.TXT. This is a distribution capacitor switching transient. Use voltage and current waveforms to estimate the frequency of the ringing transient. If the capacitor is 1800kVAr (three phase), what is the system inductance? Comment on the damping level in the underdamped ringing transient.

4. File CAP_TRANS.TXT. This is a transmission capacitor switching transient. Ignore the current signals. Estimate the frequency of the ringing transient. Comment on the damping level.

The positive-sequence one-line diagram for a network is shown below. The ground ties at busses 1 and 4 represent the subtransient impedances of machines. Prefault voltages are all 1.0pu.



	1	2	3
1			
2			
3			

a. Use the definition $z_{jk} = \dfrac{\partial V_j}{\partial I_k} = \left.\dfrac{V_j}{I_k}\right|_{I_m = 0, m \neq k}$ to fill in column 1 of the Z matrix.

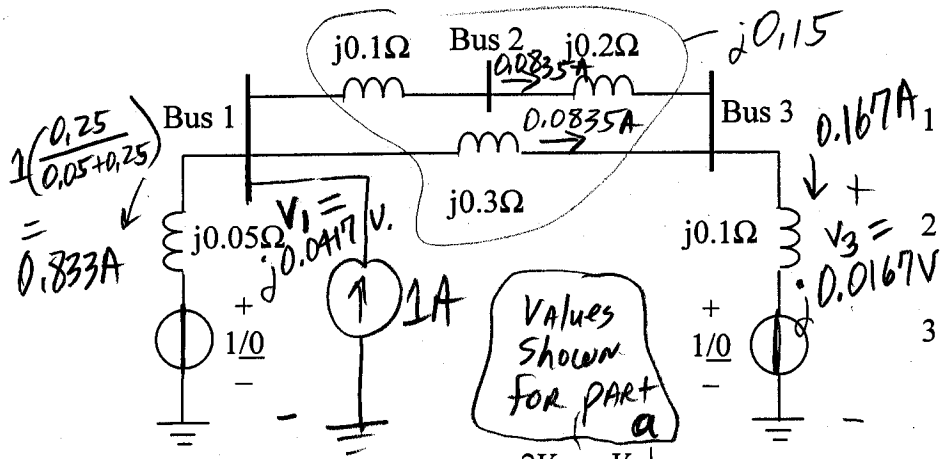
Now, a solidly-grounded three-phase fault occurs at bus 1.

b. Compute the fault current
c. Use the fault current and Z matrix terms to compute the voltages at busses 2 and 3.
d. Find the magnitude of the current flowing in the line connecting busses 2 and 3.

EE369, Final Exam, 2-5pm, Dec. 9, WRW102. Name: KEY

Six equally-weighted problems. Three sheets of notes permitted. Show all work.

Problem 2. The positive-sequence one-line diagram for a network is shown below. ~~The ground ties at busses 1 and 4 represent the subtransient impedances of machines~~. Prefault voltages are all 1.0pu.



$1\left(\frac{0.25}{0.05+0.25}\right) = 0.833A$

$V_1 = j0.0417$ V.

$V_3 = j0.0167$ V

1	2	3
$Z_{11} = j0.0417$		
$Z_{21} = j0.0334$		
$Z_{31} = j0.0167$		

Part a

a. Use the definition $z_{jk} = \frac{\partial V_j}{\partial I_k} = \frac{V_j}{I_k}\Big|_{I_m=0, m\neq k}$ to fill in column 1 of the Z matrix.

Now, a solidly-grounded three-phase fault occurs at bus 1.

Z_{11} Check, $j0.05 \| (j0.15 + j0.10) = j0.0417$ (OK)

b. Compute the fault current

c. Use the fault current and Z matrix terms to compute the voltages at busses 2 and 3.

d. Find the magnitude of the current flowing in the line connecting busses 2 and 3.

$V_2 = V_3 + j0.2\left(\frac{0.167}{2}\right) = j0.0334$

b. $I_{1a}^F = \frac{V_{1a}^{PRE}}{Z_{11}} = \frac{1}{j0.0417}$

$I_{1a}^F = -j24.0A$

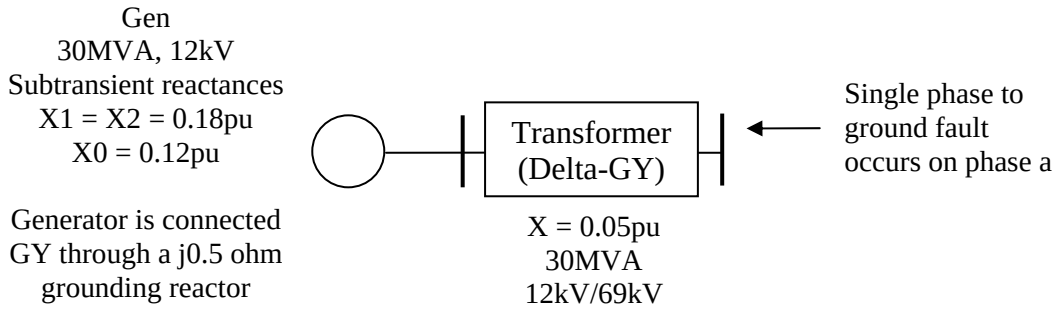
c. $V_{2a}^F = V_{2a}^{PRE} - Z_{21} I_{1a}^F = 1.0 - (j0.0334)(-j24) = 1.0 - 0.800 = 0.200V$

$V_{3a}^F = V_{3a}^{PRE} - Z_{31} I_{1a}^F = 1.0 - (j0.0167)(-j24) = 1.0 - 0.401 = 0.600V$

d. $I_{23}^F = \frac{V_2^F - V_3^F}{j0.2} = \frac{0.2 - 0.6}{j0.2} = j2.0$

A 30MVA, 12kV generator is connected to a delta - grounded wye transformer. The generator and transformer are isolated and not connected to a "power grid." Impedances are given on equipment bases.

A single-phase to ground fault, with zero impedance, suddenly appears on phase a of the 69kV transformer terminal. Find the resulting a-b-c generator currents (magnitude **in amperes** and phase). Regarding reference angle, assume that the pre-fault phase a voltage on the transformer's 69kV bus has angle = 0.



EE369, Test 9, Nov. 5. Name: KEY

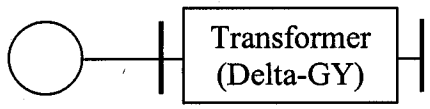
One sheet of notes permitted. Show all work.

A 30MVA, 12kV generator is connected to a delta - grounded wye transformer. The generator and transformer are isolated and not connected to a "power grid." Impedances are given on equipment bases.

A single-phase to ground fault, with zero impedance, suddenly appears on phase a of the 69kV transformer terminal. Find the resulting a-b-c generator currents (magnitude **in amperes** and phase). Regarding reference angle, assume that the pre-fault phase a voltage on the transformer's 69kV bus has angle = 0.

$$I_B = \frac{\frac{30\times10^6}{3}}{\frac{12\times10^3}{\sqrt{3}}} = \frac{10\times10^6}{6928} = 1443A$$

Gen
30MVA, 12kV
Subtransient reactances
X1 = X2 = 0.18pu
X0 = 0.12pu



Single phase to ground fault occurs on phase a

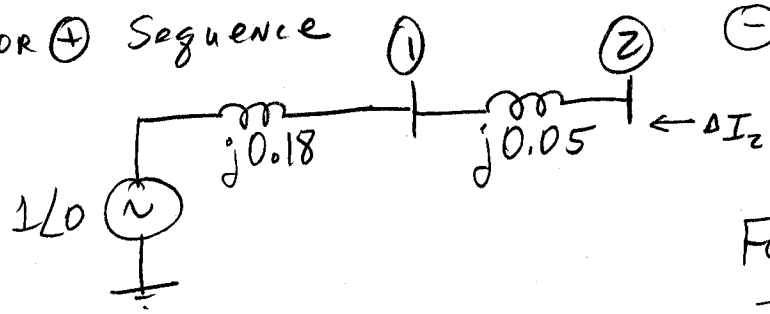
Generator is connected GY through a j0.5 ohm grounding reactor

X = 0.05pu
30MVA
12kV/69kV

$$Z_B = \frac{(12)^2}{30} = \frac{144}{30} = 4.8\Omega$$

So, $j0.5\Omega = j0.1042\Omega$

For ⊕ Sequence



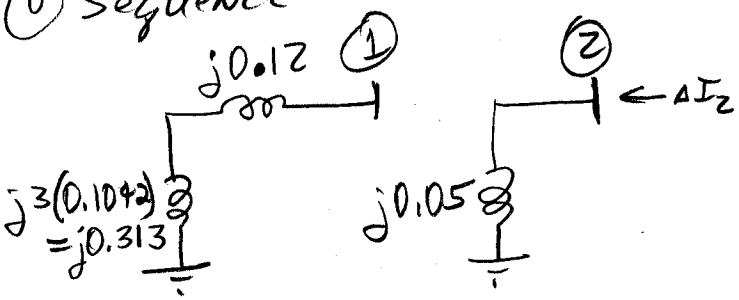
⊖ Sequence like ⊕, but source voltage is zero.

For ⊕ and ⊖ sequences,

$$Z_{22}^{(+,-)} = j(0.18 + 0.05) = j0.23$$

$$Z_{12}^{(+,-)} = \frac{\partial V_1}{\partial I_2} = \frac{j0.18\, I_2}{I_2} = j0.18$$

⓪ Sequence



For ⓪ Sequence

$$Z_{22}^{(0)} = j0.05. \quad Z_{12}^{(0)} = 0$$

Now, for the fault current,

$$I_{2a}^F = \frac{3V_{2a}^{pre}}{Z_{22}^{(0)} + Z_{22}^{(1)} + Z_{22}^{(2)} + 3Z_F^{(0)}} = \frac{3\angle 0}{j0.05 + 2j(0.23)} = -j5.88pu$$

For a single phase fault, we know that

$$I_{K0}^F = I_{K1}^F = I_{K2}^F = \frac{1}{3} I_{Ka}^F, \text{ so } I_{2,0}^F = I_{2,1}^F = I_{2,2}^F = \frac{-j5.88}{3}$$

$$= -j1.960pu$$

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Test 9, 11/5/04, continued

Now, ⊕ and ⊖ sequence currents are the same at the generator (except for phase shift) because we have a simple series path. Furthermore, the zero sequence current does not flow through the transformer. Thus, on the generator side, we have

ANSI shifts

$$I^F_{gen,1} = -j\,1.960\,(1\angle -30) = 1.960\angle -120^\circ$$

$$I^F_{gen,2} = -j\,1.960\,(1\angle +30) = 1.960\angle -60$$

$$I^F_{gen,0} = 0.$$

Converting to abc

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1.960\angle -120^\circ \\ 1.960\angle -60^\circ \end{bmatrix} = \begin{bmatrix} 1.960\angle -120^\circ + 1.960\angle -60^\circ \\ 1.960\angle 120^\circ + 1.960\angle 60^\circ \\ 1.960\angle 0 + 1.960\angle -180 \end{bmatrix}$$

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = 1.960 \begin{bmatrix} 1\angle -120 + 1\angle -60 \\ 1\angle 120 + 1\angle 60 \\ 0 \end{bmatrix} = 1.960 \begin{bmatrix} \sqrt{3}\angle -90 \\ \sqrt{3}\angle 90 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} -j\,3.39 \\ j\,3.39 \\ 0 \end{bmatrix}. \text{ In amps, } \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} -j\,4892 \\ j\,4892 \\ 0 \end{bmatrix}$$

Note – in the general case, you use the off-diagonal Z elements. Doing the problem in the way ...

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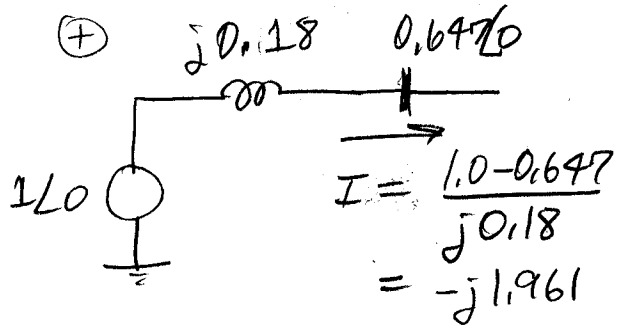
$$V^F_{1,0} = \cancel{V^{pre}_{1,0}}^{\,0} - \cancel{Z_{12,0}}^{\,0}\, I^F_{2,0} = 0$$

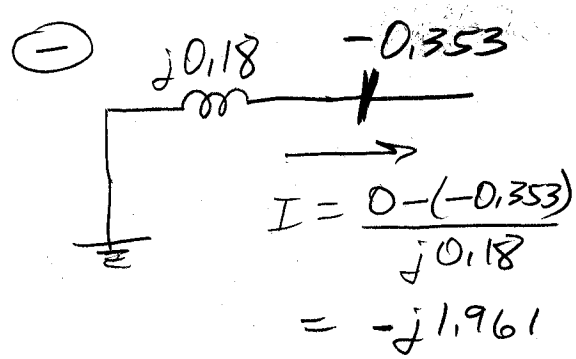
$$V^F_{1,1} = V^{pre}_{1,1} - Z_{12,1}\, I^F_{2,1} = 1.0 - j0.18(-j1.960)$$

$$= 1.0 - 0.353 = 0.647$$

$$V^F_{1,2} = \cancel{V^{pre}_{1,2}}^{\,0} - Z_{12,2}\, I^F_{2,2} = -j0.18(-j1.960) = -0.353$$

Then, using the network





You can see these currents match the previous ones. And so on.

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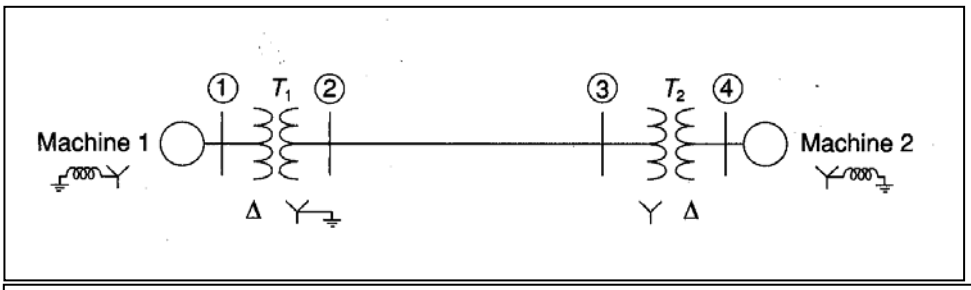
A one-line diagram for a two-machine system is shown below.



All values in pu on equipment base

Machines 1 and 2:	100 MVA	20 kV	$X_1 = X_2 = 0.2$
	$X_0 = 0.04$	$X_n = 0.05$	
Transformers T_1 and T_2:	100 MVA	20Δ/345Y kV	
	$X_1 = X_2 = X_0 = 0.08$		

The transmission line between busses 2 and 3 has X1 = X2 = 0.12pu, X0 = 0.40pu on a 100MVA, 345kV base.

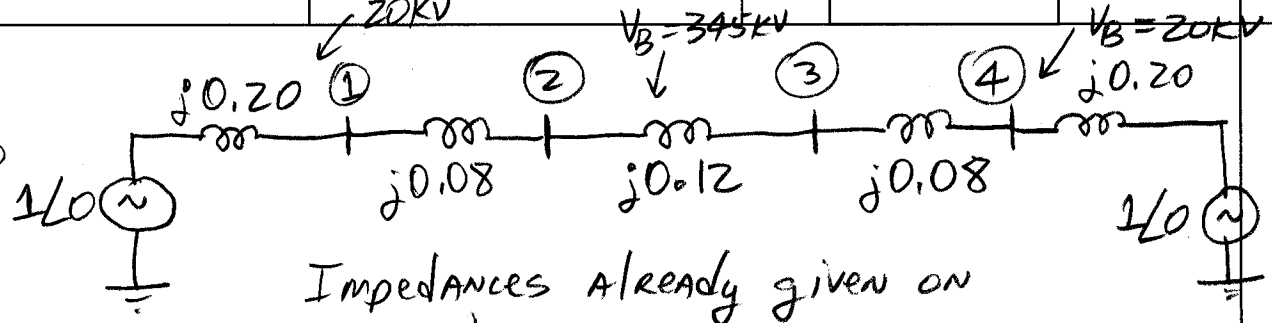
Using a base of 100MVA, 345kV in the transmission line, draw one line diagrams in per unit for positive, negative, and zero-sequences.

Using the results from above,

a. Compute the phase a fault current (in pu) for a three-phase bolted fault at bus 2.
b. Compute the phase a fault current (in pu) for a line-to-ground fault at bus 2, phase a.

#1

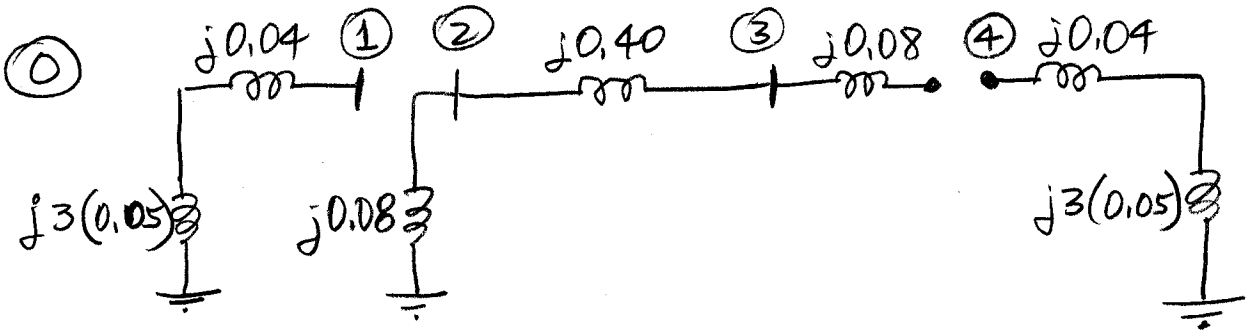
(+)



Impedances already given on system base.

(−) Like above, but with voltage sources turned off (i.e., shorted through)

(0)



#2

(a) $Z_{22}^{+} = Z_{22}^{-} = (j0.20 + j0.08) \,||\, (j0.12 + j0.08 + j0.20)$

$= j0.28 \,||\, j0.40 = j0.1647 \text{ pu}$

$$I_{2a}^{F} = \frac{1\angle 0}{j0.1647} = -j6.07 \text{ pu}$$

(b) $Z_{22}^{0} = j0.08$ (there is no path to ground going right)

$$I_{2a}^{F} = \frac{3\angle 0}{Z_{22}^{0} + Z_{22}^{+} + Z_{22}^{-} + 3Z_F^{0}} = \frac{3}{j(0.08 + 0.1647 + 0.1647)}$$

$$I_{2a}^{F} = \frac{3}{j0.409} = -j7.33 \text{ pu}$$

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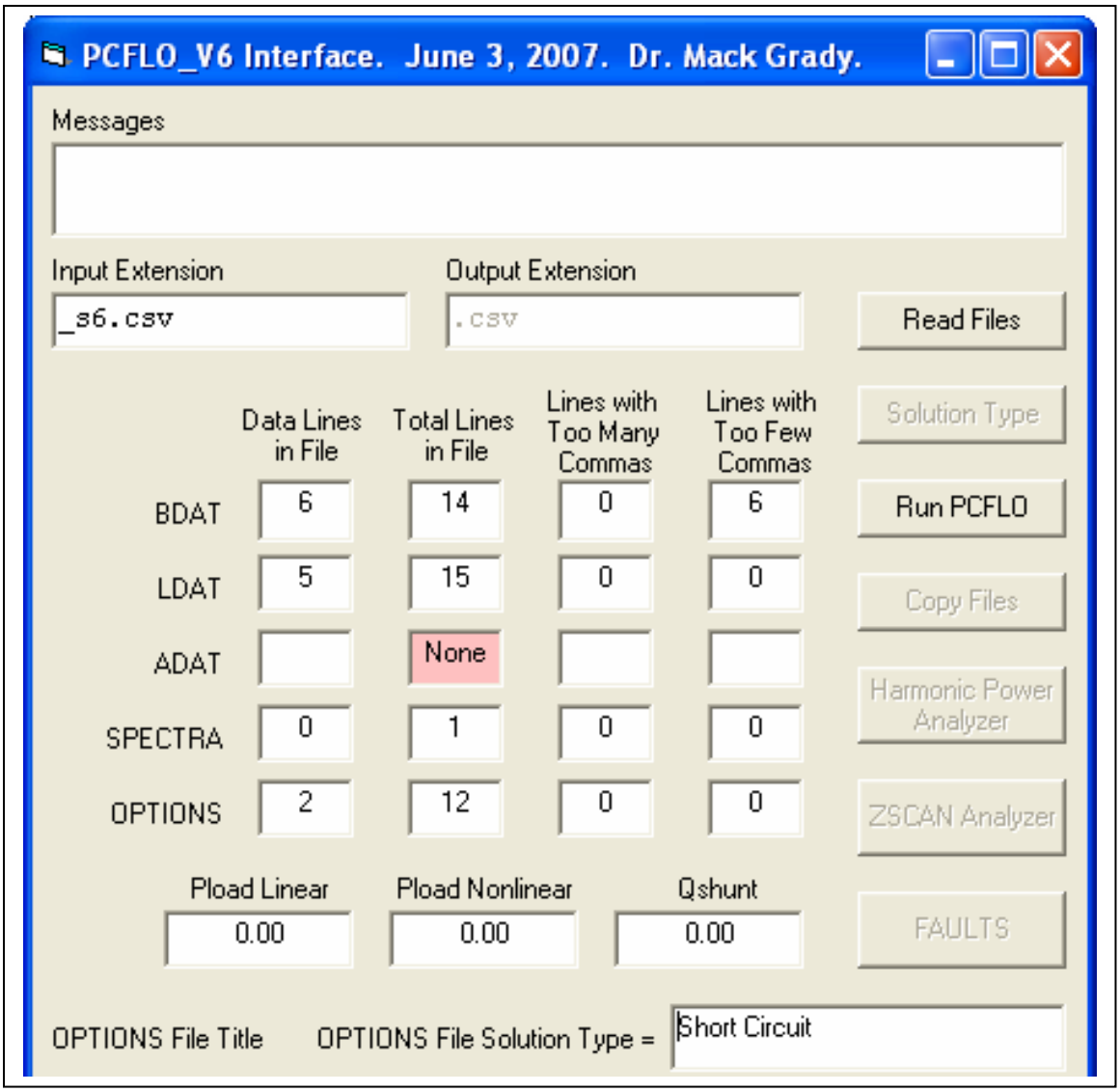
STAEDTLER® No. 937 811E Engineer's Computation Pad

Stevenson Six-Bus Short Circuit Example. Fault at Bus 4.



```
u.t. austin power system engineering                                          , faults version =      6.0
 capabilities =  5000 busses, 12500 lines and transformers
 "

 ---------------------------------------------------------------------------
 three phase fault at bus =     4, name = bus#4
 (per unit impedances in rectangular form)
 (per unit voltages and currents in polar form)

 012 system impedance (pu) =
    0.00000E+00   0.15248E+00
   -0.51938E-08   0.22411E+00
    0.51938E-08   0.22411E+00

 fault impedance (pu) =    0.00000E+00   0.00000E+00

 012 voltage =        0.00000    0.0       0.00000    0.0       0.00000    0.0
 abc voltage =        0.00000    0.0       0.00000    0.0       0.00000    0.0

 012 current =        0.00000    0.0       4.46200  -90.0       0.00000    0.0
 abc current =        4.46200  -90.0       4.46200  150.0       4.46200   30.0

 from subtransient impedance
 012 current =        0.00000    0.0       0.00000    0.0       0.00000    0.0
 abc current =        0.00000    0.0       0.00000    0.0       0.00000    0.0

 at neighboring bus =     1, name = gen#1
 012 voltage =        0.00000    0.0       0.28571  -30.0       0.00000    0.0
 abc voltage =        0.28571  -30.0       0.28571 -150.0       0.28571   90.0
 fault contribution from circuit =  1 at bus =     4
 012 current =        0.00000    0.0       1.42857  -90.0       0.00000    0.0
 abc current =        1.42857  -90.0       1.42857  150.0       1.42857   30.0
 v-i impedance ratio for circuit =  1 at bus =     1
    012 impedance ratio =            abc impedance ratio =
    0.00000E+00   0.00000E+00        0.17321E+00   0.10000E+00
    0.17321E+00   0.10000E+00        0.17321E+00   0.10000E+00
    0.00000E+00   0.00000E+00        0.17321E+00   0.10000E+00
```

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```
at neighboring bus =     5, name = bus#5
012 voltage =        0.00000    0.0       0.25056    0.0       0.00000    0.0
abc voltage =        0.25056    0.0       0.25056 -120.0       0.25056  120.0
fault contribution from circuit =  1 at bus =     4
012 current =        0.00000    0.0       3.03343  -90.0       0.00000    0.0
abc current =        3.03343  -90.0       3.03343  150.0       3.03343   30.0
v-i impedance ratio for circuit =  1 at bus =     5
   012 impedance ratio =            abc impedance ratio =
   0.00000E+00   0.00000E+00       -0.21386E-15   0.82600E-01
   0.51241E-08   0.82600E-01        0.65021E-08   0.82600E-01
   0.00000E+00   0.00000E+00        0.88702E-08   0.82600E-01

end of three phase fault report

-------------------------------------------------------------------------
line-to-line fault at bus =     4, name = bus#4
(per unit impedances in rectangular form)
(per unit voltages and currents in polar form)

012 system impedance (pu) =
   0.00000E+00   0.15248E+00
  -0.51938E-08   0.22411E+00
   0.51938E-08   0.22411E+00

fault impedance (pu) =    0.00000E+00   0.00000E+00

012 voltage =        0.00000    0.0       0.50000    0.0       0.50000    0.0
abc voltage =        1.00000    0.0       0.50000 -180.0       0.50000  180.0

012 current =        0.00000    0.0       2.23100  -90.0       2.23100   90.0
abc current =        0.00000    0.0       3.86421  180.0       3.86421    0.0

from subtransient impedance
012 current =        0.00000    0.0       0.00000    0.0       0.00000    0.0
abc current =        0.00000    0.0       0.00000    0.0       0.00000    0.0

at neighboring bus =     1, name = gen#1
012 voltage =        0.00000    0.0       0.64286  -30.0       0.35714   30.0
```

```
abc voltage =        0.87773   -9.4       0.87773 -170.6       0.28571   90.0
fault contribution from circuit =  1 at bus =     4
012 current =        0.00000    0.0       0.71429  -90.0       0.71429   90.0
abc current =        0.00000    0.0       1.23718 -180.0       1.23718    0.0
v-i impedance ratio for circuit =  1 at bus =     1
   012 impedance ratio =            abc impedance ratio =
   0.00000E+00   0.00000E+00        0.12124E+01  -0.20000E+00
   0.77942E+00   0.45000E+00        0.60622E+00   0.10000E+00
   0.43301E+00  -0.25000E+00        0.84905E-07   0.40000E+00

at neighboring bus =     5, name = bus#5
012 voltage =        0.00000    0.0       0.62528    0.0       0.37472    0.0
abc voltage =        1.00000    0.0       0.54506 -156.5       0.54506  156.5
fault contribution from circuit =  1 at bus =     4
012 current =        0.00000    0.0       1.51672  -90.0       1.51672   90.0
abc current =        0.00000    0.0       2.62703 -180.0       2.62703    0.0
v-i impedance ratio for circuit =  1 at bus =     5
   012 impedance ratio =            abc impedance ratio =
   0.00000E+00   0.00000E+00        0.00000E+00   0.00000E+00
   0.22552E-07   0.41226E+00        0.19033E+00   0.82600E-01
  -0.29754E-08  -0.24706E+00       -0.19033E+00   0.82600E-01

end of line-to-line fault report

-------------------------------------------------------------------------------
line-to-ground fault at bus =     4, name = bus#4
(per unit impedances in rectangular form)
(per unit voltages and currents in polar form)

012 system impedance (pu) =
   0.00000E+00   0.15248E+00
  -0.51938E-08   0.22411E+00
   0.51938E-08   0.22411E+00

fault impedance (pu) =    0.00000E+00   0.00000E+00

012 voltage =        0.25383  180.0       0.62691    0.0       0.37309  180.0
abc voltage =        0.00000    0.0       0.94603 -113.7       0.94603  113.7
```

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```
012 current =        1.66471  -90.0       1.66471  -90.0       1.66471  -90.0
abc current =        4.99412  -90.0       0.00000    0.0       0.00000    0.0

from subtransient impedance
012 current =        0.00000    0.0       0.00000    0.0       0.00000    0.0
abc current =        0.00000    0.0       0.00000    0.0       0.00000    0.0

at neighboring bus =     1, name = gen#1
012 voltage =        0.00000    0.0       0.73351  -30.0       0.26649 -150.0
abc voltage =        0.64310  -51.0       0.64310 -129.0       1.00000   90.0
fault contribution from circuit =  1 at bus =     4
012 current =        1.26915  -90.0       0.53298  -90.0       0.53298  -90.0
abc current =        2.33511  -90.0       0.73617  -90.0       0.73617  -90.0
v-i impedance ratio for circuit =  1 at bus =     1
   012 impedance ratio =            abc impedance ratio =
   0.00000E+00   0.00000E+00        0.54163E+00   0.43812E+00
   0.11919E+01   0.68812E+00        0.00000E+00   0.00000E+00
   0.43301E+00  -0.25000E+00        0.10833E+01  -0.39620E-07

at neighboring bus =     5, name = bus#5
012 voltage =        0.16799  180.0       0.72040    0.0       0.27960  180.0
abc voltage =        0.27280    0.0       0.94913 -114.2       0.94913  114.2
fault contribution from circuit =  1 at bus =     4
012 current =        0.39556  -90.0       1.13173  -90.0       1.13173  -90.0
abc current =        2.65902  -90.0       0.73617   90.0       0.73617   90.0
v-i impedance ratio for circuit =  1 at bus =     5
   012 impedance ratio =            abc impedance ratio =
  -0.47320E-24  -0.42470E+00        0.17018E-24   0.10259E+00
   0.15477E-07   0.63654E+00       -0.11764E+01   0.52758E+00
  -0.29754E-08  -0.24706E+00        0.11764E+01   0.52758E+00

end of line-to-ground fault report
```

Loadflow

Formulation of the loadflow problem. Gauss-Seidel, Newton-Raphson, and Stott's algorithm. Calculation of line flows, system losses, and area interchange.

1 Formulation of the Problem

The loadflow problem is one of the classic power system engineering problems. During the early days of digital computers, many advances in techniques for solving large sets of equations were brought about specifically to help solve the loadflow problem.

In most electrical circuit analyses, the network consists of known impedances, voltage sources, and current sources. However, in the loadflow problem, active and reactive powers, rather than shunt impedances, are specified at most network busses, because most loads behave, on average, as constant power loads (active and reactive power), as long as their applied voltage remains within reasonable ranges. Consider, for example, the air conditioning load of a building. A certain amount of energy is required to maintain ΔT between inside and outside temperatures. Even though the air conditioner cycles on-and-off, and the voltage may change slightly, the air conditioning load appears, on the average, as a fixed power load, rather than as a fixed impedance load.

Power system loads are closely monitored at substations, at large customers, and for total electric utility companies. Loads tend to have predictable daily, weekly, and seasonal patterns. Annual peak demands and energies for electric utilities are forecasted for generation and planning purposes.

The purpose of the loadflow program is to compute bus voltages and line/transformer/cable power flows once network topology, impedances, loads, and generators have been specified. Ideally, the computed bus voltages for the study system should remain within acceptable ranges, and line/transformer/cable power flows should be below their rated values, for a reasonable set of outage contingencies.

From a loadflow perspective, there are four parameters at every bus - voltage magnitude V, voltage angle δ, active power P, and reactive power Q. Two may be specified, and the other two calculated. For most busses, P and Q are specified, and V and δ are calculated. Obviously, P and Q cannot be specified at all busses because that would imply that system losses are known a priori. Therefore, the loadflow problem must include one "swing bus" at which the P can assume any value so that it "makes up" system losses. The swing bus is usually a centrally-located large generator whose voltage magnitude and phase angle (usually δ = zero) are specified.

Although any two of the four parameters can be specified, the usual way in which power system busses are classified is given in Table 1.

Table 1. Loadflow Classification of Power System Busses

Classification	Knowns	Unknowns
PQ (Load Bus)	P, Q	V, δ
PV (Generator Bus)	P, V	Q, δ
Vδ (Swing Bus)	V, δ	P, Q

The loadflow program solves for the set of unknowns that produces power balance at all busses, or as illustrated for bus *i* in Figure 1,

$$P_i^{spec} + jQ_i^{spec} = P_i^{calc} + jQ_i^{calc} \ ,$$

where

$$P_i^{calc} + jQ_i^{calc} = V_i I_i^* \ .$$

In other words, the power specified at each bus must equal the power flowing into the system. Note in Figure 1 that specified power is drawn as positive generation, to be consistent with KCL equation *YV=I*.

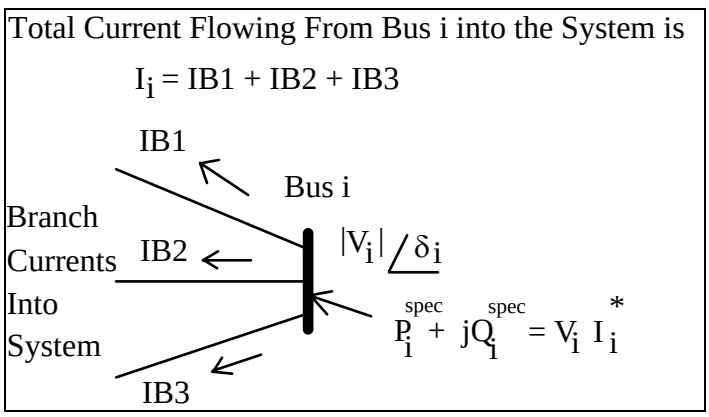


Figure 1. Power Balance for Bus *i*

Since there are two unknowns at every bus, the size of the loadflow problem is 2*N*, where *N* is the number of busses. Obviously, to solve the problem, there must be two equations for every bus. These come from KCL, which for any bus *i* have the form

$$P_i^{spec} + jQ_i^{spec} = P_i^{calc} + jQ_i^{calc} = V_i I_i^* = V_i \left[\sum_{j=1}^{N} y_{i,j} V_j \right]^* \ .$$

Separating into real and imaginary components yields two equations for bus *i*,

$$P_i^{spec} = \sum_{j=1}^{N} |V_i| |y_{i,j}| |V_j| \cos\left(\delta_i - \delta_j - \theta_{i,j}\right),$$

$$Q_i^{spec} = \sum_{j=1}^{N} |V_i| |y_{i,j}| |V_j| \sin\left(\delta_i - \delta_j - \theta_{i,j}\right),$$

where $V_i = |V_i| \angle \delta_i$, $V_i = |V_i| \angle \delta_i$, $y_{i,j} = |y_{i,j}| \angle \theta_{i,j}$.

The problem now is now to find the set of bus voltages that satisfies the above 2*N* equations.

2. Gauss-Seidel Method

Gauss-Seidel is an early formulation of the loadflow problem that requires little memory and it is easily programmed. However, it is usually slower than other methods. It is based upon the idea of expanding the complex form of the power balance equation as follows:

$$P_i^{spec} + jQ_i^{spec} = V_i I_i^* = V_i \left[\sum_{j=1}^{N} y_{i,j} V_j \right]^* = V_i \left[y_{i,i} V_i + \sum_{j=1,\, j \neq i}^{N} y_{i,j} V_j \right]^*,$$

or

$$P_i^{spec} - jQ_i^{spec} = V_i^* y_{i,i} V_i + V_i^* \sum_{j=1,\, j \neq i}^{N} y_{i,j} V_j ,$$

so that

$$V_i = \frac{1}{y_{i,i}} \left[\frac{P_i^{spec} - jQ_i^{spec}}{V_i^*} - \sum_{j=1,\, j \neq i}^{N} y_{i,j} V_j \right].$$

The solution procedure is to:

1. Initialize the bus voltages. For load busses, use $V = 1 + j0$. For generator busses (including the swing bus), use $V = V_{spec} + j0$.

2. One-by-one, update the individual bus voltages using
$$V_i = \frac{1}{y_{i,i}} \left[\frac{P_i^{spec} - jQ_i^{spec}}{V_i^*} - \sum_{j=1,\, j \neq i}^{N} y_{i,j} V_j \right].$$

   For PV busses, update the voltage angle, while holding the voltage magnitude constant at the specified value. Do not update the swing bus.

3. Check the mismatch P and Q at each bus. If all are within tolerance (typical tolerance is 0.00001 pu), a solution has been found. Otherwise, return to Step 2.

Convergence is usually faster if an acceleration factor is used. For example, assume that the voltage at bus i at iteration m is V_i^m , and that the updating equation in Step 2 computes V_j^{new} . Instead of using V_j^{new} directly, accelerate the update with

$$V_i^{m+1} = V_i^m + \alpha\left(V_i^{new} - V_i^m\right),$$

where acceleration factor α is in the range of 1.2 to 1.6.

3 Newton-Raphson Method

The Newton-Raphson method is a very powerful loadflow solution technique that incorporates first-derivative information when computing voltage updates. Normally, only 3 to 5 iterations are required to solve the loadflow problem, regardless of system size. Newton-Raphson is the most commonly used loadflow solution technique.

An easy way to illustrate the Newton-Raphson technique is to solve a simple equation whose answers are already known. For example, consider

$$(x-1)(x-99) = 0\,,$$

which when expanded becomes

$$x^2 - 100x + 99 = 0\,.$$

The objective is to find x so that

$$f(x) = x^2 - 100x + 99 = 0\,.$$

Of course, in this case, the two solutions are known a priori as $x = 1$, and $x = 99$.

The Newton-Raphson procedure is based on Taylor's expansion, truncated past the first derivative, which gives

$$f(x+\Delta x) \approx f(x) + \left.\frac{\partial f(x)}{\partial x}\right|_x (\Delta x)\,.$$

Clearly, the above equation gives a straight-line approximation for $f(x+\Delta x)$.

The objective is to find Δx so that $f(x+\Delta x)$ is the desired value (which in this example is zero). Solving for Δx yields

$$\Delta x = \frac{f(x+\Delta x) - f(x)}{\left.\frac{\partial f(x)}{\partial x}\right|_x} \ ,$$

which for this example is

$$\Delta x = \frac{0 - f(x)}{\left.\frac{\partial f(x)}{\partial x}\right|_x} = \frac{-f(x)}{\left.\frac{\partial f(x)}{\partial x}\right|_x} \ .$$

The update equation for iteration ($m + 1$) is then

$$x^{(m+1)} = x^{(m)} + \Delta x = x^{(m)} - \frac{f(x^{(m)})}{\left.\frac{\partial f(x)}{\partial x}\right|_{x^{(m)}}} \ ,$$

where in this example

$$\frac{\partial f(x)}{\partial x} = 2x - 100 \ .$$

If a starting point of $x = 2$ is chosen, then the solution proceeds as follows:

Iteration - m	x	$f(x)$	$\frac{\partial f(x)}{\partial x}$
0	2	-97	-96
1	0.9896	1.0193	-98.02
2	0.9999	0.0098	-98.00

Additional iterations can be performed if tighter solution tolerance is needed. Note that if a starting point of $x = 50$ had been chosen, the partial derivative would have been zero, and the method would have failed.

If $x = 80$ is the starting point, then the process yields the following:

Iteration - m	x	$f(x)$	$\frac{\partial f(x)}{\partial x}$
0	80	-1501	60
1	105.02	626.2	110.04
2	99.33	32.45	98.66
3	99.00	0	98.00

Therefore, the starting point greatly affects the ability of a Newton-Raphson method to converge, and the answer to which it converges. Fortunately, in the loadflow problem, most voltages are near 1.0 pu in magnitude and 0.0 degrees, so that we are able to accurately estimate starting values.

For the loadflow problem, the Newton-Raphson method is expanded in matrix form. For example, consider a set of *N* nonlinear equations and *N* unknowns,

$$\begin{array}{rcl} f_1(x_1, x_2, \cdots, x_N) & = & y_1 \\ f_2(x_1, x_2, \cdots, x_N) & = & y_2 \\ \vdots & \vdots & \vdots \\ f_N(x_1, x_2, \cdots, x_N) & = & y_N \end{array} .$$

The task is to find the set of unknown $x_1, x_2, \cdots, x_N$, given the known set $y_1, y_2, \cdots, y_N$, and given a starting point $x_1^{(0)}, x_2^{(0)}, \cdots, x_N^{(0)}$.

Applying Taylor's theorem as before, truncated after the first derivative, yields for Row *i*

$$y_i \approx f_i\left(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \cdots, x_N^{(0)} + \Delta x_N\right) = f_i\left(x_1^{(0)}, x_2^{(0)}, \cdots, x_N^{(0)}\right) +$$

$$+ \Delta x_1 \left.\frac{\partial f_i}{\partial x_1}\right|_{x(0)} + \Delta x_2 \left.\frac{\partial f_i}{\partial x_2}\right|_{x(0)} + \cdots + \Delta x_N \left.\frac{\partial f_i}{\partial x_N}\right|_{x(0)} ,$$

where *x(0)* represents the starting estimate set $x_1^{(0)}, x_2^{(0)}, \cdots, x_N^{(0)}$.

N similar equations in matrix form are

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_N \end{bmatrix} + \begin{bmatrix} f_1\left(x_1^{(0)}, x_2^{(0)}, \cdots, x_N^{(0)}\right) \\ f_2\left(x_1^{(0)}, x_2^{(0)}, \cdots, x_N^{(0)}\right) \\ \vdots \\ f_N\left(x_1^{(0)}, x_2^{(0)}, \cdots, x_N^{(0)}\right) \end{bmatrix} ,$$

or

$$\begin{bmatrix} y_1 - f_1\left(x_1^{(0)}, x_2^{(0)}, \cdots, x_N^{(0)}\right) \\ y_2 - f_2\left(x_1^{(0)}, x_2^{(0)}, \cdots, x_N^{(0)}\right) \\ \vdots \\ y_N - f_N\left(x_1^{(0)}, x_2^{(0)}, \cdots, x_N^{(0)}\right) \end{bmatrix} \approx [J] \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_N \end{bmatrix},$$

where J is an N x N matrix of partial derivatives, known as the Jacobian matrix. Therefore, in an iterative procedure, the above equation is used to update the X vector according to

$$X^{(m+1)} = X^{(m)} + \Delta X^{(m)} = X^{(m)} + \left[J^{(m)}\right]^{-1}\left[Y_{spec} - Y^{(m)}\right],$$

where

$$Y_{spec} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, Y^{(m)} = \begin{bmatrix} f_1\left(x_1^{(m)}, x_2^{(m)}, \cdots, x_N^{(m)}\right) \\ f_2\left(x_1^{(m)}, x_2^{(m)}, \cdots, x_N^{(m)}\right) \\ \vdots \\ f_N\left(x_1^{(m)}, x_2^{(m)}, \cdots, x_N^{(m)}\right) \end{bmatrix}.$$

In the loadflow problem, the matrix update equation is symbolically written in mixed rectangular-polar form as

$$\begin{bmatrix} P_{1spec} - P_{1calc}^{(m)} \\ P_{2spec} - P_{2calc}^{(m)} \\ \vdots \\ P_{Nspec} - P_{Ncalc}^{(m)} \\ --- \\ Q_{1spec} - Q_{1calc}^{(m)} \\ Q_{2spec} - Q_{2calc}^{(m)} \\ \vdots \\ Q_{Nspec} - Q_{Ncalc}^{(m)} \end{bmatrix} = \begin{bmatrix} J_1 = \dfrac{\partial P}{\partial \delta} & J_2 = \dfrac{\partial P}{\partial |V|} \\ J_3 = \dfrac{\partial Q}{\partial \delta} & J_4 = \dfrac{\partial Q}{\partial |V|} \end{bmatrix} \begin{bmatrix} (\delta_1)^{(m+1)} - (\delta_1)^{(m)} \\ (\delta_2)^{(m+1)} - (\delta_2)^{(m)} \\ \vdots \\ (\delta_N)^{(m+1)} - (\delta_N)^{(m)} \\ --- \\ |V_1|^{(m+1)} - |V_1|^{(m)} \\ |V_2|^{(m+1)} - |V_2|^{(m)} \\ \vdots \\ |V_N|^{(m+1)} - |V_N|^{(m)} \end{bmatrix}$$

or, in abbreviated form,

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta |V| \end{bmatrix}.$$

The dimension of the above problem is actually $\left(2N - \left(\text{Number of PV busses}\right) - 2\right)$ since $|V|$ updates at PV busses are not required, and since $|V|$ and δ updates at the swing bus are not required.

In highly inductive power systems, P is closely related to voltage angles, and Q is closely related to voltage magnitudes. Therefore, in the above mixed rectangular-polar formulation, the terms in J_1 and J_4 tend to have larger magnitudes than those in J_2 and J_3. This feature makes the Jacobian matrix more diagonally dominant, which improves robustness when Gaussian eliminating or LU decomposing *J*.

The above formulation of the Jacobian matrix is often modified to take advantage of symmetry in the partial derivatives. This modification is

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} \frac{1}{|V|}\frac{\partial P}{\partial \delta} & \frac{\partial P}{\partial |V|} \\ \frac{1}{|V|}\frac{\partial Q}{\partial \delta} & \frac{\partial Q}{\partial |V|} \end{bmatrix} \begin{bmatrix} |V|\Delta\delta \\ \Delta |V| \end{bmatrix} .$$

The partial derivatives are derived from

$$P_i^{calc} = \sum_{j=1}^{N} |V_i| |y_{i,j}| |V_j| \cos\left(\delta_i - \delta_j - \theta_{i,j}\right) ,$$

$$Q_i^{calc} = \sum_{j=1}^{N} |V_i| |y_{i,j}| |V_j| \sin\left(\delta_i - \delta_j - \theta_{i,j}\right) ,$$

and have the following form

For J_1 ,

$$\frac{\partial P_i}{\partial \delta_i} = -\sum_{j=1,\, j\neq i}^{N} |V_i| |y_{i,j}| |V_j| \sin\left(\delta_i - \delta_j - \theta_{i,j}\right)$$

$$\frac{\partial P_i}{\partial \delta_k} = |V_i| |y_{i,k}| |V_k| \sin\left(\delta_i - \delta_k - \theta_{i,k}\right), k \neq i \ .$$

For J_2 ,

$$\frac{\partial P_i}{\partial |V_i|} = \sum_{j=1,\, j\neq i}^{N} |y_{i,j}||V_j|\cos(\delta_i - \delta_j - \theta_{i,j}) + 2|V_i||y_{i,i}|\cos(-\theta_{i,i})$$

$$\frac{\partial P_i}{\partial |V_k|} = |V_i||y_{i,k}|\cos(\delta_i - \delta_k - \theta_{i,k}),\ k \neq i \ .$$

For J_3 ,

$$\frac{\partial Q_i}{\partial \delta_i} = \sum_{j=1,\, j\neq i}^{N} |V_i||y_{i,j}||V_j|\cos(\delta_i - \delta_j - \theta_{i,j})$$

$$\frac{\partial Q_i}{\partial \delta_k} = -|V_i||y_{i,k}||V_k|\cos(\delta_i - \delta_k - \theta_{i,k}),\ k \neq i \ .$$

For J_4 ,

$$\frac{\partial Q_i}{\partial |V_i|} = \sum_{j=1,\, j\neq i}^{N} |y_{i,j}||V_j|\sin(\delta_i - \delta_j - \theta_{i,j}) + 2|V_i||y_{i,i}|\sin(-\theta_{i,i})$$

$$\frac{\partial Q_i}{\partial |V_k|} = |V_i||y_{i,k}|\sin(\delta_i - \delta_k - \theta_{i,k}),\ k \neq i \ .$$

Note the symmetry in the J terms. If $|V|\delta$ is used as an updating parameter rather than δ , then the expressions for J_1 are

$$\frac{\partial P_i}{|V_i|\partial \delta_i} = -\sum_{j=1,\, j\neq i}^{N} |y_{i,j}||V_j|\sin(\delta_i - \delta_j - \theta_{i,j})$$

$$\frac{\partial P_i}{|V_k|\partial \delta_k} = |V_i||y_{i,k}|\sin(\delta_i - \delta_k - \theta_{i,k}),\ k \neq i \ .$$

and for J_3 ,

$$\frac{\partial Q_i}{|V_i|\partial \delta_i} = \sum_{j=1,\, j\neq i}^{N} |y_{i,j}||V_j|\cos(\delta_i - \delta_j - \theta_{i,j})$$

$$\frac{\partial Q_i}{|V_i|\partial \delta_k} = -|y_{i,k}||V_k|\cos(\delta_i - \delta_k - \theta_{i,k}),\ k \neq i \ .$$

and there is even more symmetry in *J*.

The diagonal dominance of J_1 and J_4 can be observed by the examining the partial derivatives as follows: the differences between voltage angles at adjacent busses is usually small, so that the

$(\delta_i - \delta_j)$ terms are small. The angles found in the admittance matrix are usually large (i.e. near 90°) because most power systems are reactive. Therefore, the sine terms in the matrix update equation tend to be near unity, while the cosine terms tend to be near zero. Decoupled loadflow programs use only J_1 and J_4, treating the *P* and *Q* problems separately.

So that the benefits of optimal bus ordering can be fully exploited, non-decoupled loadflow Jacobian matrices are usually formulated in the following alternating-row form, rather than that described symbolically above:

$$\begin{bmatrix} \Delta P_1 \\ \Delta Q_1 \\ \Delta P_2 \\ \Delta Q_2 \\ \vdots \\ \Delta P_N \\ \Delta Q_N \end{bmatrix} = [J] \begin{bmatrix} |V_1| \Delta\delta_1 \\ \Delta |V_1| \\ |V_2| \Delta\delta_2 \\ \Delta |V_2| \\ \vdots \\ |V_N| \Delta\delta_N \\ \Delta |V_N| \end{bmatrix} .$$

The solution procedure for the Newton-Raphson loadflow proceeds with:

1. Initialize the bus voltages. For load busses, use $V = 1 + j0$. For generator busses (including the swing bus), use $V = V_{spec} + j0$.

2. Form the Jacobian matrix, and update all bus voltage magnitudes and phase angles, except for those at the swing bus, and except for the voltage magnitudes at PV busses.

3. Check the mismatch *P* and *Q* at each bus. If all are within tolerance (typical tolerance is 0.00001 pu), a solution has been found. Otherwise, return to Step 2.

4 Stott's Algorithm

Stott's algorithm takes advantage of the strong decoupling of *P* and *Q* that occurs in most high-voltage power systems. His formulation is based upon the assumption that *P* is primarily related to δ, and that *Q* is primarily related to $|V|$.

He begins by writing the expression for power at bus *i*, which is

$$P_i^{calc} + jQ_i^{calc} = V_i \left[\sum_{j=1}^{N} y_{i,j} V_j \right]^* .$$

He then defines

$$y_{i,j} = G_{i,j} + jB_{i,j} \ .$$

Expanding the power expression yields

$$P_i^{calc} + jQ_i^{calc} = \left[\sum_{j=1}^{N} \left(|V_i|\angle\theta_i\right)\left(|V_i|\angle-\theta_j\right)\left(G_{i,j} - jB_{i,j}\right) \right],$$

or, expanding in real and imaginary form

$$P_i^{calc} = \sum_{j=1}^{N} |V_i||V_j|\left(G_{i,j}\cos\delta_{i,j} + B_{i,j}\sin\delta_{i,j}\right)$$

$$Q_i^{calc} = \sum_{j=1}^{N} |V_i||V_j|\left(G_{i,j}\sin\delta_{i,j} - B_{i,j}\cos\delta_{i,j}\right) ,$$

where $\delta_{i,j} = \delta_i - \delta_j$.

From the above equations, the necessary loadflow partial derivatives are

$$\left.\frac{\partial P_i}{\partial \delta_j}\right|_{j\neq i} = |V_i||V_j|\left(G_{i,j}\sin\delta_{i,j} - B_{i,j}\cos\delta_{i,j}\right) ,$$

and

$$\frac{\partial P_i}{\partial \delta_i} = \sum_{j=1,\, j\neq i}^{N} |V_i||V_j|\left(-G_{i,j}\sin\delta_{i,j} + B_{i,j}\cos\delta_{i,j}\right) ,$$

which, by comparing to the Q equation simplifies to

$$\frac{\partial P_i}{\partial \delta_i} = -\left[Q_i - |V_i|^2\left(G_{i,i}\sin\delta_{i,i} - B_{i,j}\cos\delta_{i,i}\right)\right] = -Q_i - |V_i|^2 B_{i,i} \ .$$

Similarly, for Q,

$$\left.\frac{\partial Q_i}{\partial |V_j|}\right|_{j\neq i} = |V_i|\left(G_{i,j}\sin\delta_{i,j} - B_{i,j}\cos\delta_{i,j}\right) = \frac{1}{|V_j|}\left.\frac{\partial P_i}{\partial \delta_j}\right|_{j\neq i} ,$$

and

$$\frac{\partial Q_i}{\partial |V_i|} = \sum_{j=1,\, j\neq i}^{N} |V_j| \left(G_{i,j} \sin\delta_{i,j} - B_{i,j} \cos\delta_{i,j} \right) + 2|V_i| \left(G_{i,i} \sin\delta_{i,i} - B_{i,i} \cos\delta_{i,i} \right) .$$

After simplifying, the above equation becomes

$$\frac{\partial Q_i}{\partial |V_i|} = \frac{Q_i}{|V_i|} - |V_i| B_{i,i} \ .$$

The form of the Jacobian update equation is now

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \frac{1}{V}\Delta V \end{bmatrix} ,$$

where

$$H_{i,j} = L_{i,j} = |V_i||V_j| \left(G_{i,j} \sin\delta_{i,j} - B_{i,j} \cos\delta_{i,j} \right), i \neq j \ ,$$

$$H_{i,i} = -Q_i - |V_i|^2 B_{i,i} \ ,$$

$$L_{i,i} = Q_i - |V_i|^2 B_{i,i} \ .$$

Now, since the angular differences are small, then $|\cos\delta_{i,j}| >> |\sin\delta_{i,j}|$. In reactive power systems, $|B_{i,j}| > |G_{i,j}|$, so that $|B_{i,j} \cos\delta_{i,j}| >> |G_{i,j} \sin\delta_{i,j}|$. Also, in most cases, $|B_{i,i}| >> |Q_i|$, so that $|V_i^2 B_{i,i}| >> |Q_i|$. Substituting these approximations into the above expressions for H and L yields

$$H_{i,j} = L_{i,j} \approx -|V_i||V_j| B_{i,j} \ ,$$

$$H_{i,i} \approx L_{i,i} \approx -|V_i|^2 B_{i,i} \ .$$

These lead to the following simple form of the update equation:

$$[\Delta P] = \begin{bmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_N \end{bmatrix} \begin{bmatrix} -B_{1,1} & -B_{1,2} & \cdots & -B_{1,N} \\ -B_{2,1} & -B_{2,2} & \cdots & -B_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ -B_{N,1} & -B_{N,2} & \cdots & -B_{N,N} \end{bmatrix} \begin{bmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_N \end{bmatrix} \begin{bmatrix} \Delta\delta_1 \\ \Delta\delta_2 \\ \vdots \\ \Delta\delta_N \end{bmatrix} =$$

$$= VB'V[\Delta\delta],$$

$$[\Delta Q] = \begin{bmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_N \end{bmatrix} \begin{bmatrix} -B_{1,1} & -B_{1,2} & \cdots & -B_{1,N} \\ -B_{2,1} & -B_{2,2} & \cdots & -B_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ -B_{N,1} & -B_{N,2} & \cdots & -B_{N,N} \end{bmatrix} \begin{bmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_N \end{bmatrix} \begin{bmatrix} \frac{\Delta V_1}{V_1} \\ \frac{\Delta V_2}{V_2} \\ \vdots \\ \frac{\Delta V_N}{V_N} \end{bmatrix} =$$

$$= VB''V\left[\frac{\Delta V}{V}\right].$$

Stott proposes the following additional simplifications:

1. Omit from B' those network elements which primarily affect MVAr flows (i.e. shunt reactances, off-nominal transformer taps, etc.)

2. Omit from B'' the angle-shifting effects of phase-shifting transformers, which primarily affect MW flows.

3. Since the voltage magnitudes are close to unity, write

$$\left[\frac{\Delta P}{|V|}\right] = \left[B'|V|\right][\Delta\delta] \approx \left[B'\right][\Delta\delta] \text{ (first one-half iteration),}$$

and

$$\left[\frac{\Delta Q}{|V|}\right] = \left[B''|V|\right]\left[\frac{\Delta|V|}{|V|}\right] \approx \left[B''\right][\Delta V] \text{ (second one-half iteration).}$$

4. Neglect series resistances when calculating B'.

At this point, B' and B'' remain constant throughout the solution. Therefore, B' and B'' should be LU decomposed once, and re-used for each half iteration. The solutions steps are

1. LU decompose B' and B''.

2. Calculate $\left[\frac{\Delta P}{|V|}\right]$.

3. Update δ using $\left[\frac{\Delta P}{|V|}\right] = \left[B'\right][\Delta\delta]$.

4. Check convergence, and continue if not converged.

5. Calculate $\left[\frac{\Delta Q}{|V|}\right]$.

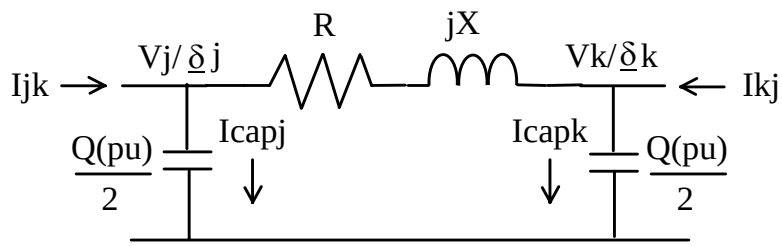
6. Update $\Delta|V|$ using $\left[\frac{\Delta Q}{|V|}\right] = \left[B''\right]\left[\Delta V\right]$

7. Check convergence, and return to Step 2 if not converged.

5. Other Considerations

Current and Power Flow in Transmission Lines and Cables

Once the bus voltages throughout the system have been calculated, then the loadflow program must calculate power flows through lines/transformers/cables. The standard pi-equivalent models are used for this purpose as follows:



,

so from side j, $$I_{jk} = \frac{|V_j|\angle\delta_j - |V_k|\angle\delta_k}{R + jX} + |V_j|\frac{Q}{2}\angle\left(\delta_j + 90^0\right),$$

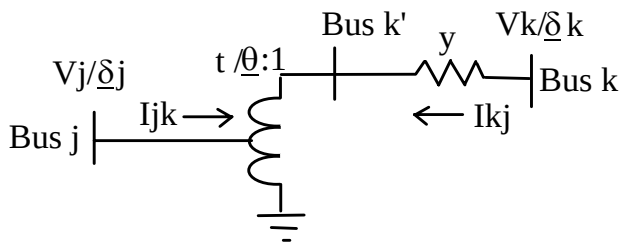
and from side k, $$I_{kj} = \frac{|V_k|\angle\delta_k - |V_j|\angle\delta_j}{R + jX} + |V_k|\frac{Q}{2}\angle\left(\delta_k + 90^0\right).$$

Note that the current on the two opposite ends of the line are not exactly the same due to the fact that the capacitor currents are not, in general, equal to each other.

The corresponding power flows for the two ends of the line are $S_{jk} = V_{jk} I_{jk}^*, S_{kj} = V_{kj} I_{kj}^*$.

Current and Power Flow in Transformers

The transformer equivalent circuit taken from Section 3 is



,

so for side k, $$I_{kj} = \left(|V_k| \angle \delta_k - \frac{|V_j| \angle \delta_j}{t \angle \theta} \right) y ,$$

and for side j, $$I_{jk} = -\frac{I_{kj}}{t \angle \theta} .$$

The corresponding power flows for the two ends of the transformer are $S_{jk} = V_{jk} I_{jk}^{*}, S_{kj} = V_{kj} I_{kj}^{*}$.

Area Interchange

Large-scale power systems usually consist of several, or perhaps many, individual electric utility companies. In these cases, each area may have a desired net input or output power to satisfy sales and purchase agreements.

The area interchange feature in a loadflow program sums the tie-line flows into each area for a net area power input. If these nets are not the desired values, to within a few megawatts, then an area-control generator within the area is adjusted by the error amount. The sum of all input powers for all areas must, of course, be zero.

Convergence Criteria

Most loadflow data cases are developed using a 100 MVA base, so that a 1% load corresponds to 1 MW. A typical convergence criteria is that the highest individual bus P and Q mismatches are within 0.001 - 0.01%, or 0.00001 - 0.0001 pu.

1. Summary of PCFLO Files

Input (note – any lines in input files that begin with a colon in column 1 are treated as comments and are skipped)

- ADAT.CSV: Loadflow area input data. (PCFLO also creates temporary file ADAT.TMP)
- BDAT.CSV: Bus data.
- LDAT.CSV: Line and transformer data.
- OPTIONS.CSV: Solution options.
- SPECTRA.CSV: User-specified harmonic current injection spectra. (PCFLO also creates temporary file SPECTRA.TMP)

Output

- ASOLN.CSV: Solved area data for loadflows.
- BORDER.CSV: File built by PCFLO that lists the busses in optimal order.
- EXLOG.CSV: Echo print of screen messages.
- FREP.TXT: Output of program FAULTS.
- HPA_LAST_CASE.CSV, HPA_SUMMARY.CSV. Output of harmonic power analyzer program HPA.
- ISOLN.CSV, VSOLN.CSV: Solved branch currents and bus voltages for loadflow and harmonics studies. (Sine series format for Fourier series)
- OUT1.CSV, OUT2.CSV, OUT3.CSV: Echo print of input data for loadflow, short circuit, and harmonics, along with pertinent messages and errors.
- OUT4.CSV: Full loadflow output data (if requested).
- OUT5.CSV: Loadflow summary output used for analyzing the impact of power transactions across a power grid.
- THDV.CSV, THDI.CSV: Solved total harmonic voltage and current distortions for harmonics studies.
- ZBUS0.CSV, ZBUS1.CSV, ZBUS2.CSV: Solved zero/positive/negative impedance matrix elements for short circuit (in rectangular form) or harmonic studies (in polar form), in per unit.

Temporary Files Created During Execution

- BDAT.TMP: Unformatted bus data file built by PCFLO and read by FAULTS.
- FFREP1.TXT, FFREP2.TXT, FFREP3.TXT. Temporarly files used during short circuit studies (produced by program FAULTS)
- LDAT.TMP: Unformatted line and transformer data file built by PCFLO and read by FAULTS.
- SPECTRA.TMP: Temporary file built by PCFLO during harmonic studies.

2. Prepared Cases, Ready to Run

Loadflow

- 5 Bus Stevenson* Loadflow Example, pp. 200-205. File Extension _S5.
- 4860 Bus Loadflow Case. File Extension _SCREWBEAN. Used for power grid studies.

Short Circuit

- 6 Bus Grainger-Stevenson** Short Circuit Example (Prob. 3.12, p. 139, and continued with Prob. 11.17, p. 469). File Extension _S6
- 9 Bus Grainger-Stevenson** Short Circuit Example (Prob. 3.13, pp. 139-140, and continued with Prob. 11.18, p. 469). File Extension _S9.

Harmonics

- 5 Bus Tutorial. File Extensions _FIVE and _FIVE_FILTER.
- 33 Bus ASD Example. File Extension _DFW.
- 18 Bus Oil Pipeline ASD Example. File Extension _PIPELINE.
- 111 Bus 3rd Harmonic Resonance Example. File Extension _TV.
- 17 Bus Small Ski Area Example. File Extensions _SKIA, _SKIB, _SKIC.
- 454 Bus Large Ski Area Example. File Extensions _OLYMPICS_A (unfiltered), _OLYMPICS_D (filtered).
- 87 Bus Distribution Example with 3rd Harmonic Problem Due to Single-Phase Loads. File Extension _PC.

* William D. Stevenson, Jr., *Elements of Power System Analysis*, Fourth Edition, McGraw-Hill, New York, 1982.
** John J. Grainger, William D. Stevenson, Jr., *Power System Analysis*, McGraw-Hill, New York, 1994.

3. Content of Input Data Files

BUS DATA
(File = BDAT.CSV, one record per bus. CSV format)

Variable	**Comments**
Number	Integer
Name	Up to 12 characters
Type	1 = Swing Bus 2 = PV Bus 3 = PQ Bus
Linear P Generation	Percent
Linear Q Generation	Percent
Linear P Load	Percent
Linear Q Load	Percent
Desired Voltage	Per unit
Shunt Reactive Q Load	Percent @ voltage = 1 per unit
Maximum Q Generation	Percent
Minimum Q Generation	Percent
Control Area	Integer

Remote-Controlled Bus Number	Used for controlling voltage at a remote bus. For these cases, the desired voltage specified above applies to the remote bus.
Connection Type for Shunt Reactive Q Load	0 or 1 = Grounded Wye. Otherwise, ungrounded wye or delta (i.e. no zero sequence path)
Subtransient R, X (Pos. Sequence)	Series impedance of motor or generator, in per unit
Subtransient R, X (Neg. Sequence)	Series impedance of motor or generator, in per unit
Subtransient R, X (Zero Sequence)	Series impedance of motor or generator, in per unit (ignoring connection type and grounding impedances) (do not multiply by 3)
Connection Type for Subtransient Impedances	0 or 1 = Grounded Wye. Otherwise, ungrounded wye or delta (i.e. no zero sequence path)
Grounding Impedance R, X for Sub-Transient Impedances	Series impedance from wye point to ground, in per unit. (do not multiply by 3).
Nonlinear Device P Generation	P Generation that comes from the nonlinear device, in percent.
Nonlinear Device P Load	P Load of the nonlinear device, in percent.
Nonlinear Device Displacement Power Factor	Per Unit (positive for lagging is positive, negative for leading).

Nonlinear Device Type	1 = 2-pulse current source 2 = 2-pulse voltage source 3 = 6-pulse current source 4 = 6-pulse voltage source 5 = 12-pulse current source 6 = 12-pulse voltage source 7 = 18-pulse current source 8 = 18-pulse voltage source 9 = Diversified 6-pulse current source (like type 3, but with the 11 & 13th harmonics multiplied by 0.75, the 17 & 19th harmonics by 0.50, and all higher harmonics by 0.25) 10 = Single-phase electronic GY-GY 11 = Single-phase electronic Delta-GY 12 = Magnetic fluorescent GY-GY 13 = Magnetic fluorescent Delta-GY 14 = User-specified type 14 . . . . 33 = User-specified type 33
Nonlinear Device Phase Shift	Degrees for positive sequence, on system side with respect to device side. This is the additional phase shift by which the current injection phase angles will be advanced for positive sequence, and delayed for negative sequence.
Connection Type for Harmonics Model of Linear Load	0 or 1 = Grounded Wye. Otherwise, ungrounded wye or delta (i.e. no zero sequence path)

CSV Header and Structure for Input File BDAT.CSV (using sample file BDAT_HEADER.csv)

:Bus Data
:
:

:Bus :Number :(I)	Bus Name (A)	Bus Type (I)	Linear P Generation (%) (F)	Linear Q Generation (%) (F)	Linear P Load (%) (F)	Linear Q Load (%) (F)	Desired Voltage (pu) (F)	Shunt Reactive Q Load (%) (F)	Maximum Q Generation (%) (F)

continuing across,

Minimum Q Generation (%) (F)	Bus Control Area (I)	Remote-Controlled Bus No. (I)	Connection Type for Shunt Reactive Q Load (I)	Positive Sequence Subtransient R (pu) (F)	Positive Sequence Subtransient X (pu) (F)	Negative Sequence Subtransient R (pu) (F)	Negative Sequence Subtransient X (pu) (F)	Zero Sequence Subtransient R (pu) (F)	Zero Sequence Subtransient X (pu) (F)

continuing across,

Connection Type for Subtransient R and X (I)	Grounding Impedance R for Subtransient Impedances (pu) (F)	Grounding Impedance X for Subtransient Impedances (pu) (F)	Nonlinear Device P Gen (%) (F)	Nonlinear Device P Load (%) (F)	Nonlinear Device Displacement Power Factor (pu) (F)	Nonlinear Device Type (I)	Nonlinear Device Phase Shift (Degrees) (F)	Linear Load Connection Type (I)

LINE AND TRANSFORMER DATA
(File = LDAT.CSV, one record per branch. CSV format)

Variable	Comments
FROM BUS Number	Integer (or blank if neutral)
TO BUS Number	Integer (or blank, if neutral)
Circuit Number	Integer (or blank)
R, X (Positive/Negative Sequence)	Series impedance, in per unit
Charging (Positive/Negative Sequence)	Percent, for entire length of line
Rating	Percent
Minimum Tap, or Minimum Phase Shift Angle	Per unit tap, or degrees, FROM BUS side wrt. TO BUS side
Maximum Tap, or Maximum Phase Shift Angle	Per unit tap, or degrees, on FROM BUS side wrt. TO BUS side
Tap Step Size, or Phase Shift Step Size	Per unit, or degrees
Tap	Per unit, on FROM BUS side
Phase Shift	Degrees, FROM BUS side wrt. TO BUS side
Voltage-Controlled Bus Number	Used for controlling voltage at a remote bus. For these cases, the desired voltage specified applies to the remote bus.

Voltage-Controlled Bus Side	When controlling the voltage at a remote bus, enter 1 when the remote bus is on the FROM BUS side of the transformer. Enter 2 when the remote bus is on the TO BUS side of the transformer.
Desired Voltage for Voltage-Controlled Bus, or Desired Active Power Flow for Phase Shifting Transformer	Per unit Voltage, or Percent Active Power (FROM BUS toward TO BUS)
R, X (Zero Sequence)	Series impedance, in per unit (ignoring connection type and grounding impedances), (do not multiply by 3)
Charging (Zero Sequence)	Percent, for entire length of line
Connection Type for Transformers and Shunt Elements	For transformers:

Type	FROM BUS	TO BUS
0 or 1	GY	GY
2	GY	Y
3	Y	GY
4	Y	Y
5	Δ	Δ
6	GY	Δ
7	Y	Δ
8	Δ	GY
9	Δ	Y

For shunt elements:
0 or 1 = Grounded Wye.
Otherwise, ungrounded wye or delta (i.e. no zero sequence path)

Grounding Impedance R, X	Series impedance from wye point to ground, in per unit Applies to wye-connected transformers and shunt

	elements. (do not multiply by 3)
Resistive Skin Effect Factor for Positive/Negative Sequence	Harmonic h (h $\geq$ 2, fractional values OK) at which the conductor resistance is double the fundamental frequency resistance.
Resistive Skin Effect Factor for Zero Sequence	Harmonic h (h $\geq$ 2, fractional values OK) at which the conductor resistance is double the fundamental frequency resistance. This value applies to the combined conductor and grounding resistance.

Header and Structure for Input File LDAT.CSV (using sample file LDAT_HEADER.csv)

:Line and Transformer Data
:
:
:
:

:From :Bus :(I)	To Bus (I)	Circuit Number (I)	Positive Sequence R (pu) (F)	Positive Sequence X (pu) (F)	Pos/Neg Sequence Charging (%) (F)	Rating (%) (F)	Minimum Tap (pu) (F)	Maximum Tap (pu) (F)	Tap Step Size (pu) (F)	Fixed Tap (pu) (F)

continuing across,

Phase Shift (Degrees) (F)	Voltage-Cont. Bus Number (I)	Voltage-Cont. Bus Side (I)	Desired Voltage at Voltage Cont. Bus (pu) or Desired P for Phase Shifter (%) (F)	Zero Sequence R (pu) (F)	Zero Sequence X (pu) (F)	Zero Sequence Charging (%) (F)	Connect. Type for Trans. and Shunt Elements (I)	Series Grounding R (pu) (F)	Series Grounding X (pu) (F)

AREA INTERCHANGE DATA
(File = ADAT.CSV, one record per loadflow area. CSV format)

Variable	**Comments**
Number	Integer
Tie-Line Loss Assignment	If non-zero, then power losses on tie lines are assigned equally between the two areas. If zero, the TO BUS area for each tie line is assigned the loss (i.e., meter at the FROM BUS).
Control Bus Number	Integer
Desired Export Power	Percent
Solution Tolerance for Export	Percent
Name	Up to 20 characters

Header and Structure for Input File ADAT.CSV (using sample file ADAT_HEADER.csv)

```
:Area Interchange Data
:          Tie Line
:          Loss
:          Assignm.
:          (1 for
:          Splitting,                Desired   Solution
:          0 for To     Control      Export    Tolerance
:Area      Bus          Bus          Power     for Export  Area
:Number    Ownership    Number       (%)       (%)         Name
:(I)       (I)          (I)          (F)       (F)         (A)
```

USER-SPECIFIED HARMONIC CURRENT SPECTRAL DATA
(File = SPECTRA.CSV, one record per harmonic per nonlinear load type. CSV format)

Variable	**Comments**
Type of Series	Must be sin for a sine series, cos for a cosine series. All entries in this file must be either sin, or cos, and cannot be mixed.
Nonlinear Load Type	Must be 14, 15, 16, . . . , 33.
Harmonic Order	1, 2, 3, etc.
Current Harmonic Magnitude	Per unit. If the fundamental is given, its magnitude must be 1.0, and the other harmonic magnitudes for the same nonlinear load type are assumed to be relative to 1.0. The actual injection currents will be scaled according to bus load/generation. **Important:** If the fundamental is not given for a nonlinear load type, then the harmonic magnitudes are assumed to be given on the system base, rather than as a fraction of the P for that bus.
Current Harmonic Phase Angle	Degrees, using load current convention. If the fundamental angle is given, it must be 0.0. The actual phase angles will be adjusted internally according to bus power factor and fundamental voltage angle. If the fundamental is not given, the phase angles are assumed to be given with respect to the bus fundamental frequency voltage phase angle.

Header and Structure for Input File SPECTRA.CSV (using sample file SPECTRA_HEADER.csv)

```
:Harmonic Current Spectral Data
:
:Type of                                  Current
:Series    Nonlinear  Harmonic  Current   Harmonic
:(SIN or   Load       Order     Harmonic  Phase
:COS)      Type       (Integer) Mag.      Angle
:(A)       (I)        (I)       (pu)      (Degrees)
:(A)       (I)        (I)       (F)       (F)
```

SOLUTION OPTIONS

(File = OPTIONS.CSV. CSV format.)

For Loadflow (using sample file OPTIONS_HEADER_LOADFLOW.csv)

Loadflow Study Case - User Title Goes on This Line

:Loadflow Solution Options

:									
:Optimal :Bus :Ordering :Method :(Integer) :(1-2-3) :(I)	Gauss-Seidel Start? (T or F) (L)	P & Q Mismatch for Gauss-Seidel Start (0.5 pu) (F)	Accel. Factor for Gauss-Seidel Start (1.2 pu) (F)	Voltage Update Cap for Gauss-Seidel Start (0.005 pu) (F)	P & Q Mismatch Solution Tolerance (5E-06 pu) (F)	Disable Remote Volt. Reg. by PV Busses? (T or F) (L)	Disable Area Intrchnge P Adjust? (T or F) (L)	Ignore Q Limits on PV Busses? (T or F) (L)	Disable Transf. Tap Adjust. for Voltage Control? (T or F) (L)
2	T	0.5	1.2	0.005	5.00E-06	F	F	F	F

continuing across,

Limit Output to This Control Area? (Integer) (I)	and This Control Area? (Integer) (I)	and This Control Area? (Integer) (I)	and This Control Area? (Integer) (I)	and This Control Area? (Integer) (I)
0	0	0	0	0

For Short Circuit (using sample file OPTIONS_HEADER_SHORT_CIRCUIT.csv)

```
Short Circuit Study Case - User Title Goes on This Line
:Short Circuit Solution Options
:Optimal
:Bus
:Ordering
:Method     Enter T for Diagonal and Neighbor ZBUS Elements Only (recommended)
:(Integer)  Enter F for All ZBUS Elements (not recommended and not to be followed by FAULTS)
:(1-2-3)    (T or F)
:(I)        (L)
          2 T
```

For Full Harmonic Solution (using sample file OPTIONS_HEADER_FULL_HARMONIC_SOLUTION.csv)

```
Full Harmonic Solution Study Case - User Title Goes on This Line
:Full Harmonic Solution Options
:
:
:Optimal                                                                  Harmonic Load Model
:Bus                    P & Q    Accel.   Update                  Highest  for PQ Linear Loads                    Global
:Ordering  Gauss-       Mismatch Factor   Cap      P & Q          Harmonic 0 or 1:  Resistive-only (recommended)  Linear   Global
:Method    Seidel       for Gauss- for Gauss- for Gauss- Mismatch of       2:  Parallel R & L Model            Motor    Resistance
:(Integer) Start?       Seidel   Seidel   Seidel   Solution       Interest 3:  Series R & L Model              Load     Doubling
:(1-2-3)   (T or F)     Start    Start    Start    Tolerance      (Integer) 4:  Ignore PQ Loads (i.e. No Model)  Modeling Harmonic
:(I)       (L)          (0.5 pu) (1.2 pu) (0.005 pu) (5E-06 pu)   (1 - 49) (0 - 4)                             Fraction (I)
:(I)       (L)          (F)      (F)      (F)      (F)            (I)      (I)                                 (F)      (I)
          2 T                0.5      1.2    0.005  5.00E-06         25                                   1        0        15
```

For Harmonic Impedance Scan (using sample file OPTIONS_HEADER_HARMONIC_IMPEDANCE_SCAN.csv)

Harmonic Impedance Scan Study Case - User Title Goes on This Line
:Harmonic Impedance Scan
:

:Optimal :Bus :Ordering :Method :(Integer) :(1-2-3) :(I)	Gauss-Seidel Start? (T or F) (L)	P & Q Mismatch for Gauss-Seidel Start (0.5 pu) (F)	Accel. Factor for Gauss-Seidel Start (1.2 pu) (F)	Update Cap for Gauss-Seidel Start (0.005 pu) (F)	P & Q Mismatch Solution Tolerance (5E-06 pu) (F)	Lowest Harmonic of Interest (Integer) (1 - 49) (I)	Highest Harmonic of Interest (Integer) (1 - 49) (I)	Number of Steps per Harmonic (Integer) (1 - 100) (I)	Limit the Output to Diagonal Elements Only? (T or F) (L)

continuing across,

Scan This Bus (Integer) (I)	and This Bus (Integer) (I)	and This Bus (Integer) (I)	and This Bus (Integer) (I)	and This Bus (Integer) (I)	Harmonic Load Model for PQ Linear Loads 0 or 1: Resistive-only (recommended) 2: Parallel R & L Model 3: Series R & L Model 4: Ignore PQ Loads (i.e. No Model) (0 - 4) (I)	Global Linear Motor Load Modeling Fraction (F)	Global Resistance Doubling Harmonic (I) (I)

4. Harmonic-Related and Short-Circuit Related Output Files

(Note - for loadflow studies, the formats of ISOLN and VSOLN are different from below but are self-explanatory when viewing the files. For short circuit studies, the ZBUS files are similar to below, but written in rectangular form)

Commas separate the fields shown below to facilitate their use with Microsoft Excel.

ISOLN.CSV (for harmonics)

Data Field (starting from the left)	Description
1	Harmonic number
2	From bus number
3	To bus number
4	Circuit number
5	Current magnitude - per unit
6	Current phase angle (sine reference) - degrees
7	From bus name (at the first opportunity only)
8	To bus name (at the first opportunity only)
9	Loading level - percent of line rating (for fundamental frequency only)

VSOLN.CSV (for harmonics)

Data Field (starting from the left)	Description
1	Harmonic number
2	Bus number
3	Voltage magnitude - per unit

4	Voltage phase angle (sine reference) - degrees
5	Nonlinear device load current magnitude - per unit
6	Nonlinear device load current phase angle (sine reference) - degrees
7	Bus name (at the first opportunity only)

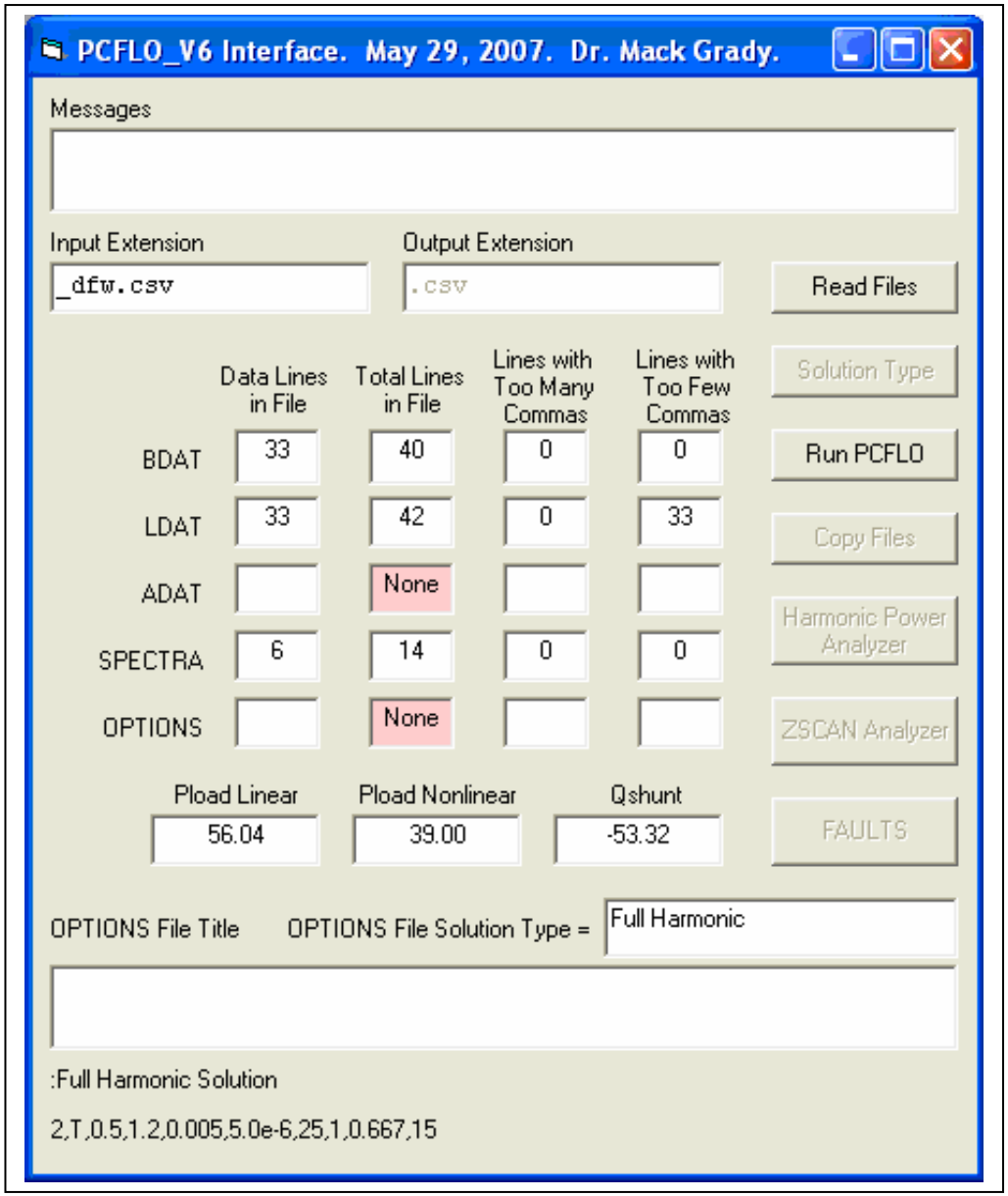
ZBUS0.CSV, ZBUS1.CSV, ZBUS2.CSV (for harmonics and short circuit)

Data Field (starting from the left)	Description
1	Harmonic number
2	From bus number
3	To bus number
4	Impedance magnitude - per unit
5	Impedance phase angle - degrees
6	From bus name (at the first opportunity only)
7	To bus name (at the first opportunity only)

THDV.CSV (for harmonics)

Contains a list of bus numbers with their corresponding names and voltage distortions.

Example Screens for the _DFW Case



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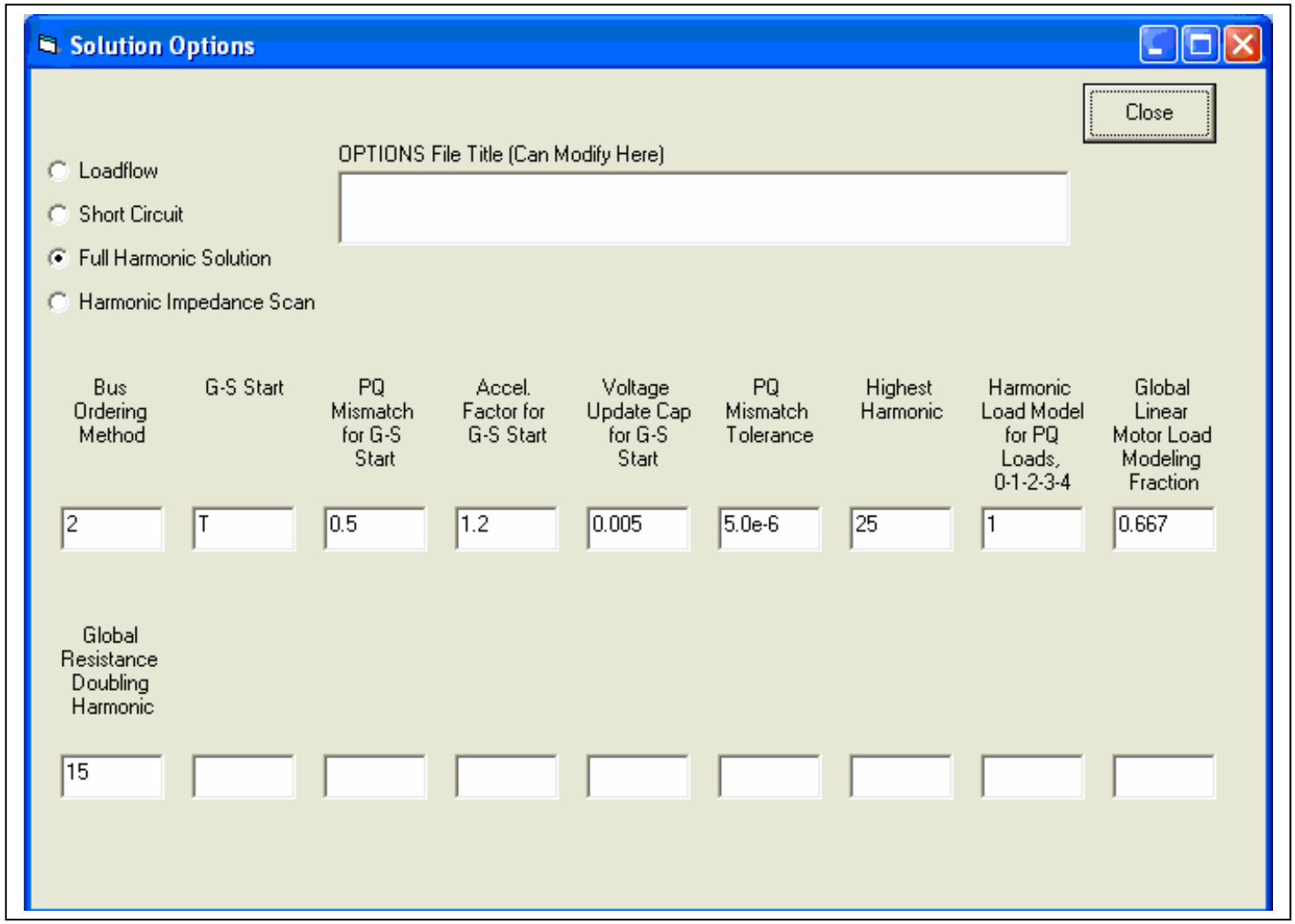
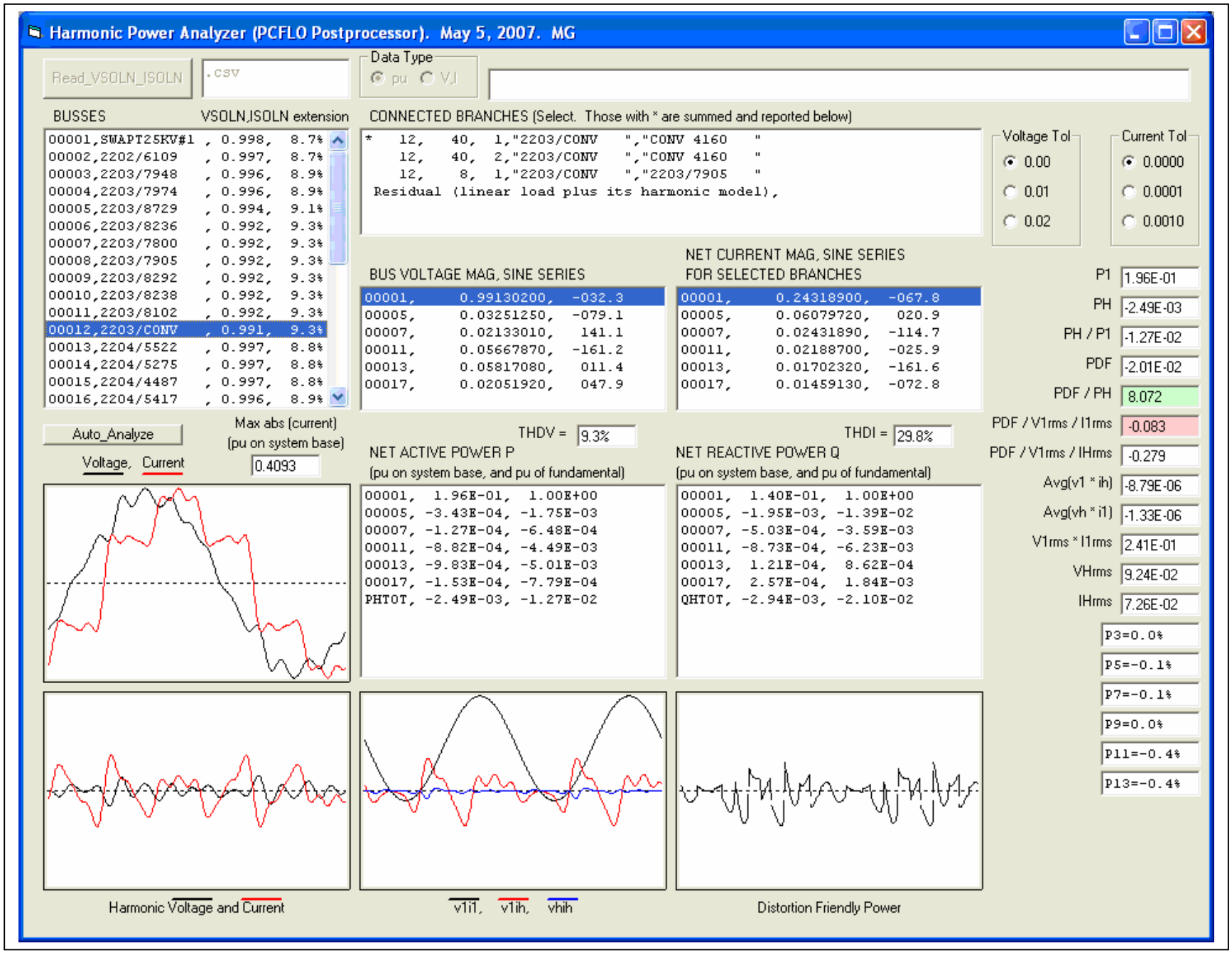

Solution Options
Close
OPTIONS File Title (Can Modify Here)
Loadflow
Short Circuit
Full Harmonic Solution
Harmonic Impedance Scan
Bus Ordering Method
G-S Start
PQ Mismatch for G-S Start
Accel. Factor for G-S Start
Voltage Update Cap for G-S Start
PQ Mismatch Tolerance
Highest Harmonic
Harmonic Load Model for PQ Loads, 0-1-2-3-4
Global Linear Motor Load Modeling Fraction
2
T
0.5
1.2
0.005
5.0e-6
25
1
0.667
Global Resistance Doubling Harmonic
15



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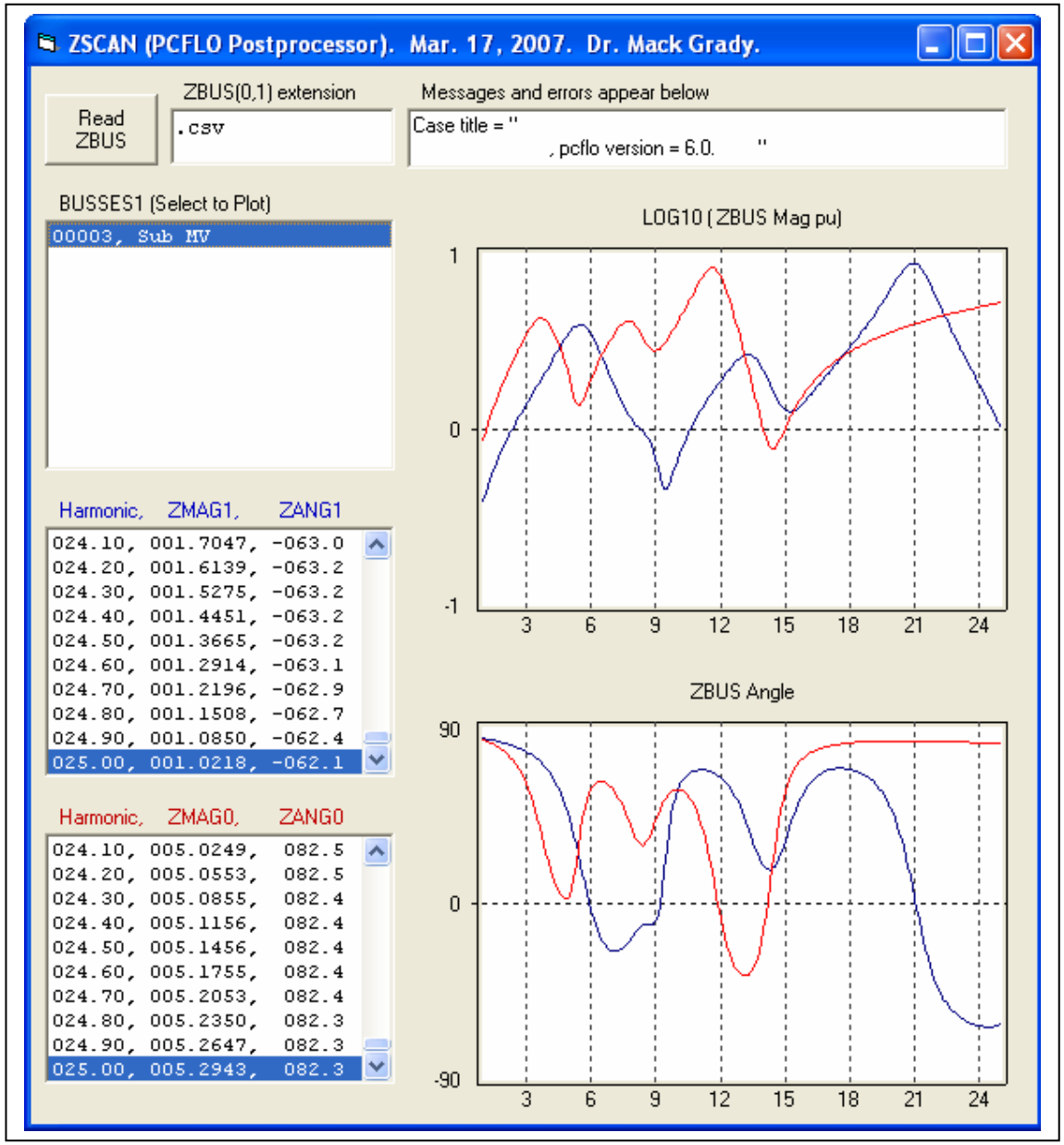
Example Impedance Scan for Bus 3 of the _PC case



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Five Bus Stevenson Loadflow Example

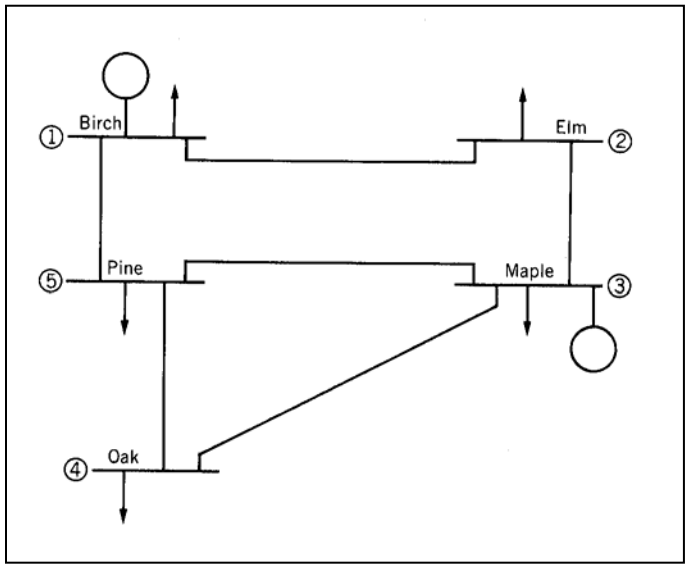

Table 8.1

Line, bus to bus	Length km	Length mi	R Ω	X Ω	R per unit	X per unit	Charging Mvar†
1–2	64.4	40	8	32	0.042	0.168	4.1
1–5	48.3	30	6	24	0.031	0.126	3.1
2–3	48.3	30	6	24	0.031	0.126	3.1
3–4	128.7	80	16	64	0.084	0.336	8.2
3–5	80.5	50	10	40	0.053	0.210	5.1
4–5	96.5	60	12	48	0.063	0.252	6.1

† At 138 kV.

Table 8.2

Bus	Generation P, MW	Generation Q, Mvar	Load P, MW	Load Q, Mvar	V, per unit	Remarks
1	.....	.....	65	30	$1.04\angle 0°$	Swing bus
2	0	0	115	60	$1.00\angle 0°$	Load bus (inductive)
3	180	.....	70	40	$1.02\angle 0°$	Voltage magnitude constant
4	0	0	70	30	$1.00\angle 0°$	Load bus (inductive)
5	0	0	85	40	$1.00\angle 0°$	Load bus (inductive)

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Five Bus Stevenson Loadflow Example

SAVE THE TREES POWER COMPANY - LOAD FLOW STUDY

DATE 04/26/79 TIME 00.00.12

REPORT OF POWER FLOW CALCULATIONS FOR AREA 1.

3 ITERATIONS, SWING BUS IS 1.

B U S - D A T A

BUS	NAME	VOLTS	ANGLE	GENERATION MW	GENERATION MVAR	LOAD MW	LOAD MVAR	CAP/REAC MVAR
1	BIRCH	1.040	0.0	234.7	100.1	65.0	30.0	
2	ELM	0.961	-6.3	0.0	0.0	115.0	60.0	
3	MAPLE	1.020	-3.7	180.0	110.3R	70.0	40.0	
4	OAK	0.920	-10.9	0.0	0.0	70.0	30.0	
5	PINE	0.968	-6.2	0.0	0.0	85.0	40.0	
	AREA TOTALS			414.7	210.4	405.0	200.0	0.0

L I N E - F L O W

BUS	TO BUS	NAME	MW	MVAR	TAP
1	2	ELM	73.98	31.55	
	5	PINE	95.68	38.56	
2	1	BIRCH	-71.41	-25.39	
	3	MAPLE	-43.59	-34.60	
3	2	ELM	44.59	35.65	
	4	OAK	40.46	18.06	
	5	PINE	24.95	16.58	
4	3	MAPLE	-38.74	-18.90	
	5	PINE	-31.25	-11.09	
5	1	BIRCH	-92.59	-29.14	
	3	MAPLE	-24.44	-19.62	
	4	OAK	32.03	8.77	

SOLUTION TIME 0.10 SECONDS.
TOTAL TIME 0.54 SECONDS.

Figure 8.2 Digital-computer solution of load-flow for the system of Example 8.1. Base is 100 MVA.

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:Five Bus Stevenson Loadflow Case												
:Bus Data												
:												
:			Linear	Linear	Linear	Linear		Shunt	Maximum	Minimum		
:			P	Q	P	Q	Desired	Reactive	Q	Q	Bus	Remote-
:Bus	Bus	Bus	Generation	Generation	Load	Load	Voltage	Q Load	Generation	Generation	Control	Controlled
:Number	Name	Type	(%)	(%)	(%)	(%)	(pu)	(%)	(%)	(%)	Area	Bus No.
:(I)	(A)	(I)	(F)	(F)	(F)	(F)	(F)	(F)	(F)	(F)	(I)	(I)
1	birch	1	0	0	65	30	1.04	0	0	0	1	0
2	elm	3	0	0	115	60	0	0	0	0	1	0
3	maple	2	180	0	70	40	1.02	0	0	0	1	0
4	oak	3	0	0	70	30	0	0	0	0	1	0
5	pine	3	0	0	85	40	0	0	0	0	1	0

:Five Bus Stevenson Loadflow Case														
:Line and Transformer Data														Desired Voltage
:														at Voltage
:														Cont. Bus
:														(pu)
:			Positive	Positive	Pos/Neg							Voltage-	Voltage-	or Desired P
:			Sequence	Sequence	Sequence		Minimum	Maximum	Tap	Fixed	Phase	Cont.	Cont.	for
:From	To	Circuit	R	X	Charging	Rating	Tap	Tap	Step Size	Tap	Shift	Bus	Bus	Phase Shifter
:Bus	Bus	Number	(pu)	(pu)	(%)	(%)	(pu)	(pu)	(pu)	(pu)	(Degrees)	Number	Side	(%)
:(I)	(I)	(I)	(F)	(F)	(F)	(F)	(F)	(F)	(F)	(F)	(F)	(I)	(I)	(F)
1	2	1	0.042	0.168	4.1	0	0	0	0	0	0	0	0	0
1	5	1	0.031	0.126	3.1	0	0	0	0	0	0	0	0	0
2	3	1	0.031	0.126	3.1	0	0	0	0	0	0	0	0	0
3	4	1	0.084	0.336	8.2	0	0	0	0	0	0	0	0	0
3	5	1	0.053	0.21	5.1	0	0	0	0	0	0	0	0	0
4	5	1	0.063	0.252	6.1	0	0	0	0	0	0	0	0	0

Screwbean Wind Farm Study

The Screwbean 138kV substation is located in west Texas, halfway between Midland/Odessa and El Paso, near Guadalupe Mountains National Park. It is about 400 miles from Austin. This is prime wind country, and several wind farms are already located in the area.

Your job is to examine the feasibility of transporting 50MW of power from a new wind farm near Screwbean to the U.T. Austin campus. In particular, you are to determine the impact of this transaction on the losses in individual control areas, and also determine if any high or low voltages, or line overloads, are created by your transaction.

To perform the analysis, you will use a 5000 bus version of PCFLO, together with a summer peak loadflow case (in which most bus names have been disguised). You should prepare a ½ to 1 page summary report of your study, as if you were going to submit it to your client. Tables should be attached as an appendix.

Explain to your client how many MW must be generated at Screwbean to deliver 50MW to U.T. Austin. Quantify the MW needed by each negatively-impacted control area to pay back for their increased losses.

Here are the steps:

1. Go to www.ece.utexas.edu/~grady, and click the PCFLO_V6 link. Follow the download and unzip instructions on the PCFLO page.

2. The _SCREWBEAN case is your "base case." Solve it using PCFLO_V6_Interface.exe. Using Excel, examine the output files produced, notably
   - exlog_SCREWBEAN.csv
   - asoln_SCREWBEAN.csv
   - vsoln_SCREWBEAN.csv
   - isoln_SCREWBEAN.csv and
   - out5_SCREWBEAN.csv.

3. Print out asoln_SCREWBEAN.csv, using the landscape option. To verify your loadflow result, check your power loss in asoln_SCREWBEAN.csv. It should be about 1215 MW.

4. Find the Screwbean 138kV substation (SCRWBEAN 138, bus 1095) and the U.T. Austin Harris 69kV substation (HARRIS 69, bus 9204) in the out5_SCREWBEAN.csv file. Note their voltage magnitudes and phase angles, and the P and Q flows in lines/transformers attached to these busses.

5. Copy files bdat_SCREWBEAN.csv to bdat_mod, ldat_SCREWBEAN.csv to ldat_mod, and adat_SCREWBEAN.csv to adat_mod.

6. Add new PV bus SB WIND as bus 2 to bdat_mod.csv, using 20 for its control area. Put 50MW (i.e., 50% on 100MVA base) of generation on this new bus, with a Max Q Gen of 25MVAr, and a Min Q Gen of negative 12.5MVAr. For the desired voltage, put a value that is 0.005pu higher than the base case voltage at SCRWBEAN 138.

7. Add new PQ bus UT CAMPUS as bus 3 to bdat_mod.csv, using 21 for its control area. Put 50MW, 25MVAr of load on this new bus.

8. Connect new bus SB WIND to SCRWBEAN 138 through a line with impedance R = 0.001pu, X = 0.01pu, B = 0%.

9. Connect new bus UT CAMPUS to HARRIS 69 through a line with impedance R = 0.001pu, X = 0.01pu, B = 0%.

10. Add control area SB as area 20 to adat_mod.csv, with a desired export of 50MW (i.e., 50%). The area control bus number will be that of SB WIND. Use an export solution tolerance of 0.1%.

11. Add control area UT as area 21 to adat_mod.csv, with a desired import of 50MW (i.e., negative 50MW export). The area control bus number will be that of UT CAMPUS. Use an export solution tolerance of 0.1%.

12. Re-run PCFLO_V6_Interface.exe, using _mod as the input case. Print out the new asoln.csv file (using the landscape option), and using the new asoln file, tabulate area by area the increase/decrease in each control area's losses compared to the base case. The areas with increased losses may reasonably expect MW payment from the wind power company. This can be accomplished by putting in generator larger than 50MW, and exporting some power to the control areas that are negatively impacted.

13. Use the loss increases from Step 12 to estimate how much actual generation would be needed at SB WIND to deliver 50MW to UT CAMPUS and payback the extra losses to the negatively-impacted control areas.

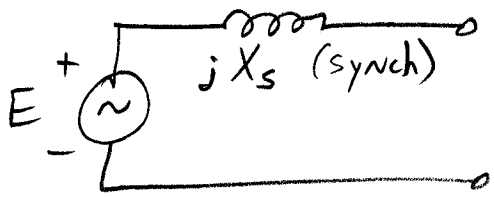
14. Check for any line overloads and high/low voltages created in the vicinity of SCRWBEAN 138 and HARRIS 69. (In an actual study, there would have to be described and remedies proposed. However, do not investigate remedies in your study.)

15. Repeat the above process, but this time reverse the transaction by putting a wind generator at UT CAMPUS, and a 50MW load at SB WIND.

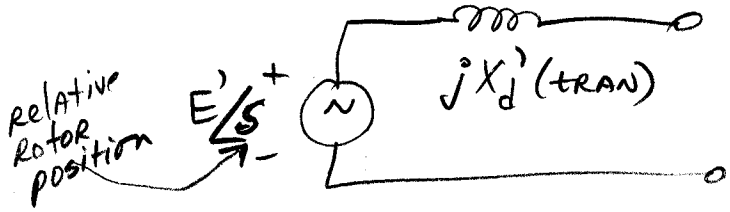
16. Describe the impacts of both transactions in your report.

Generator Representation

Accurately modeled using transformation of variables, into a reference frame rotating at synch speed. Internal impedances are actually in operational form $X(s)$. Time constants involved. Sometimes simplified as



Steady State



Transient
5 to 30 cycles

Note constant E, E'. Constant voltage behind synch / trans reactance

Time constants for E' ~ 1/2 second

Assume voltage regulator, governor, boiler controls unchanged.

51A on back

Swing Equation

Rotating Mass with no damping

$$J\alpha = T_A = T_m - T_e \qquad (1)$$

J: Moment of inertia — $kg\text{-}m^2$

α: Angular acceleration — rad/s^2

T_A: Net accelerating torque - Newt-m

T_m: Applied mechanical torque (from external sys)

T_e: Produced electrical torque (to external sys)

$$(kg\ m^2)\frac{rad}{s^2} = N\text{-}m$$

$$kg\ \frac{m^2}{s^2} \rightarrow \left(\frac{kg\ m}{s^2}\right)m$$

units of Newton

Define $P = T\omega$ $\quad (N\text{-}m)\left(\frac{R}{s}\right) \rightarrow \frac{N\text{-}M}{s}$ rate of doing work

$$\omega = \underset{\frac{d\theta}{dt}}{\dot{\theta}}, \quad \alpha = \underset{\frac{d\omega}{dt}}{\dot{\omega}} = \underset{\frac{d^2\theta}{dt^2}}{\ddot{\theta}}$$

$$J\frac{d^2\theta}{dt^2} = T_m - T_e, \quad J\omega\frac{d^2\theta}{dt^2} = P_m - P_e$$

P_m: Mech input power

P_e: Generated Elect Power

Define $H = \dfrac{\text{Stored kinetic energy at synch speed}}{\text{machine rating in MVA}}$

$$= \frac{\frac{1}{2}J\omega_s^2}{S_{MB}} = (J\omega_s)\frac{\omega_s}{2S_{MB}} = H$$

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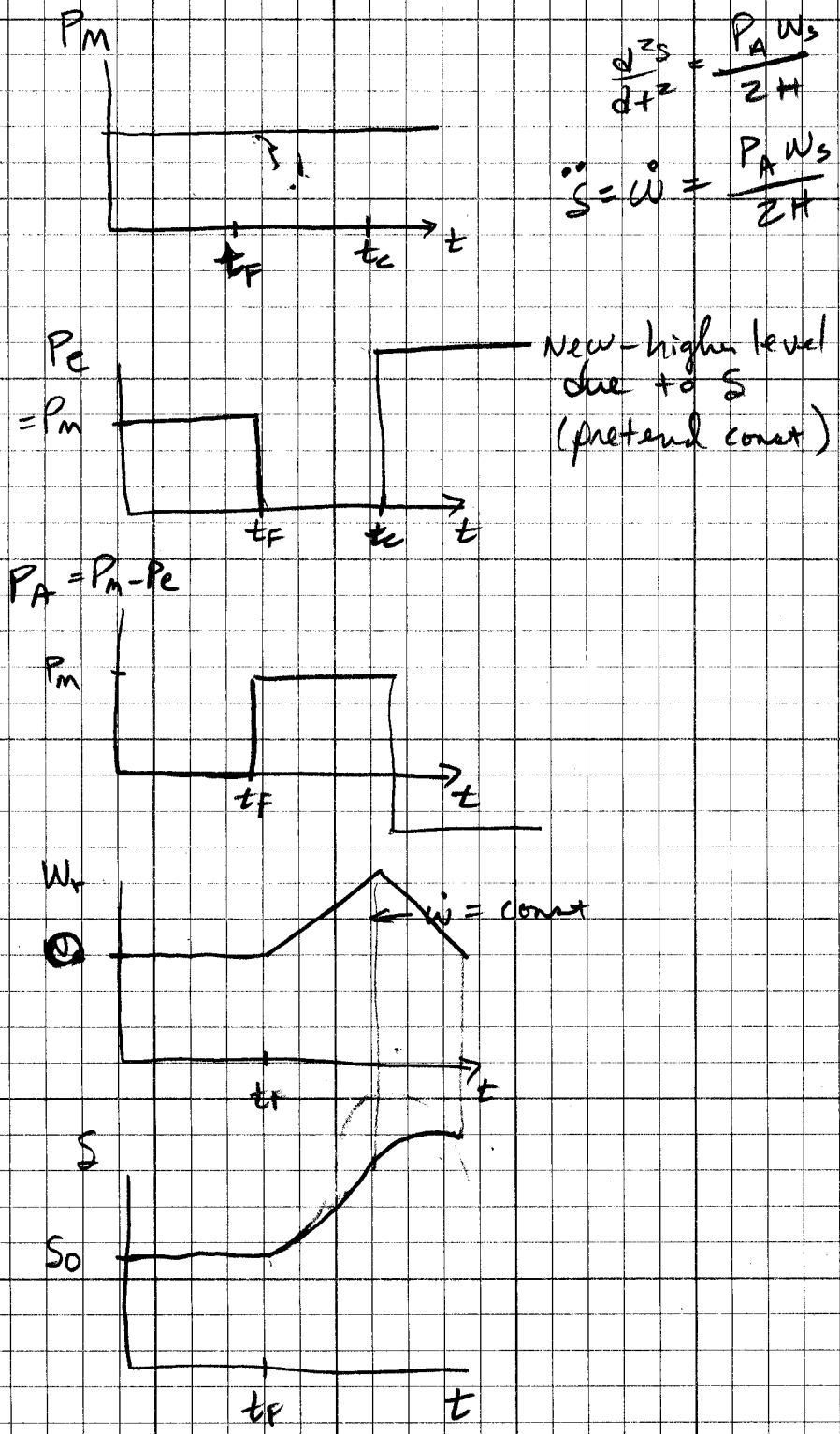

$$\frac{d^2\delta}{dt^2} = \frac{P_A \omega_s}{2H}$$

$$\ddot{\delta} = \dot{\omega} = \frac{P_A \omega_s}{2H}$$

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~~AA2/2~~

Actually, $\omega \neq \omega_s$, but % variation very small.

Let $J\omega = J\omega_s \rightarrow J\omega = \dfrac{2HS_{MB}}{\omega_s}$

Now, $$\frac{2HS_{MB}}{\omega_s}\frac{d^2\theta}{dt^2} = P_m - P_e$$

$$\frac{2H}{\omega_s}\frac{d^2\theta}{dt^2} = \frac{P_m}{S_{MB}} - \frac{P_e}{S_{MB}} = P_m - P_e \Big] \text{ per unit}$$

a sec rated power

325MW
93 ft long, 178 tons, on 7 bearings, 3RPM turning gear for 72 hours before stop

H for turbines (on own base) 4–9

Takes a long time to roll to a stop ~ 45min

Spins in Hydrogen
↳ 7% of air density
Better cooling — Would burn up in air – red hot tips
13ft DIAM wheel @ 3600
Pressurized system — no moisture, oxygen
Not explosive if >70% hydrogen
Check > 94% H

$\omega = 60(2\pi) = 377$ r/s
$R = 6\tfrac{1}{2}' = 1.98$m
$\omega R = 746.5$ m/s
$= 1700$ MPH
>MACH 2

Of course, θ is always increasing

$$\theta(t) = \int \underset{\substack{\uparrow \\ \text{synch}}}{\omega} dt + \theta_0$$

Steady state $\rightarrow P_m = P_e$, $\dfrac{d^2\theta}{dt} = 0$

Important quantity is movement of θ from synch θ.

Let $\theta(t) = \underbrace{\omega_s t + \theta_0}_{\text{steady state}} + \delta(t)$

$$\frac{d\theta(t)}{dt} = \omega_s + \frac{d\delta(t)}{dt}, \quad \frac{d^2\theta(t)}{dt^2} = \frac{d^2\delta(t)}{dt^2}$$

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$$P_{MECH} = P_{ELEC} + \frac{d}{dt} W_{KINETIC} + \cancel{P_{FRICTION}} \text{ SMALL} \quad (1)$$

$$W_{KINETIC} = \frac{1}{2} J \omega^2 = \frac{1}{2} J (\dot{\theta})^2$$

$$\frac{dW_{KINETIC}}{dt} = J\omega\dot{\omega} = J\omega\ddot{\delta} \approx J\omega_0\ddot{\delta}$$

So $P_{MECH} = P_{ELEC} + J\omega_0\ddot{\delta}$

$$\ddot{\delta} = \frac{P_{MECH} - P_{ELEC}}{J\omega_0}$$

$$\underbrace{\frac{P_{MECH}}{S_{B3\phi}}}_{P_{M,pu}} = \underbrace{\frac{P_{ELEC}}{S_{B3\phi}}}_{P_{E,pu}} + \frac{J\omega_0\ddot{\delta}}{S_{B3\phi}}$$

$$\frac{2\left(\frac{1}{2}J\omega_0^2\right)}{\omega_0 S_{B3\phi}} = \frac{2 W_{KINETIC}}{\omega_0 S_{B3\phi}} \leftarrow H \text{ sec.}$$

$$= \frac{2H}{\omega_0} = \frac{H}{\omega_0/2}$$

$$= \frac{H}{2\pi f_0/2} = \frac{H}{\pi f_0} = M$$

So

$$P_{MECH\,pu} - P_{ELEC\,pu} = M\ddot{\delta}$$

$$\ddot{\delta} = \frac{P_{MECH\,pu} - P_{ELEC\,pu}}{M}$$ The swing Equation

$$\omega_{REL} = \omega - \omega_0$$

$$\dot{\omega} = \dot{\omega}_{REL}$$

Before a disturbance, ω_{REL} is zero.

After a disturbance, ω_{REL} becomes zero again if stable (see curve)

$$\int_0^{t_{SMAX}} \omega_{REL}\, dt = 0$$

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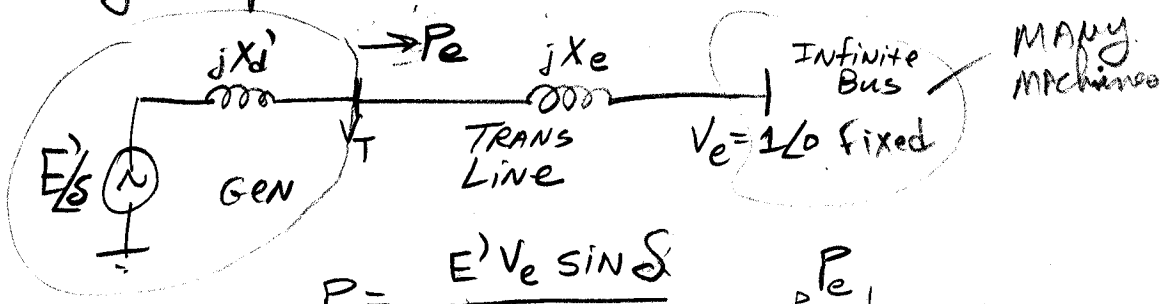
STAEDTLER® No. 937 811E Engineer's Computation Pad

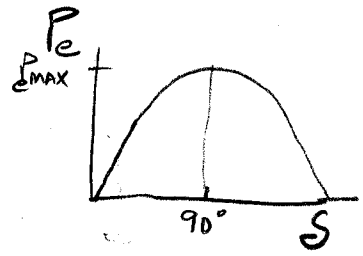
Then $$\boxed{\frac{2H}{\omega_s}\frac{d^2\delta}{dt^2} = P_M - P_e}$$ Per Unit swing eq. (No damping)

δ is departure from synch reference value

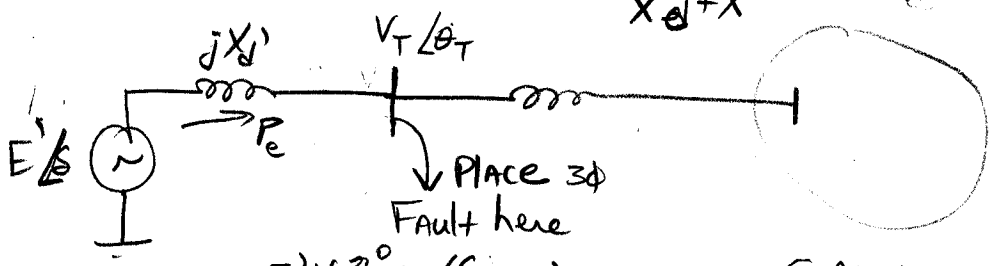
Why important?



$$P_e = \frac{E' V_e \sin\delta}{X_d' + X_e}$$



If E', V_e const, $P_{emax} = \dfrac{E' V_e}{X_d' + X}$



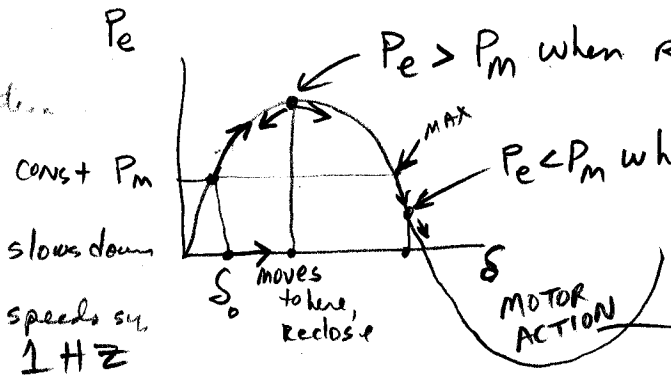
$$P_e = \frac{E' \cancel{V_T}^{\,0} \sin(\delta - \theta_T)}{X_d'} = 0$$

[Note - very large current, but no real power during fault]

Swing Equation

$$\frac{d^2\delta}{dt^2} = \frac{\omega_s}{2H}\left[P_M - \cancel{P_e}^{\,0}\right] = \frac{\omega_s P_M}{2H}$$ Large pos number

show perturbat...



$P_e > P_m$ when reclose, $\dot{\omega} < 0$, $\dfrac{d^2\delta}{dt^2} < 0$, slows down, but overshoots

$P_e < P_m$ when reclose too late, $\dot{\omega}$, $\dfrac{d^2\delta}{dt^2} > 0$, keeps accelerating

$P_e > P_m$ slows down

$P_M > P_e$ speeds up

Freq $\simeq$ 1 Hz

MACHINE TRIPS OFF TO AVOID SERIOUS over currents

trip 108% overspeed

ω_r: dep from ω_s

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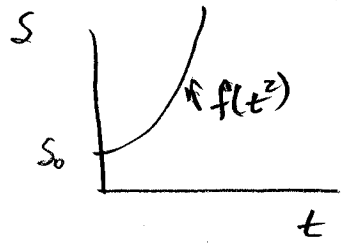
What is the MAX reclose δ for stability?

$$\omega_r = \frac{d\delta}{dt} = \int_0^t \frac{d^2\delta}{dt^2} = \frac{\omega_s P_m t}{2H} \quad \text{(Above synch)}$$

$$\delta = \int \omega\, dt = \frac{\omega_s P_m t^2}{4H} + \delta_0 \quad \text{(Above synch)}$$



rapid increase

$$\frac{d\omega_r}{dt} = \frac{\omega_s P_A}{2H}$$

$$\omega_r = \frac{\omega_s}{2H}\int P_A\, dt$$

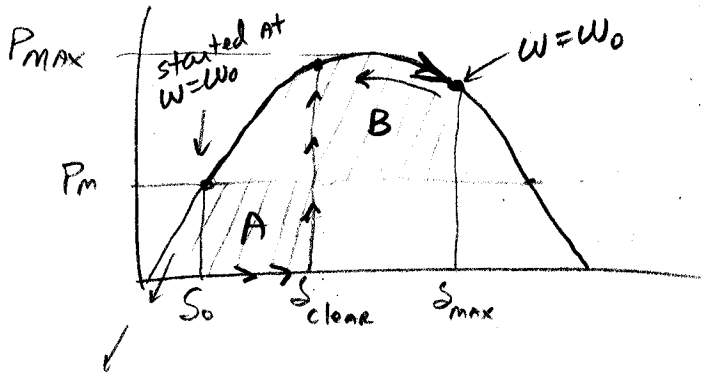
Starts & stops where $\omega_r = 0$

$$\int P_A\, dt = 0$$

$$d\delta = \omega_r\, dt$$

$$\int \frac{P_A\, d\delta}{\omega_r} =$$

EQUAL AREA criteria



accel power

$$\frac{d\omega_r}{dt} = \frac{\omega_s P_A}{2H} \rightarrow d\omega_r = \frac{\omega_s P_A}{2H}\, dt$$

Also, $d\delta = \omega_r\, dt \rightarrow dt = \frac{d\delta}{\omega_r} \rightarrow d\omega_r = \frac{\omega_s P_A}{2H\omega_r}\, d\delta$

start & finish at $\omega_r = 0$, $\omega_r = 0$

$$\omega_r\, d\omega_r = \frac{\omega_s P_A}{2H}\, d\delta \qquad \text{Integrate both sides}$$

$\int \omega_r\, d\omega_r = 0$ since ω_r same integration limits

$$\therefore \int_{\delta_0}^{\delta_{max}} P_A\, d\delta = 0 \qquad \text{Equal area criteria}$$

$$A = B$$

$$A = P_m(\delta_{CLEAR} - \delta_0)$$

$$B = \int_{\delta_{CLEAR}}^{\delta_{max}} (P_e - P_m)\, d\delta$$

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But $P_e = P_{MAX} \sin \delta$

$$B = P_{MAX} \int_{\delta_{CLEAR}}^{\delta_{MAX}} \sin \delta \, d\delta - P_M (\delta_{MAX} - \delta_{CLEAR})$$

$$= P_{MAX} \left[\cos(\delta_{CLEAR}) - \cos(\delta_{MAX}) \right] - P_M \left[\delta_{MAX} - \delta_{CLEAR} \right]$$

$A = B$

$$\cancel{P_M \delta_{CLEAR}} - P_M \delta_0 = P_{MAX} \left[\cos(\delta_{CLEAR}) - \cos(\delta_{MAX}) \right] - P_M \delta_{MAX} + \cancel{P_M \delta_{CLEAR}}$$

$$\cos(\delta_{CLEAR}) = \frac{P_M}{P_{MAX}} \left[\delta_{MAX} - \delta_0 \right] + \cos(\delta_{MAX})$$

WHAT IS δ_{MAX} allowed? (Intersection of P_M)

$= \pi - \delta_0$

Also, $P_M = P_{MAX} \sin \delta_0$

$$\cos(\delta_{CLEAR}) = \sin \delta_0 \left[\pi - \delta_0 - \delta_0 \right] + \underbrace{\cos(\pi - \delta_0)}_{= -\cos \delta_0}$$

$$\boxed{\delta_{CLEAR} = \cos^{-1} \left[(\pi - 2\delta_0) \sin \delta_0 - \cos \delta_0 \right]}$$ critical clearing

How long?

$$\delta_{CLEAR} = \frac{\omega_s P_M t_{CLEAR}^2}{4H} + \delta_0$$

$$\boxed{t_{CLEAR} = \sqrt{\frac{4H(\delta_{CLEAR} - \delta_0)}{\omega_s P_M}}}$$ critical clearing time

≤ 10 ~~s~~

$\approx .166$ sec

RAD/sec

H, P_M, δ_0 important

MJ/MVA, pu, RAD

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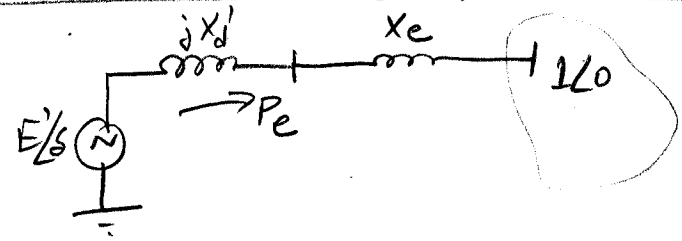
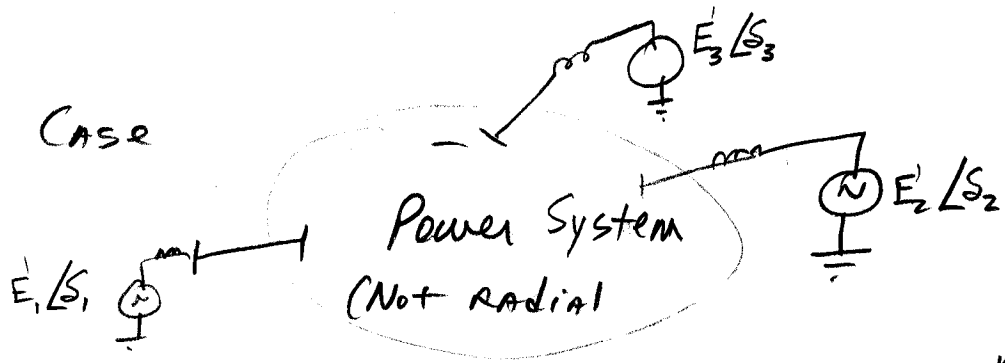
Solution of Swing Equation



$$\underbrace{\frac{d^2\delta}{dt^2} = \frac{\omega_s}{2H}\left[P_m - P_e\right] = \frac{\omega_s P_m}{2H} - \frac{\omega_s}{2H} P_{MAX} \sin\delta}_{\text{Nonlinear D.E.}}$$

Gen Case



$$\frac{2H^N}{\omega_s}\frac{d^2\delta_N^N}{dt^2} = P_M^N - P_e^N$$

Function of E, $\angle\delta$ ← controlled by voltage regulator

P_M^N: either constant or controlled by steam/governor system

How to get P_e

If loads are represented as <u>constant Z's</u> to gnd, knowing E's, $\angle\delta$'s will yield all network voltages & currents ⟶ powers P_e (w/o loadflow)

<u>Constant Power Loads</u>

Run loadflow with E, $\angle\delta$ known. Solve for powers P_e

<u>Constant Current Loads</u> ?

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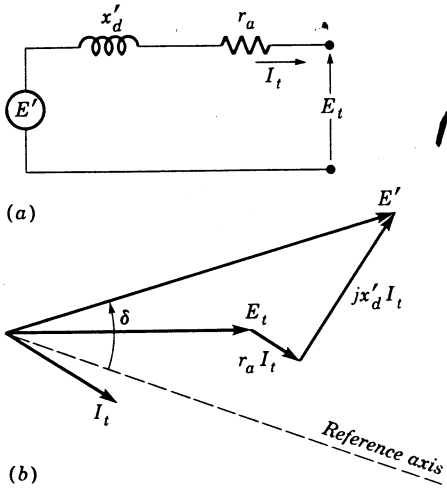


Mod 1

Fig. 10.1 Simplified representation of a synchronous machine. (a) Equivalent circuit; (b) phasor diagram.

synchronous reactance and is determined from

$$E_q = E_t + r_a I_t + jx_q I_t$$

where E_q = voltage back of quadrature-axis synchronous reactance
x_q = quadrature-axis synchronous reactance

The representation of the synchronous machine used for network solutions and the corresponding phasor diagram are shown in Fig. 10.2.

The sinusoidal flux produced by the field current acts along the direct axis. The voltage induced by field current lags this flux by 90° and, therefore, is on the quadrature axis. This voltage can be determined by adding to the terminal voltage E_t the voltage drop across the armature resistance and the voltage drops representing the demagnetizing effects along the direct and quadrature axes. Then neglecting saturation,

$$E_I = E_t + r_a I_t + jx_d I_d + jx_q I_q$$

where E_I = voltage proportional to field current
x_d = direct-axis synchronous reactance
x_q = quadrature-axis synchronous reactance, Grady 2007, p. 323 Saliency
I_d = direct-axis component of machine terminal current
I_q = quadrature-axis component of machine terminal current

Mod 2

The phasor diagram showing E_I as well as the voltage back of transient reactance is shown in Fig. 10.3.

The quadrature component of voltage back o. ansient reactance from the phasor diagram is

$$E'_q = E_q - j(x_q - x'_d)I_d$$

where E'_q is the voltage proportional to the field flux linkages resulting from the combined effect of the field and armature currents. Since the field flux linkages do not change instantaneously following a disturbance, E'_q also does not change instantaneously. The rate of change of E'_q along the quadrature axis is dependent on the field voltage controlled by the regulator and exciter, the voltage proportional to the field current, and the direct-axis transient open circuit time constant and is given by

$$\frac{dE'_q}{dt} = \frac{1}{T'_{d0}}(E_{fd} - E_I)$$ ← Mod 2

where E_{fd} = term representing the field voltage acting along the quadrature axis

T'_{d0} = direct-axis transient open circuit time constant

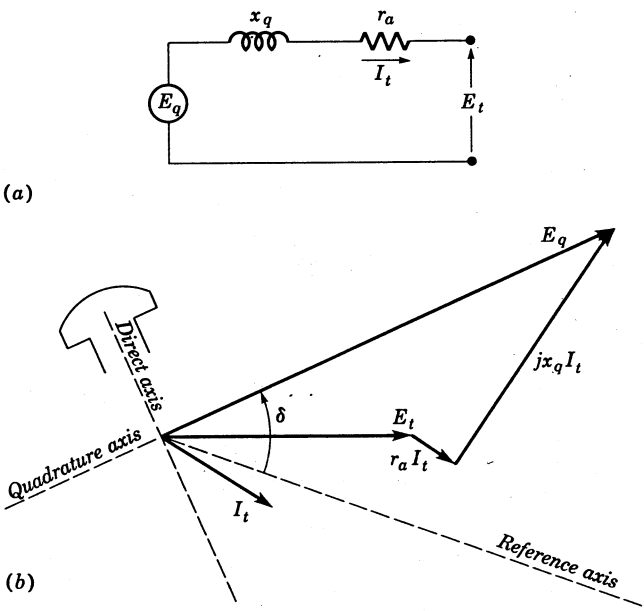
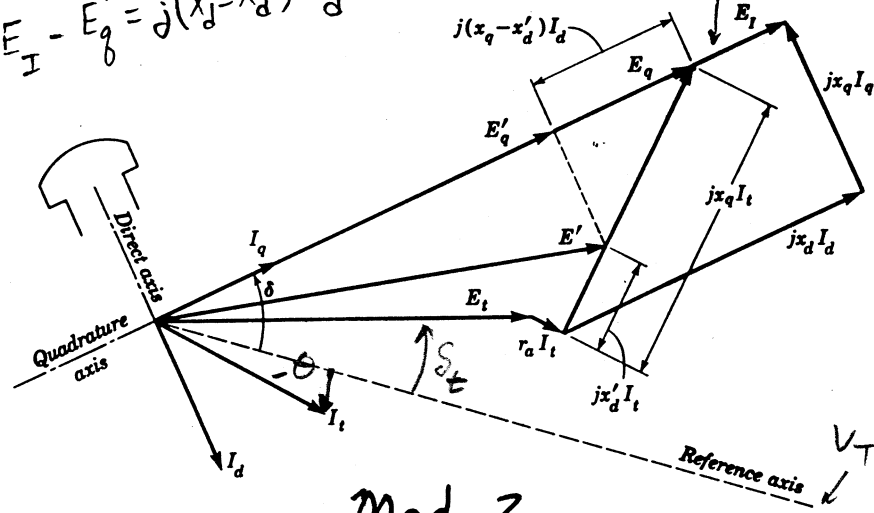


Fig. 10.2 Representation of a synchronous machine for determining E_q. (a) Equivalent circuit; (b) phasor diagram.

$E_I - E_q' = j(X_d - X_d')I_d$

$(X_d - X_q) \cdot I_d$



$V_{TH} \angle 0$

Mod 2

Fig. 10.3 Phasor diagram for determining the quadrature-axis component of voltage behind transient reactance.

Induction machines

In power system transient stability studies loads, including induction motors, usually can be represented adequately by shunt impedances. However, in studies involving large induction motor loads it is necessary frequently to represent the induction motors in a more detailed manner. Induction motors are used extensively in industrial processes and can have significant effects on the transient response of a power system.

A reasonable linear representation of an induction machine can be obtained by taking into account the effects of mechanical transients and rotor electrical transients. The effects of stator electrical transients on system response usually can be neglected. The equivalent circuit shown in Fig. 10.4 has been used to represent the transient behavior of an induction motor including the effects of mechanical transients and rotor electrical transients with a single time constant.

The differential equation describing the rate of change of the voltage behind transient reactance X' is

$$\frac{dE'}{dt} = -j2\pi f s E' - \frac{1}{T_0}\{E' - j(X - X')I_t\}$$

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where the rotor open circuit time constant T_0 in seconds is

$$T_0 = \frac{x_r + x_m}{2\pi f r_r}$$

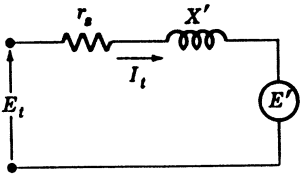


Fig. 10.4 Simplified represenla induction machine for transient a

and the terminal current is

$$I_t = (E_t - E')\frac{1}{r_s + jX'}$$

The reactances X and X' can be obtained from the steady state equivalent circuit of an induction machine Fig. 10.5, where

r_s is the stator resistance in per unit
x_s is the stator reactance in per unit
r_r is the rotor resistance in per unit
x_r is the rotor reactance in per unit
x_m is the magnetizing reactance in per unit
s is the rotor slip in per unit

The resistances and reactances are all on the same kva base. of the base voltages of the stator and rotor is equal to the voltage ratio at standstill. The per unit slip is

$$s = \frac{\text{Synchronous speed} - \text{actual speed}}{\text{Synchronous speed}}$$

Since the rotor resistance r_r is small compared with the react be neglected in the calculation of X and X'. From the equivalent circuit, then, the open circuit reactance is appro

$$X = x_s + x_m$$

The blocked rotor reactance is approximately

$$X' = x_s + \frac{x_m x_r}{x_m + x_r}$$

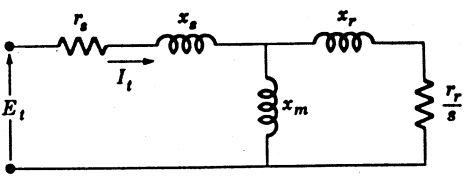
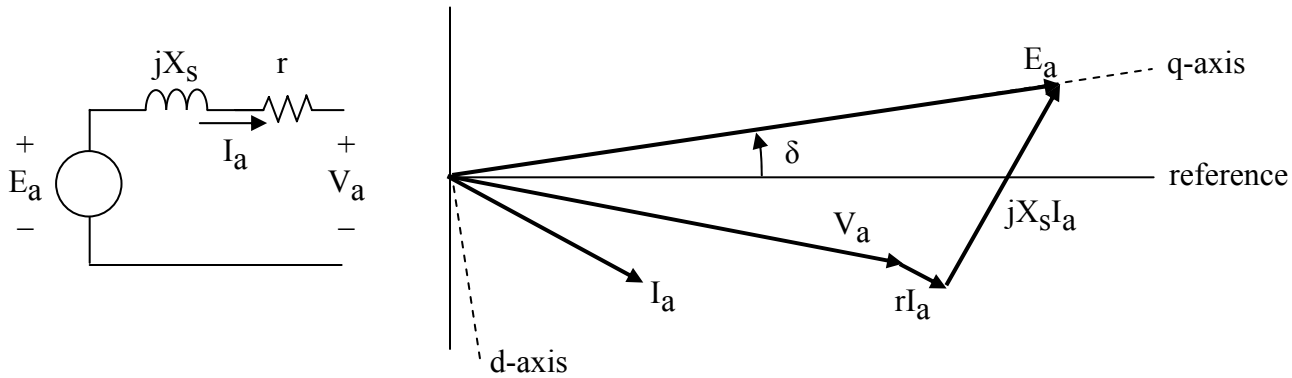


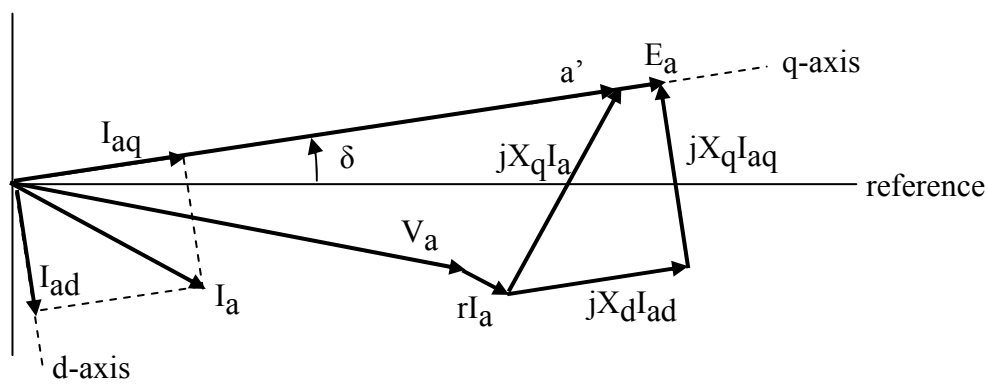
Fig. 10.5 Steady sequence equivalı an induction mac

Steady State Model for Round Rotor Machine

V_a and I_a are the terminal voltage and current. X_s is the synchronous reactance. Resistance r is the stator resistance. E_a is the Thevenin equivalent voltage behind synchronous reactance.



Steady-State Model for Salient Pole Machine
(No equivalent circuit. Phasor diagram only)



Steps when V_a and I_a are known:

- With V_a, I_a, r, and X_q, find phasor a' to locate the q-axis.
- Then, compute I_a projections I_{aq} and I_{ad} where

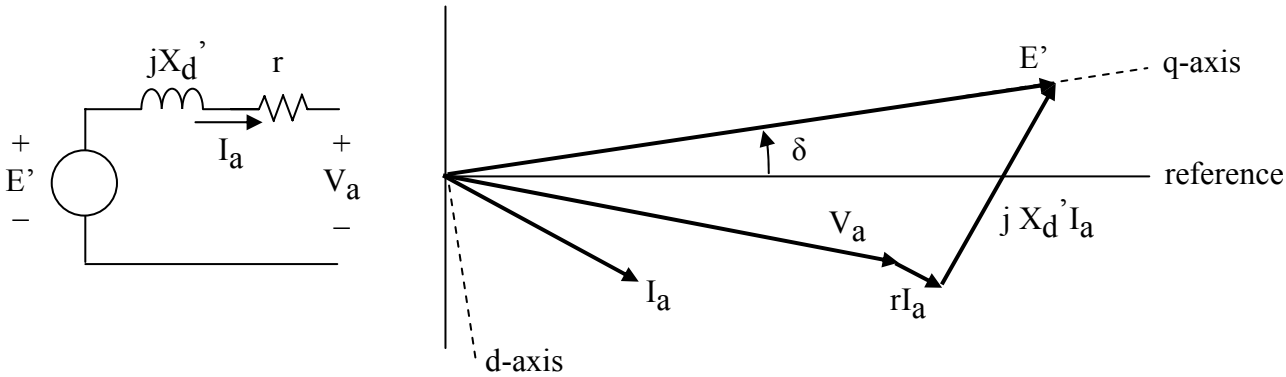
  I_{aq} has magnitude $|I_a|\cos(\delta - \angle I_a)$ and has phase angle δ,

  I_{ad} has magnitude $|I_a|\sin(\delta - \angle I_a)$ and has phase angle $(\delta - 90°)$,

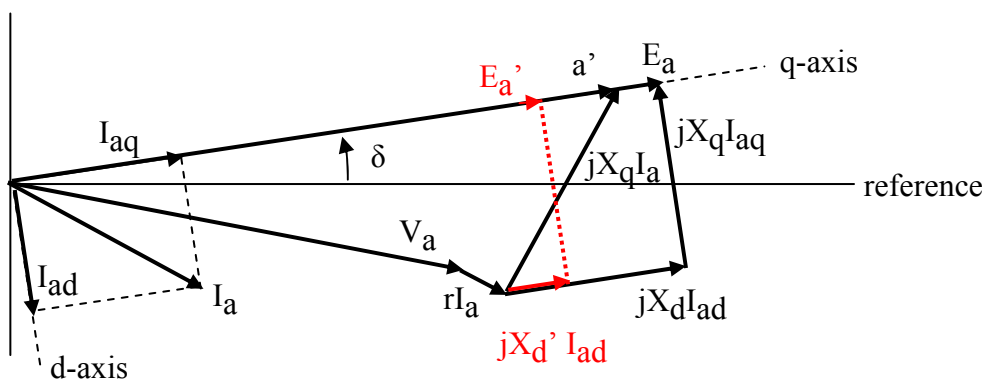
  and where $\angle I_a$ is the angle of I_a.
- Then, find E_a.

Transient Stability Machine Model 1
Constant Voltage Magnitude E' Behind Transient Reactance X_d'

This model is like the round rotor synchronous model, except that the transient reactance is used instead of the synchronous reactance.



Transient Stability Machine Model 2
Salient Pole Rotor



Steps when V_a and I_a are known: Begin with same steps used in the salient pole synchronous machine model. Then, use X_d' and I_{ad} to find E_a'.

The magnitude of E_a' varies according to $\frac{d\left|E_a'\right|}{dt} = \frac{1}{T_{do}'}\left(E_{fd} - \left|E_a\right|\right)$, where T_{do}' is the direct-axis transient open circuit time constant, and E_{fd} is the field voltage (as seen from the stator). As a first approximation, E_{fd} is treated as a constant whose initial value is the same as the initial value of $\left|E_a\right|$.

A synchronous generator is operating in steady-state with the following conditions (in per unit): $V_a = 1.0\underline{/0°}$, $I_a = 0.8\underline{/-30°}$, $X_d = 1.0$, $X_q = 0.6$, $r = 0$. Find S (at the machine terminals), and phasors I_{ad}, I_{aq}, E_a.

EE369, Test 10, Nov. 19. Name: KEY

One sheet of notes permitted. Show all work.

A synchronous generator is operating in steady-state with the following conditions (in per unit): $V_a = 1.0\angle 0°$, $I_a = 0.8\angle -30°$, $X_d = 1.0$, $X_q = 0.6$, $r = 0$. Find S (at the machine terminals), and phasors I_{ad}, I_{aq}, E_a.

$$S = V_a I_a^* = (1.0\angle 0)(0.8\angle -30)^* = 0.8\angle 30° \text{ pu}$$

Find $a' = V_a + \cancel{r}^{\,0} I_a + jX_q I_a = 1.0 + j0.6(0.8\angle -30°)$

$$a' = 1.0 + 0.48\angle 60° = 1.0 + 0.24 + j0.417$$

$$\boxed{a' = 1.308\angle 18.6°}. \text{ Thus, } \boxed{\delta = 18.6°}$$

Find I_{aq}, I_{ad}

$$|I_{aq}| = |I_a|\cos(\delta - \angle I_a) = 0.8\cos(18.6° + 30°) = 0.529$$

$$|I_{ad}| = |I_a|\sin(\delta - \angle I_a) = 0.8\sin(18.6 + 30°) = 0.600$$

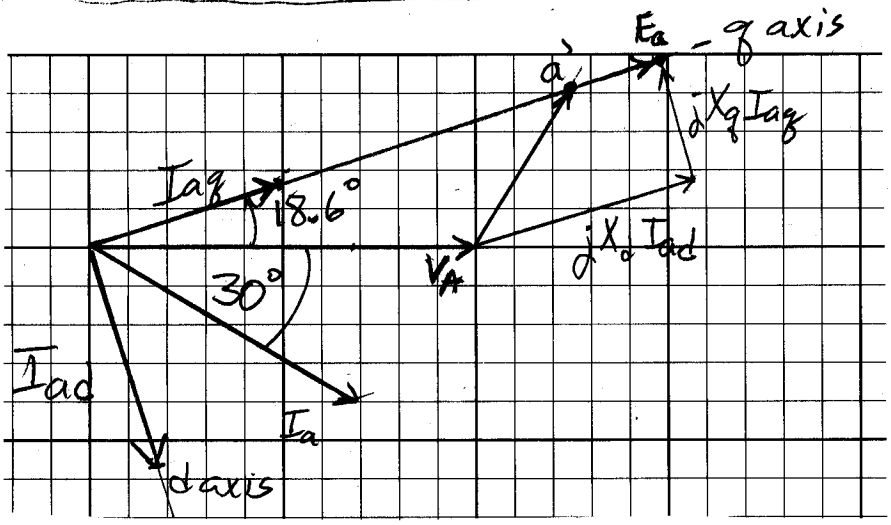
So, $\boxed{I_{aq} = 0.529\angle 18.6°}$, $\boxed{I_{ad} = 0.600\angle 18.6 - 90 = 0.600\angle -71.4°}$

Find $E_a = V_a + \cancel{r}^{\,0} I_a + jX_d I_{ad} + jX_q I_{aq}$

$$= 1.0 + (1.0)(0.600)\angle -71.4 + 90° + (0.6)(0.529)\angle 18.6° + 90°$$

$$= 1.0 + 0.569 + j0.191 - 0.101 + j0.301$$

$$\boxed{E_a = 1.468 + j0.492 = 1.548\angle 18.5°}$$ OK, Angle checks with δ.



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Bergen Problem 6.2. A generator with reactances $X_d = 1.6$ and $X_q = 0.9$ delivers $S_G = 1.0\angle 45^\circ$ to a bus with voltage $V_a = 1.0\angle 0^\circ$. Find I_a, I_{ad}, I_{aq}, E_a, and the rotor angle θ_o (i.e., the rotor angle θ at $t = 0$).

Bergen Problem 6.4. Given a generator with $V_a = 1.0\angle 0^\circ$, $I_a = 1.0\angle 60^\circ$, $X_d = 1.0$, $X_q = 0.6$, and $r = 0.1$, find S_G, I_{ad}, I_{aq}, and E_a.

6.2

$V_a I_a^* = S_g$, $I_a^* = \frac{S_g}{V_a} = \frac{1\angle 45}{1\angle 0} = 1\angle 45$, $I_a = 1\angle -45$

Find $a' = V_a + r_a I_a + jX_q I_a$ to locate the q-axis

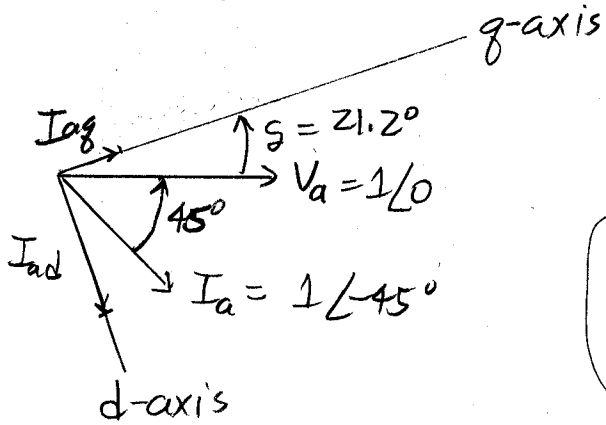
$= 1\angle 0 + (0)I_a + j0.9(1\angle -45) = 1\angle 0 + 0.9\angle 45$

$= (1 + j0) + (0.636 + j0.636) = 1.636 + j0.636$

$\boxed{a' = 1.755\angle 21.2^\circ \text{ V.}}$

So, since V_a has Ref. angle 0°, $\boxed{\delta = 21.2^\circ}$

Find I_{ad}, I_{aq}. $I_{ad} = |I_a| \sin(\overset{(21.2+45)}{\delta - \angle I_a}) = \boxed{0.915_{pu} = I_{ad}}$



$I_{aq} = |I_a| \cos(\delta - \angle I_a)$
$= 0.404_{pu}$

So, $I_{ad} = 0.915\angle \delta - 90^\circ$
$= 0.915\angle -68.8^\circ$
$I_{aq} = 0.404\angle \delta = 21.2^\circ$

Find $E_a = V_a + rI_a + jX_d I_{ad} + jX_q I_{aq}$

$= 1 + j1.6(0.915\angle 68.8) + j0.9(0.404\angle 21.2)$

$= 1 + 1.464\angle 21.2 + 0.364\angle 111.2$

$= 1 + 1.365 + j0.529 - 0.1316 + j0.339$

$= 2.23 + j0.868$

$\boxed{E_a = 2.39\angle 21.2 \text{ pu}}$ (Angle checks with δ)

overexcited
because $\angle I_a$ is lagging, so E_a is leading

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6.4

$S_g = V_a I_a^* = (1\angle 0)(1\angle +60)^* = 1\angle -60^\circ$

$a' = V_a + r_a I_a + jX_q I_a = 1 + (0.1)(1\angle 60) + j(0.6)(1\angle 60)$

$a' = 1 + 0.1\angle 60 + 0.6\angle 150 = 1 + 0.05 + j0.0866 - 0.520 + j0.3$

$a' = 0.53 + j0.387 = 0.656\angle 36.1^\circ$

So, the q-axis is at 36.1°

$|I_{aq}| = |I_a|\cos(\delta - \angle I_a) = 1\cos(36.1 - 60) = 1\cos(-23.9)$

$\boxed{I_{aq} = 0.914\angle \delta = 0.914\angle 36.1}$

$|I_{ad}| = |I_a|\sin(\delta - \angle I_a) = 1\sin(36.1 - 60) = -0.405$

$I_{ad} = -0.405\angle 36.1 - 90^\circ = -0.405\angle -53.9 = 0.405\angle 126.1$

$I_{ad} = 0.405\angle 126.1$

$\hat{E}_a = V_a + r_a I_a + jX_d I_{ad} + jX_q I_{aq}$

$= 1 + (0.1)(1\angle 60) + j(1.0)(0.405\angle 126.1) + j(0.6)(0.914\angle 36.1)$

$= 1 + 0.1\angle 60 + 0.405\angle 216.1 + 0.548\angle 126.1$

$= 1 + 0.05 + j0.0866 - 0.327 - j0.239 - 0.323 + j0.443$

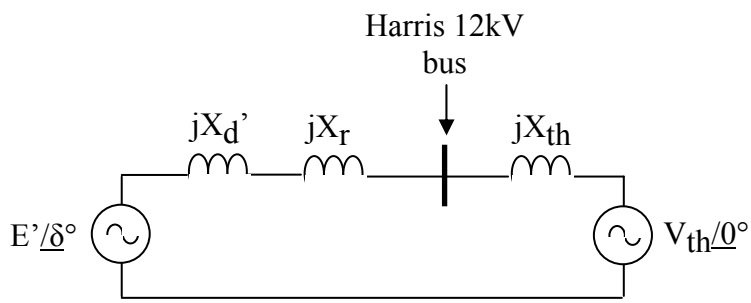
$= 0.400 + j0.291$

$= 0.495\angle 36.1$ ← Underexcited because $\angle I_a$ is leading so E_a is small

Checks with δ

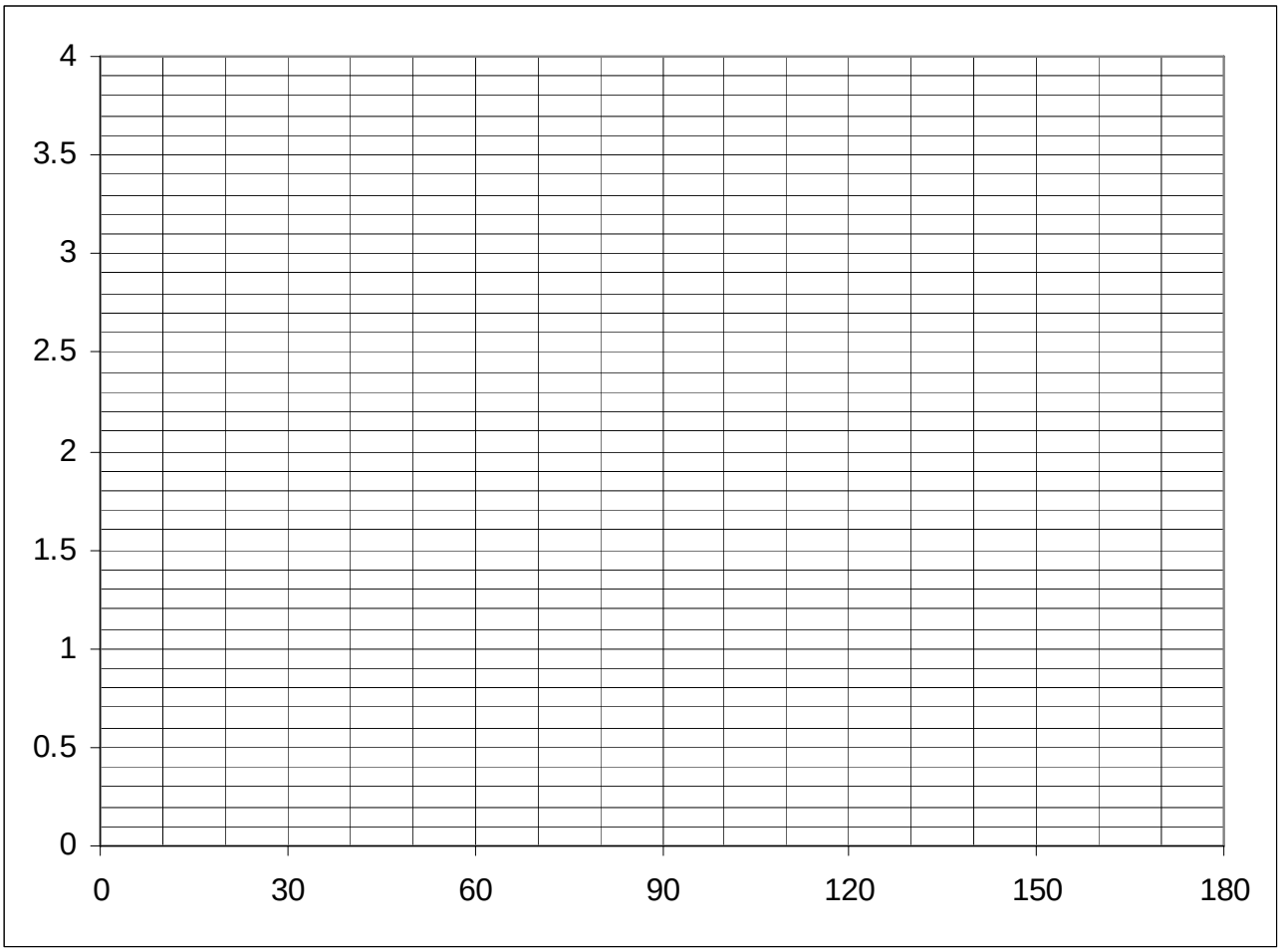
A new 32MVA, 60Hz steam generator was recently installed at the UT power plant. The generator is connected through a large reactor to ERCOT at the Harris 12kV bus.

The electrical diagram and internal voltage (for the transient condition) that exists when the generator is producing (25MW + j12.4MVAr) is shown below. Suddenly, at t = 0, a three-phase fault with zero impedance occurs at the Harris 12kV bus. Assuming that E', V_{th}, and P_{mech} are constant, **determine the critical clearing time (in cycles)**. Use the attached graph to show important P and δ points and lines on the power curve, and show clearly which areas are equal.



Machine E' = 1.100pu
Machine δ = 16.5°
Machine P = 25MW (i.e., 0.781pu)
Machine jX_d' = j0.22pu
Machine H = 1.4seconds
Reactor jX_r = j0.17pu
Thevenin jX_{th} = j0.01pu
Thevenin V_{th} = (1.0 + j0)pu
Values given on the generator base

Remember that during the fault, the electrical power output is zero, and thus the rotor accelerates according to $\delta(t) = \frac{\omega_s P_{mech}}{4H} t^2 + \delta_o$, where $\delta_o = \delta(t=0)$. Also, note that ω_s = 377rad/sec, δ is in radians, and P_{mech} is in pu. The formula for critical clearing angle is $\delta_{crit} = \cos^{-1}\left[(\pi - 2\delta_o)\sin(\delta_o) - \cos(\delta_o)\right]$.



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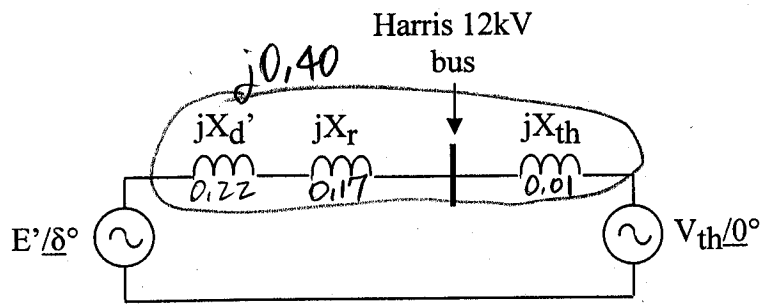
EE369, Final Exam, 2-5pm, Dec. 9, WRW102. Name: KEY

Six equally-weighted problems. Three sheets of notes permitted. Show all work.

Problem 6. A new 32MVA, 60Hz steam generator was recently installed at the UT power plant. The generator is connected through a large reactor to ERCOT at the Harris 12kV bus.

The electrical diagram and internal voltage (for the transient condition) that exists when the generator is producing (25MW + j12.4MVAr) is shown below. Suddenly, at t = 0, a three-phase fault with zero impedance occurs at the Harris 12kV bus. ~~The fault is successfully removed in 6 cycles (of 60Hz).~~ Assuming that E', V_{th}, and P_{mech} are constant, ~~use the equal area criterion to find the maximum value that δ achieves~~. Use the attached graph to show important P and δ points and lines on the power curve, and show clearly which areas are being set equal. determine the critical clearing time (in cycles).



Machine E' = 1.100pu
Machine δ = 16.5°
Machine P = 25MW (i.e., 0.781pu)
Machine jX_d' = j0.22pu
Machine H = 1.4seconds
Reactor jX_r = j0.17pu
Thevenin jX_{th} = j0.01pu
Thevenin V_{th} = (1.0 + j0)pu
Values given on the generator base

Remember that during the fault, the electrical power output is zero, and thus the rotor accelerates according to $\delta(t) = \frac{\omega_s P_{mech}}{4H} t^2 + \delta(t=0)$. Also, note that ω_s = 377rad/sec, δ is in radians in the above formula, and P_{mech} is in pu. $\delta_{crit} = \cos^{-1}\left[(\pi - 2\delta_0)\sin(\delta_0) - \cos(\delta_0)\right]$

$\delta_0 = 16.5°$, so $\delta_{crit} = \cos^{-1}\left[\underbrace{(\pi - 2(0.288))\sin(16.5°)}_{0.729} - \underbrace{\cos(16.5°)}_{0.959}\right]$

$= 0.288$ rad

$\boxed{\delta_{crit} = 103.3°}$

$= 1.803$ rad

Use $\delta(t) = \frac{\omega_s P_{mech}}{4H} t^2 + \delta_0$ to get $t^2 = \frac{4H}{\omega_s P_{mech}}(\delta_{crit} - \delta_0)$

1.803 0.288

$t^2 = \frac{4(1.4)}{(377)(0.781)}(1.803 - 0.288)$

On curve,

$P_{max} = \frac{E' V_{TH}}{j0.40}\sin 90° = 2.75$

$\boxed{t = 0.1697 \text{ sec} = 10.18 \text{ cycles of } 60\text{Hz}}$

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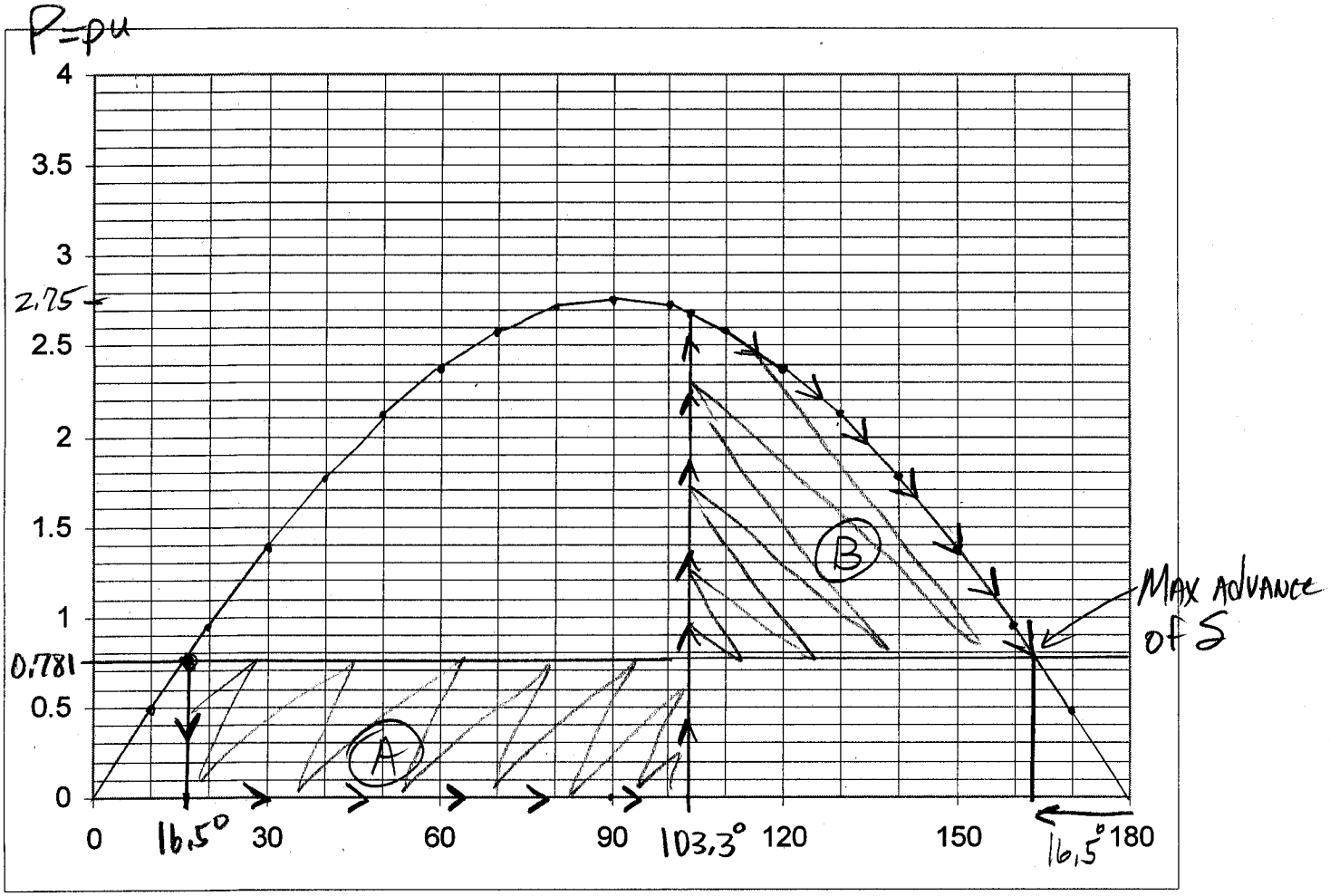
EE369, Final Exam, 2-5pm, Dec. 9, WRW102. Name: KEY

Six equally-weighted problems. Three sheets of notes permitted. Show all work.



→ δ movement

Areas (A) = (B)

δ decelerating

δ Accelerating

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TRANSMISSION AND DISTRIBUTION OF ELECTRICAL ENERGY

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1817

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Chapter 2

Transmission Line Equations, Parameters, and Solutions

The lines and cables that deliver electrical energy have characteristic values of circuit parameters determined by their sizes, shapes, and material constitution. As suggested in the last chapter, for many diverse reasons it is important to know the specific values of these parameters. In this chapter, the calculation of these parameters is presented in probably the easiest and shortest correct method. The inherent relationship between the several circuit parameters is brought out by a consideration of the theory from both a distributed circuit and an electromagnetic field approach. In this way the labor of calculating the "external" line parameters is cut almost in half.

2.1 LINE EQUATIONS AND SOLUTIONS FROM A DISTRIBUTED CIRCUITS VIEWPOINT

First consider only single-phase lines. The physical arrangement of two types of such lines is suggested in Fig. 2.1(a). The circuit model for an incremental length, Δz, of such lines is suggested in Fig. 2.1(b). The transmission line (or telegrapher's) equations may be obtained by applying Kirchoff's voltage and current laws to the circuit, so as to obtain, in the

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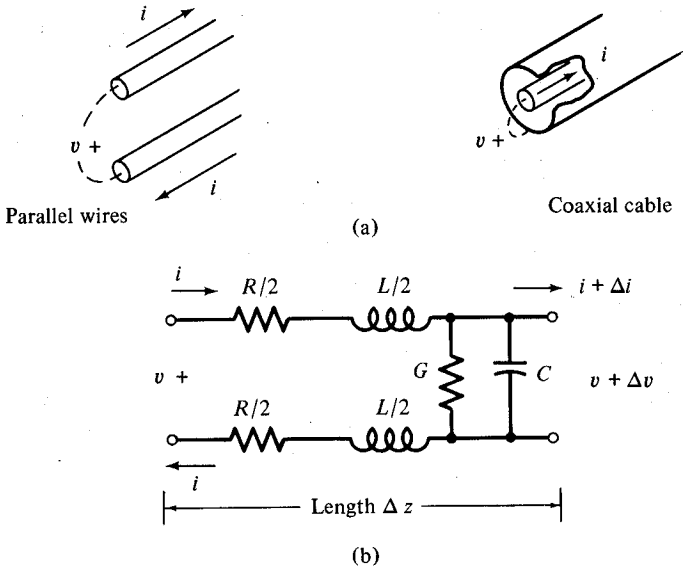

Figure 2.1 Some typical transmission lines and a circuit model for an incremental length. (a) Structures. (b) Distributed circuit model.

limit as $\Delta z \to 0$, the equations

$$\frac{\partial v}{\partial z} = -\mathcal{L}\frac{\partial i}{\partial t} - \mathcal{R}i$$

$$\frac{\partial i}{\partial z} = -\mathcal{C}\frac{\partial v}{\partial t} - \mathcal{G}v \tag{2.1}$$

In these equations $\mathcal{L}, \mathcal{R}, \mathcal{C}, \mathcal{G}$, are the per unit length values of inductance, resistance, capacitance, and conductance.

These equations can be solved with the aid of Laplace transforms to eliminate the time, and we will do this later. For now, we will simplify matters by assuming that $\mathcal{R}$ and $\mathcal{G}$ are zero. Then, combining the simplified equations by eliminating one or the other of the two variables, we obtain the equations

$$\frac{\partial^2 v}{\partial z^2} = \mathcal{L}\mathcal{C}\frac{\partial^2 v}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 i}{\partial z^2} = \mathcal{L}\mathcal{C}\frac{\partial^2 i}{\partial t^2} \tag{2.2}$$

Specific solutions to these equations depend on boundary, initial, and source conditions, but it is easy to show by trial that *any* function of the form

$$v = f(t \pm \sqrt{\mathcal{L}\mathcal{C}}\, z) \quad \text{or} \quad v = f\left(z \pm \frac{t}{\sqrt{\mathcal{L}\mathcal{C}}}\right)$$

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satisfies the equations. Equations 2.2 are called *wave equations* and the solutions are traveling waves. The "plus" sign option in the solutions gives

waves traveling in the negative z direction, the "minus" sign option gives waves traveling in the positive z direction. Each solution option describes a voltage wave that travels with a speed $c=1/\sqrt{\mathcal{L}\mathcal{C}}$. To repeat, $f(t\pm z/c)$ represents an arbitrary function; which particular function applies in a given situation is determined by the excitation.

Suppose we have a voltage function $v=f(t-z/c)$. The current function is then determined by the simplified line equations since

$$\frac{\partial i}{\partial t}=-\frac{1}{\mathcal{L}}\frac{\partial v}{\partial z}=-\frac{1}{\mathcal{L}}\left(-\frac{1}{c}\right)f'\left(t-\frac{z}{c}\right)$$

and integrating* the time function $i=(1/\mathcal{L}c)f(t-z/c)$. Note then that the ratio of v to i is a constant,

$$\frac{v}{i}=\frac{f(t-z/c)}{\frac{1}{\mathcal{L}c}f(t-z/c)}=\mathcal{L}c=\mathcal{L}\frac{1}{\sqrt{\mathcal{L}\mathcal{C}}}=\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\equiv Z_0$$

this constant ratio being called the surge impedance (or sometimes the *characteristic* impedance). Similarly, if the current is a known function,

$$i=f_1\left(t-\frac{z}{c}\right),\qquad \frac{\partial v}{\partial t}=\frac{1}{\mathcal{C}c}f_1'\left(t-\frac{z}{c}\right),\qquad v=\frac{1}{\mathcal{C}c}f_1\left(t-\frac{z}{c}\right)$$

$$\frac{i}{v}=\mathcal{C}c=\mathcal{C}\frac{1}{\sqrt{\mathcal{L}\mathcal{C}}}=\sqrt{\frac{\mathcal{C}}{\mathcal{L}}}=\frac{1}{Z_0}\equiv Y_0$$

the latter being known as the surge admittance.

Example 2.1

Suppose a current $I_g=Af(t)$ is injected into an infinitely long transmission line at its center, $z=0$, as in Fig. 2.2. The function may be thought of as a pulse of any shape. The current splits so that

$$i(t,0+)=\frac{A}{2}f(t),\qquad i(t,0-)=\frac{-A}{2}f(t)$$

Thus, for points $z>0$, the solutions to Eqs. 2.2 are

$$i=\frac{A}{2}f\left(t-\frac{z}{c}\right),\qquad v=Z_0\frac{A}{2}f\left(t-\frac{z}{c}\right)$$

This is a wave traveling to the right (i.e., the pulse propagates). For $z<0$, the solutions are

$$i=\frac{-A}{2}f\left(t+\frac{z}{c}\right),\qquad v=Z_0\frac{A}{2}f\left(t+\frac{z}{c}\right)$$

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*We can ignore the arbitrary constant associated with the integration; that constant simply allows for the possibility of a steady dc current to be present along with the time varying current that is of main interest to us.

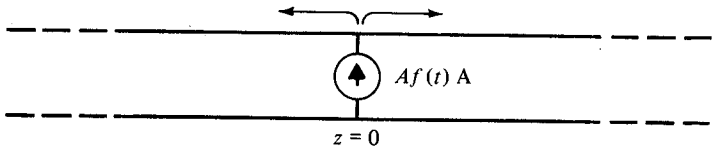

Figure 2.2 A current source injects a current into a line assumed to be infinitely long in both directions. The current splits and waves start out to the left and right as indicated.

This is a wave traveling to the left (i.e., toward negative values).

• *Note in Summary*

$$\boxed{\mathcal{L}c = Z_0, \qquad \mathcal{C}c = Y_0 = \frac{1}{Z_0}, \qquad \mathcal{L}\mathcal{C}c^2 = 1}$$

The boxed relations above are quite helpful because the quantity c, the velocity of the waves, turns out to be a constant depending only on the line material—in particular $c = 1/\sqrt{\mu\varepsilon}$ for lossless lines, as will be demonstrated next.

2.2 TRANSMISSION LINE THEORY FROM THE POINT OF VIEW OF ELECTROMAGNETIC FIELDS

There are several important transmission line properties that hold for idealized transmission structures independent of the actual type of structure. For example, the velocity of propagation is determined primarily by the properties (dielectric constant and permeability) of the insulating medium about the conductors. This fact is shown readily by a study of the field equations. We can make the connection between the transmission line variables v and i and the field variables E and H as follows: The line voltage is the (line) integral of the electric field, E, from one conductor to the other in a z-plane, that is,

$$v = \int_1^2 \mathbf{E} \cdot d\mathbf{l}_1$$

The current is the closed line integral of the magnetic field, H, around the conductor, in a plane of z, that is,

$$i = \oint \mathbf{H} \cdot d\mathbf{l}_2$$

It follows from these equations that both the z and the t variations of the circuit variable, v, and the field variable, E, must be the same and that both the z and t variations of i and H must also be the same. We will therefore study the field equations to see how E and H vary with z and t.

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Consider Maxwell's equations as they apply to the space around the transmission lines, since these are among the most reliable equations known.

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \qquad \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

Now recall that the magnetic fields generated by the currents in most idealized transmission line structures have no component along the direction of propagation. Moreover, the conductivity of most line conductors is quite high, which means that typically the component of the electric field in the direction of propagation is very small. Thus, taking z to be the direction of propagation as before, we will assume that $H_z = 0 = E_z$. Then, if we expand $\nabla \times \mathbf{H}$ in Maxwell's equation

$$\nabla \times \mathbf{H} = \begin{vmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & 0 \end{vmatrix} = \hat{\mathbf{a}}_y \frac{\partial H_x}{\partial z} + \hat{\mathbf{a}}_z \frac{\partial H_y}{\partial x} - \hat{\mathbf{a}}_z \frac{\partial H_x}{\partial y} - \hat{\mathbf{a}}_x \frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

we note that since $\partial \mathbf{E}/\partial t$ has no z-component, it follows that

$$\frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial x}$$

and more importantly,

$$-\hat{\mathbf{a}}_x \frac{\partial H_y}{\partial z} + \hat{\mathbf{a}}_y \frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{2.3}$$

or equating corresponding components

$$\frac{\partial H_y}{\partial z} = -\varepsilon \frac{\partial E_x}{\partial t}, \qquad \frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial E_y}{\partial t} \tag{2.4}$$

In similar fashion from the second Maxwell equation, it follows that

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}, \qquad \frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t} \tag{2.5}$$

That is, inspection of these equations shows that the components of E and H satisfy transmission line-like equations. Taking the first and second members of Eqs. 2.4 and 2.5 in pairs, it can be seen that the components E_x and H_y are analogous to v and i in the lossless version of the transmission line equations, (Eqs. 2.1), while the components E_y and H_x are also analogous to v and i in the same way. The parameters ε (permittivity) and μ (permeability) are analogous to $\mathcal{C}$ and $\mathcal{L}$ in the lossless transmission line equations.

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With these analogies in mind, we can combine the equations by eliminating one or the other of E or H and so obtain wave equations similar to Eqs. 2.2. Eliminating E_x first by the step of differentiating the

first of Eq. 2.4 with respect to z and the first of 2.5 with respect to t, we have

$$\frac{\partial^2 H_y}{\partial z^2} = -\varepsilon \frac{\partial^2 E_x}{\partial z\, \partial t} = -\varepsilon\left(-\mu \frac{\partial^2 H_y}{\partial t^2}\right) = \mu\varepsilon \frac{\partial^2 H_y}{\partial t^2}$$

Similarly with the other components

$$\frac{\partial^2 H_x}{\partial z^2} = \varepsilon\left(\frac{\partial^2 E_y}{\partial z\, \partial t}\right) = \mu\varepsilon \frac{\partial^2 H_x}{\partial t^2}$$

Both components of H thus satisfy the wave equations. Consequently, we can write a wave equation for the vector $\mathbf{H}$ itself.

$$\frac{\partial^2 \mathbf{H}}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

In a similar fashion, we may obtain a wave equation for both components of the vector $\mathbf{E}$

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

All of the components of E and H thus satisfy equations similar to the wave equations for v and i written as Eqs. 2.2. The solutions therefore have the delayed functional form found above for v and i. For example, the function $E_y = f(t \pm z/c)$, where $f(t)$ is any function, satisfies the wave equation for E and by analogy $c = 1/\sqrt{\mu\varepsilon}$; then from Eqs. 2.4 and 2.5 it follows that H_x must have the form $H_x = \pm\sqrt{\varepsilon/\mu}\ f(t \pm z/c)$ since

$$\frac{\partial E_y}{\partial z} = \pm \frac{1}{c} f'\left(t \pm \frac{z}{c}\right) = \mu \frac{\partial H_x}{\partial t}$$

$$\text{or} \quad H_x = \pm \frac{1}{\mu c} f\left(t \pm \frac{z}{c}\right) = \pm\sqrt{\frac{\varepsilon}{\mu}}\ f\left(t \pm \frac{z}{c}\right)$$

It is clear then that both E and H are waves that travel at the same speed* $c = 1/\sqrt{\mu\varepsilon}$ and they have the same functional form. But as pointed out above, the circuit variables v and i must have the same z (spatial) and t (time) variation as the field variables E and H, respectively; it follows then that the waves for v and i must travel at this same speed, $c = 1/\sqrt{\mu\varepsilon}$. In the MKS system of units, free space has the following parameter values.

$$\varepsilon_0 = \frac{1}{36\pi} \times 10^{-9} \text{ farad (F)/m}, \qquad \mu_0 = 4\pi \times 10^{-7} \text{ H/m},$$

$$\text{or} \quad \mu_0\varepsilon_0 = \tfrac{1}{9} \times 10^{-16}, \qquad (\mu_0\varepsilon_0)^{-1/2} = 3 \times 10^8 \text{ m/s}$$

This is what we wanted to show.

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*We have adopted the symbol c here for the speed, rather than v, in order to avoid confusion with our symbol for the time varying voltage. Here the speed c is not always equal to the speed of light in free space, that is, it depends upon μ and ε.

According to the transmission line equations presented earlier, (Eqs. 2.1), the speed of propagation on a line is $c = 1/\sqrt{\mathcal{L}\mathcal{C}}$. It must be then that $\mathcal{L}\mathcal{C} = \mu\varepsilon$. The finite conductivity of actual conductors is responsible for the fact that the velocity of propagation on actual lines is slightly less than the value given above, as we shall see later.

Summarizing once again, the voltage v and current i on a lossless transmission line must propagate with speed $c = 1/\sqrt{\mu\varepsilon}$ and also $c = 1/\sqrt{\mathcal{L}\mathcal{C}}$. It follows that

$$\boxed{\mathcal{L}_{\text{ext}}\mathcal{C}_{\text{ext}} = \mu_{\text{ext}}\varepsilon_{\text{ext}}}$$

which will prove to be a very useful relationship. The subscript "ext" has been attached to remind us that μ and ε are the permeability and permittivity values for the space outside or "external" to the conductors. $\mathcal{L}_{\text{ext}}$ is associated with the magnetic flux *outside* the conductors and $\mathcal{C}_{\text{ext}}$ is the capacitance arising from the charges on the surface of and the electric field between conductors. Also, we have for the surge impedance $Z_{0(\text{ext})} = \mathcal{L}_{\text{ext}}c = \sqrt{\mathcal{L}_{\text{ext}}/\mathcal{C}_{\text{ext}}}$, and similarly

$$Y_{0(\text{ext})} = \mathcal{C}_{\text{ext}}c = \sqrt{\mathcal{C}_{\text{ext}}/\mathcal{L}_{\text{ext}}}$$

Note that with μ, ε, and $\mathcal{C}$ given, $\mathcal{L}_{\text{ext}}$ is determined; no separate calculation is required to find the external inductance (or if $\mathcal{L}_{\text{ext}}$ is given, $\mathcal{C}_{\text{ext}}$ is determined).

The line parameters are important in that they influence stability, energy storage, and the specific values of v and i in surges. As we have seen, in a surge traveling in one direction, if the current is $i = f_1(t - z/c)$ then the voltage is $v = Z_0 f_1(t - z/c)$ or if $v = f(t - z/c)$ then $i = Y_0 f(t - z/c)$. Surges travel more or less undisturbed until they encounter a discontinuity in the line. As we will see later, they are in general reflected from discontinuities.

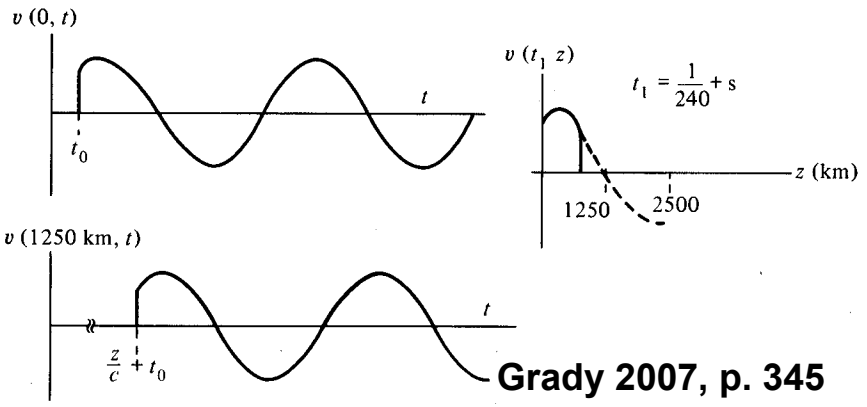

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Figure 2.3 Waveforms for the example of the 60-Hz generator that is switched on. $t_0 = \frac{1}{8}T = \frac{1}{480}$ s.

As an example, suppose we have an ideal voltage generator at $z=0$ on a line having $Z_0=400\ \Omega$. If the generator is switched on at time $t=t_0$, to a 60-cycle, rms voltage of 138 kV, we might have, ideally, $v(t,0)=\sqrt{2}\ 138{,}000u(t-t_0)\sin\omega t$, $\omega=377$. Then, until the wave meets something that forces v/i to be different from 400 Ω, the voltage and current will be $v(t,z)=\sqrt{2}\times 138{,}000u(t-z/c-t_0)\sin\omega(t-z/c)V$, $i(z,t)=\sqrt{2}\times 345u(t-z/c-t_0)\sin\omega(t-z/c)$, $c=\sqrt{1/\mu_0\varepsilon_0}$ =just under 3×10^5 km/s. The waveform as a function of both z and t is suggested in Fig. 2.3.

2.3 SURGE IMPEDANCE LOADING

If a line is terminated in a resistance $R_T=Z_0$, the pulse or surge is not reflected since the ratio v/i is the same at the resistor terminals as on the line with the surge. We will consider the question of reflections in detail in the next chapter. For now, let us simply agree that for certain terminations, a traveling wave will not be reflected. When a line has a wave that travels in one direction only, it is easy to find the power being transferred. Thus, in preliminary design consideration, planners often assume "*surge impedance loading*" (SIL),* or at least reference the transmitted power to that handled with SIL. With SIL, since we have a wave that goes in one direction only, $v=f(t-z/c)$, and $i=v/Z_0$, the power is $p=vi=v^2/Z_0$, so the power capability of different voltage classes is readily estimated. For example, with the voltage generator and line described in the previous paragraph, $p=(\sqrt{2}\times138{,}000)^2/Z_0u(t-z/c-t_0)\sin^2\omega(t-z/c)$, and, at any location $z<ct$, the average power passing after the first half-cycle goes by is $P_{av}=(138{,}000)^2/Z_0=47.6$ MW. As before, the current implied by the surge impedance loading is about 345 amperes (A), rms.

The questions that remain are: "Is 400 Ω a reasonable value for surge impedance?" "How large a conductor is required to handle 345 A (or more generally, the current implied by surge impedance loading a given voltage class)?" To answer these questions, we should know how to calculate the line parameters; this we shall do next. We will calculate the line capacitance first, since many other parameters can be found once it is known.

PROBLEMS

2.1 An approximation for the effect of lightning striking a transmission line is the insertion of a current source across the line, with the current having the form

$$I=I_0\frac{t}{t_1}\exp\left(\frac{-t}{t_2}\right),\qquad t>0$$

$$I=0,\qquad t<0$$

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*Even though lines are not usually loaded at this special level at the 60-Hz operating frequency.

Chapter 3

Transients on Transmission Lines

Having learned to calculate the parameters of single-phase lines, we now return to the subject of transient voltages and currents. The important transients in power systems are those associated with various switching operations, with faults, and with lightning (plus those associated with information carried at high frequencies along the wires).

3.1 LAPLACE TRANSFORM SOLUTION OF TRANSMISSION LINE EQUATIONS

To handle general transients on general lines, it is economical to employ a transform (Laplace or Fourier) method to eliminate the time from the transmission line equations. Recall the definition of the Laplace transform.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

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Applying this transform to the line equations and boundary conditions we

obtain

TM Line Equations		*Transformed Equations*	
$\frac{\partial v}{\partial z} = -\mathcal{L}\frac{\partial i}{\partial t} - \mathcal{R}i$	$\Rightarrow$	$\frac{dV}{dz} = -(s\mathcal{L}+\mathcal{R})I$	(3.1)
$\frac{\partial i}{\partial z} = -\mathcal{C}\frac{\partial v}{\partial t} - \mathcal{G}v$	$\Rightarrow$	$\frac{dI}{dz} = -(s\mathcal{C}+\mathcal{G})V$	(3.2)
$v(0,t) = f(t)$	$\Rightarrow$	$V(0,s) = F(s)$	
$v(z,t) \Rightarrow 0$ (infinite line) as $z \to \infty$	$\Rightarrow$	$V(z,s) \Rightarrow 0$ as $z \to \infty$	
$v(L,t) \Rightarrow i(L,t)R_L$	$\Rightarrow$	$V(L,s) = I(L,s)R_L$	

In the latter equation, it has been assumed that the line is terminated in a resistance, R_L.

In the transformed operations, it has been assumed that the quantities $i(z,0+)$, and $v(z,0+)$, the initial values of current and voltage, are zero. For certain problems it may be advantageous to insert appropriate values for such initial conditions when they are known.

The next step is to solve the transformed equations in the space variable. The actual voltages and currents are then obtained by the inverse transform operation.

For example, eliminating I between Eqs. 3.1 and 3.2

$$\frac{d^2V}{dz^2} = (\mathcal{R}+s\mathcal{L})(\mathcal{G}+s\mathcal{C})V \equiv \gamma^2 V, \qquad \gamma^2 \equiv (\mathcal{R}+s\mathcal{L})(\mathcal{G}+s\mathcal{C})$$

The solution to this second-order differential equation for V can be written in terms of sines, hyperbolic functions, or exponentials.

We will choose the exponential form. Two arbitrary constants are necessary for the solution of the second-order differential equation.

$$V(z,s) = Ae^{\gamma z} + Be^{-\gamma z} \tag{3.3}$$

where A and B are the constants to be determined. From Eq. 3.1

$$I = -\frac{1}{(\mathcal{R}+s\mathcal{L})}\frac{dV}{dz} = -\frac{1}{(\mathcal{R}+s\mathcal{L})}\gamma[Ae^{\gamma z} - Be^{-\gamma z}]$$

Define

$$Y_0 = \frac{1}{Z_0} = \frac{\gamma}{(\mathcal{R}+s\mathcal{L})} = \sqrt{\frac{\mathcal{G}+s\mathcal{C}}{\mathcal{R}+s\mathcal{L}}} \quad \text{(characteristic or surge admittance)} \tag{3.4}$$

$$I = -Y_0[Ae^{\gamma z} - Be^{-\gamma z}] \tag{3.5}$$

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If we have an ideal voltage generator connected to the line at $z = 0$, we have the conditions listed below Eq. 3.2, $v(0,t) = f(t)$ and $V(0,s) = F(s)$ and there is a source or excitation condition that gets things going. This

also provides a condition on the constants: evaluating 3.3 at $z=0$.

$$F(s)=A+B \tag{3.6}$$

One more condition is needed to determine A and B. If the line is infinite in length, $V_{z\to\infty}(z,s)\Rightarrow 0$, so from 3.3, $A=0$, and

$$V(z,s)=F(s)e^{-\gamma z}, \qquad I(z,s)=Y_0F(s)e^{-\gamma z}$$

If the line is terminated at $z=L$ by a resistor R_L,

$$\frac{V(L,s)}{I(L,s)}=R_L=\frac{Ae^{\gamma L}+Be^{-\gamma L}}{-Y_0(Ae^{\gamma L}-Be^{-\gamma L})} \tag{3.7}$$

Define a reflection coefficient $\rho(L)=(A/B)e^{2\gamma L}$; this definition, inserted into Eq. 3.7 gives

$$\frac{R_L}{Z_0}=\frac{1+\rho(L)}{1-\rho(L)}$$

or solving for $\rho(L)$,

$$\rho(L)=\frac{R_L/Z_0-1}{R_L/Z_0+1}=\frac{R_L-Z_0}{R_L+Z_0}$$

Then from Eq. 3.6, $B=F(s)-A=F(s)-B\rho(L)e^{-2\gamma L}$ or

$$B=\frac{F(s)}{1+\rho(L)e^{-2\gamma L}} \quad \text{and then} \quad A=\frac{F(s)\rho(L)e^{-2\gamma L}}{1+\rho(L)e^{-2\gamma L}} \tag{3.8}$$

3.1.1 Special Cases

• *A. Matched Load.* $R_L=Z_0\Rightarrow\rho(L)=0\Rightarrow A=0$. This is surge impedance (or "natural") loading and $V(z,s)=F(s)e^{-\gamma z}$ as in the case of the infinite line above.

• *B. Open Circuit.* $R_L=\infty\Rightarrow\rho(L)=1\Rightarrow B=F(s)/1+e^{-2\gamma L}$. From Eqs. 3.3 and 3.8,

$$V(z)=\frac{F(s)}{1+e^{-2\gamma L}}\left[e^{-\gamma z}+e^{\gamma(z-2L)}\right] \tag{3.9}$$

• *C. Short Circuit.* $R_L=0\Rightarrow\rho(L)=-1$, $B=F(s)/1-e^{-2\gamma L}$. Similarly,

$$V(z)=\frac{F(s)}{1-e^{-2\gamma L}}\left[e^{-\gamma z}-e^{\gamma(z-2L)}\right] \tag{3.10}$$

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To find the time variation of voltage, and current, we must obtain the inverse transforms of these voltages: that is, $v(t,z)=L^{-1}[V(s,z)]$.

3.2 THE PROPAGATION CONSTANT AND SIMPLE INVERSIONS

The quantity, γ, contains the s-variable

$$\gamma=\sqrt{(\mathcal{R}+s\mathcal{L})(\mathcal{G}+s\mathcal{C})} \tag{3.11}$$

and therefore it strongly influences the exact form of the inverse transformation. In general, with the square root function indicated, the inverse transform can only be done numerically. But under certain conditions, the transform can be obtained analytically to a good approximation. With one exception, the "certain conditions" are that $\mathcal{R}$ and $\mathcal{G}$ be very small, which is often a good approximation. We will examine a few common special cases.

Special Case 1

$\mathcal{G} \doteq 0$, $\mathcal{R} \ll s\mathcal{L}$. In this case

$$\gamma=\sqrt{(\mathcal{R}+s\mathcal{L})s\mathcal{C}}=s\sqrt{\mathcal{L}\mathcal{C}}\left(1+\frac{\mathcal{R}}{s\mathcal{L}}\right)^{1/2} \doteq s\sqrt{\mathcal{L}\mathcal{C}}\left(1+\frac{\mathcal{R}}{2s\mathcal{L}}\right)$$

or

$$\gamma \doteq s\sqrt{\mathcal{L}\mathcal{C}}+\frac{1}{2}\mathcal{R}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}}$$

Let us utilize this result for special case 1, surge impedance loading. In this case

$$V(s,z)=F(s)e^{-\gamma z}=F(s)e^{-s\sqrt{\mathcal{L}\mathcal{C}}\,z}e^{(-1/2)\mathcal{R}\sqrt{\mathcal{C}/\mathcal{L}}\,z}$$

or

$$v(t,z)=L^{-1}[V(s,z)]=e^{(-1/2)\mathcal{R}\sqrt{\mathcal{C}/\mathcal{L}}\,z}L^{-1}\left[F(s)e^{-s\sqrt{\mathcal{L}\mathcal{C}}\,z}\right]$$

$$=e^{-(\mathcal{R}/2)\sqrt{\mathcal{C}/\mathcal{L}}\,z}f(t-\sqrt{\mathcal{L}\mathcal{C}}\,z)u(t-\sqrt{\mathcal{L}\mathcal{C}}\,z)$$

where $u(t)$ is the familiar step function. This is an attenuated wave traveling at a speed $=1/\sqrt{\mathcal{L}\mathcal{C}}$, having value zero until $t>(\mathcal{L}\mathcal{C})^{1/2}z$. Notice that the form of this solution is the same as that we encountered in Chapter 2, for lossless lines ($\mathcal{R}=\mathcal{G}=0$), except that in the present case there is attenuation. The wave maintains its shape as it moves along at speed $v=1/\sqrt{\mathcal{L}\mathcal{C}}$ but it shrinks in amplitude.

The other common cases give similar solutions but differ in the detailed formula for the attenuation constant.

Special Case 2

$\mathcal{R} \ll s\mathcal{L}$, $\mathcal{G} \ll s\mathcal{C}$, $\mathcal{R}\mathcal{G} \ll s^2\mathcal{L}\mathcal{C}$. Then

$$\gamma=\sqrt{(\mathcal{R}+s\mathcal{L})(\mathcal{G}+s\mathcal{C})} \doteq \sqrt{0+s(\mathcal{R}\mathcal{C}+\mathcal{G}\mathcal{L})+s^2\mathcal{L}\mathcal{C}}$$

$$\doteq s\sqrt{\mathcal{L}\mathcal{C}}\left[1+\frac{(\mathcal{R}\mathcal{C}+\mathcal{G}\mathcal{L})}{s\mathcal{L}\mathcal{C}}\right]^{1/2}$$

$$\doteq s\sqrt{\mathcal{L}\mathcal{C}}\left\{1+\frac{1}{2}\frac{(\mathcal{R}\mathcal{C}+\mathcal{G}\mathcal{L})}{s\mathcal{L}\mathcal{C}}\right\}$$ **Grady 2007, p. 350**

$$=s\sqrt{\mathcal{L}\mathcal{C}}+\frac{1}{2}\mathcal{R}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}}+\frac{1}{2}\mathcal{G}\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}$$

To save writing, define

$$\alpha \equiv \frac{1}{2}\mathcal{R}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}} + \frac{1}{2}\mathcal{G}\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}$$

an attenuation constant. Then for surge impedance loading

$$V(s,z) = F(s)e^{-\gamma z} = F(s)e^{-\alpha z}e^{-s\sqrt{\mathcal{L}\mathcal{C}}\,z}$$

and performing the inverse transformation,

$$v(t,z) = e^{-\alpha z}f(t-\sqrt{\mathcal{L}\mathcal{C}}\,z)u(t-\sqrt{\mathcal{L}\mathcal{C}}\,z)$$

which is again an attenuated but undistorted wave.

In neither of the two cases is the approximation valid at dc or near zero frequency.

Special Case 3

A line may be *designed* such that $\mathcal{R}/\mathcal{L} = \mathcal{G}/\mathcal{C}$, with no other restriction on the magnitude of $\mathcal{R}$ and $\mathcal{G}$. In this case the propagation constant has a simple form as follows.

$$\gamma^2 = s^2\mathcal{L}\mathcal{C} + s(\mathcal{R}\mathcal{C} + \mathcal{G}\mathcal{L}) + \mathcal{R}\mathcal{G} = s^2\mathcal{L}\mathcal{C}\left[1 + s\frac{2\mathcal{R}\mathcal{C}}{s^2\mathcal{L}\mathcal{C}} + \frac{\mathcal{R}\mathcal{G}}{s^2\mathcal{L}\mathcal{C}}\right]$$

$$= s^2\mathcal{L}\mathcal{C}\left[1 + \frac{2\mathcal{R}}{s\mathcal{L}} + \frac{\mathcal{R}^2}{s^2\mathcal{L}^2}\right] = s^2\mathcal{L}\mathcal{C}\left[1 + \frac{\mathcal{R}}{s\mathcal{L}}\right]^2$$

or

$$\gamma = s\sqrt{\mathcal{L}\mathcal{C}}\left[1 + \frac{\mathcal{R}}{s\mathcal{L}}\right] = s\sqrt{\mathcal{L}\mathcal{C}} + \mathcal{R}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}}$$

This result is of the same form as the other two special cases except for a different attenuation rate. For a line that is surge impedance loaded, $v(t,z)$ is an attenuated but undistorted traveling wave. A line having parameters so designed is called a "distortionless" line or "heaviside" line.

3.3 INVERTING THE LAPLACE TRANSFORM SOLUTION TO GIVE THE TIME DOMAIN VALUES

Except for these special cases, the line introduces distortions that are complicated and difficult to discuss. Generally speaking, the effects are the rounding of the corners of square waves and the "stretching" of pulses.

Returning to the special cases of open circuit and short circuit termination, we will let the propagation constant be $\gamma = \alpha + s\sqrt{\mathcal{L}\mathcal{C}}$, with α not a function of s. In Sec. 3.1 we found the following result for the case of the open circuited line.

$$V(z) = \frac{F(s)}{1+e^{-2\gamma L}}\left[e^{-\gamma z} + e^{+\gamma(z-2L)}\right] = V(s,z)$$

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The easiest way to inverse transform this function is to do a little algebra first, to get a series representation of the factor $1/1+e^{-2\gamma L}$; this can be

done, for example, by synthetic division.

$$
\begin{array}{r|l}
 & 1-e^{-2\gamma L}+e^{-4\gamma L}-e^{-6\gamma L}+\cdots \\
\hline
1+e^{-2\gamma L} & 1 \\
 & \underline{1+e^{-2\gamma L}} \\
 & -e^{-2\gamma L} \\
 & \underline{-e^{-2\gamma L}-e^{-4\gamma L}} \\
 & e^{-4\gamma L} \\
 & \underline{e^{-4\gamma L}+e^{-6\gamma L}} \\
 & -e^{-6\gamma L}
\end{array}
$$

That is, from the steps shown above

$$\frac{1}{1+e^{-2\gamma L}}=1-e^{-2\gamma L}+e^{-4\gamma L}-e^{-6\gamma L}+\cdots$$

The advantage of this may not be evident at first, but note

$$V(s,z)=F(s)\left[e^{-\gamma z}+e^{+\gamma(z-2L)}\right](1-e^{-2\gamma L}+e^{-4\gamma L}-e^{-6\gamma L}+\cdots)$$

or

$$\begin{aligned}V(s,z)=F(s)e^{-\gamma z}+F(s)e^{+\gamma(z-2L)}-F(s)e^{-\gamma(z+2L)}\\ -F(s)e^{+\gamma(z-4L)}+F(s)e^{-\gamma(z+4L)}+\cdots\end{aligned}$$

Now we can inverse transform term by term ($c\equiv 1/\sqrt{\mathcal{LC}}$) to obtain the voltage on the open circuited line.

$$\begin{aligned}v(t,z)=e^{-\alpha z}f\left(t-\frac{z}{c}\right)u\left(t-\frac{z}{c}\right)+e^{-\alpha(2L-z)}f\left(t+\frac{(z-2L)}{c}\right)\\ \times u\left(t+\frac{(z-2L)}{c}\right)-e^{-\alpha(z+2L)}f\left(t-\frac{(z+2L)}{c}\right)u\left(t-\frac{(2-zL)}{c}\right)+\cdots\end{aligned}$$

This is most easily interpreted with the aid of a *bounce diagram*.

However, before discussing this bounce diagram, we should pause to point out a few more helpful details about the voltages and currents on lines. If the voltage generator at the input is not ideal but has some internal impedance Z_g, the effect is two-fold. *First*, the voltage that starts down the line is less than the ideal value and in fact, by voltage division, the voltage $(Z_0/Z_0+Z_g)f(t)$ actually starts down the line (Fig. 3.1).

Second, any wave traveling to the left and incident on the input end is reflected with a voltage reflection coefficient,

$$\rho_g=\frac{Z_g-Z_0}{Z_g+Z_0}$$

with $\rho_g=-1$ corresponding to the ideal voltage source discussed above. That is, the input end behaves like a load for those waves impinging on it from the right.

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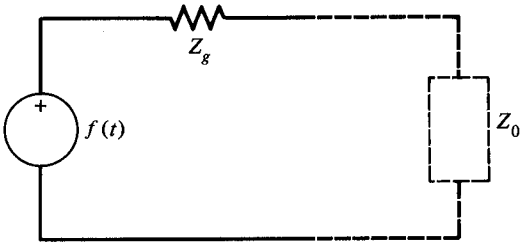

Figure 3.1 Voltage division between source and surge impedance.

We can then go back to Sec. 3.1 and work out the constants A and B that hold for the imperfect generator on a line terminated with a load R_L (reflection coefficient $\rho(L)=(R_L-Z_0)/(R_L+Z_0)$. The transformed voltage is found in this way to be

$$V(z,s)=\frac{F(s)Z_0}{Z_g+Z_0}\cdot\frac{1}{1-\rho_g\rho(L)e^{-2\gamma L}}\left[e^{-\gamma z}+\rho(L)e^{\gamma(z-2L)}\right]$$

The synthetic division discussed above can be done on the factor $1/(1-\rho_g\rho(L)e^{-2\gamma L})$ to get a series, from which $V(z,s)$ can be written

$$V(z,s)=F(s)\frac{Z_0}{Z_g+Z_0}\left[e^{-\gamma z}+\rho(L)e^{\gamma(z-2L)}\right.$$

$$\left.+\rho_g\rho(L)e^{-\gamma(z+2L)}+\rho_g\rho^2(L)e^{-\gamma(z-4L)}+\cdots\right] \tag{3.12}$$

Now this expression can be inverse transformed term by term to obtain ($c=1/\sqrt{\mathcal{LC}}$)

$$v(z,t)=\frac{Z_0}{Z_g+Z_0}\left[e^{-\alpha z}f\left(t-\frac{z}{c}\right)u\left(t-\frac{z}{c}\right)+\rho(L)e^{-\alpha(2L-z)}f\left(t+\frac{z-2L}{c}\right)\right.$$

$$\left.\times u\left(t-\frac{(2L-z)}{c}\right)+\rho_g\rho(L)e^{-\alpha(2L+z)}f\left(t-\frac{(2L+z)}{c}\right)u\left(t-\frac{2L+z}{c}\right)+\cdots\right]$$

and so on with the terms best interpreted with the aid of the *bounce diagram*.

Finally, note that we could have done the entire analysis from the start with current, $i(z,t)$, rather than voltage (by eliminating the voltage), with similar results. The current reflection coefficient would have been

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$$\rho^I(L)=\frac{Y_L-Y_0}{Y_L+Y_0}=-\rho^V(L)$$

Note: *current and voltage reflection coefficients are negatives of each other.*

3.4 BOUNCE OR LATTICE DIAGRAMS

The device known as the bounce diagram is most helpful in keeping track of *one point* on the time function representing the wave as it reflects back and forth from the ends of the line. The time function may be either voltage or current, providing the appropriate reflection coefficient is used, that is, ρ^V for voltage diagrams, ρ^I for current diagrams.

To construct the bounce diagram, Fig. 3.2, one first draws a pair of parallel lines (those running up and down on the page); a horizontal scale between these lines represents distance along the line section lengths. A vertical scale represents time elapsed, and this scale is often measured in units of transit time, that is, the time for a wave to travel the length of a line section. A diagonal zig-zag line represents the wave as it bounces back and forth between the ends or between discontinuities. The position of a given point on the wave, say the wavefront, is then easily found given the time by placing a straight edge across the page at the designated time and finding the intersection with the zig-zag; or the time of arrival at a given position is found by orienting a straight edge up and down at the given position, and noting the intersections with the zig-zag line. The reflection coefficients are indicated under the parallel lines representing the ends or discontinuities. The amplitude on successive bounces is conveniently indicated on each bounce.

The leading and trailing edge of a pulse may be tracked by thinking of the leading edge as the beginning of a "string" of length proportional to the velocity times the time difference between leading and trailing edge. The total voltage or current at any position in space and time is the superposition of all "bounces" above that point on the diagram provided the trailing edge still exists on the "bounce" line at the position under consideration.

For example, the position of leading edge and trailing edge at a time 5.75τ after the leading edge left the generator end is indicated in Fig. 3.2. At this time, at the point $\frac{1}{8}L$ distant from the generator the "field" (voltage or current) is the sum of $A\rho(L)+A\rho(L)\rho_g+A\rho^2(L)\rho_g+A\rho^2(L)\rho_g^2$, while at a point $\frac{7}{8}L$ distant from the left end, it is the sum of $A\rho(L)\rho_g+A\rho^2(L)\rho_g+A\rho^2(L)\rho_g^2+A\rho^3(L)\rho_g^2$.

If the attenuation on the line is not negligible, the amplitude following each bounce should be decreased by a factor $e^{-\alpha L}$ since the wave would be attenuated by that amount on each transit.

The bounce diagram is a very useful conceptual tool. However, in power system practice, its digital equivalent is programmed for use on a large-scale digital computer. Such programs include techniques for studying all the cases we will consider, and many others. One of the more widely known and used of these is the "Bonneville Transient Analysis Program" (from BPA).

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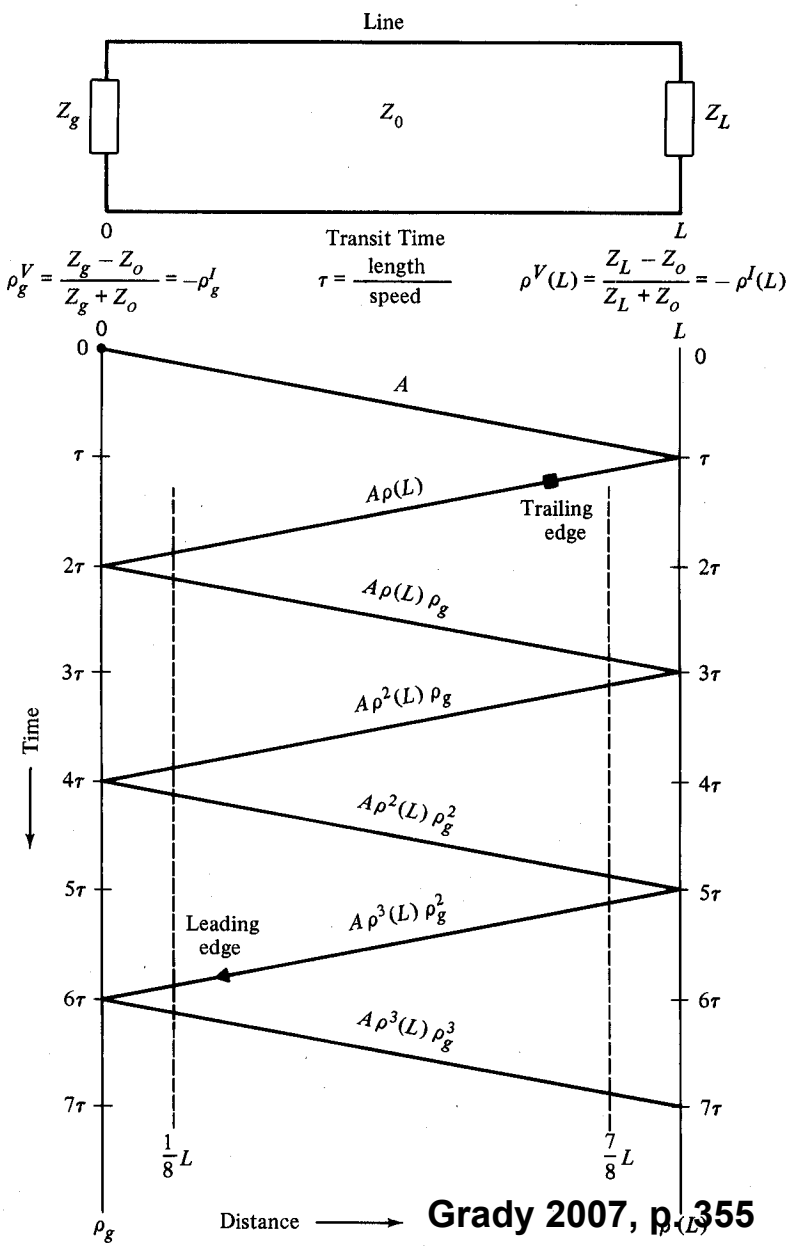

Figure 3.2 Simple bounce diagram for line shown with pulse excitation (not shown) at $z=0$ and $t=0$.

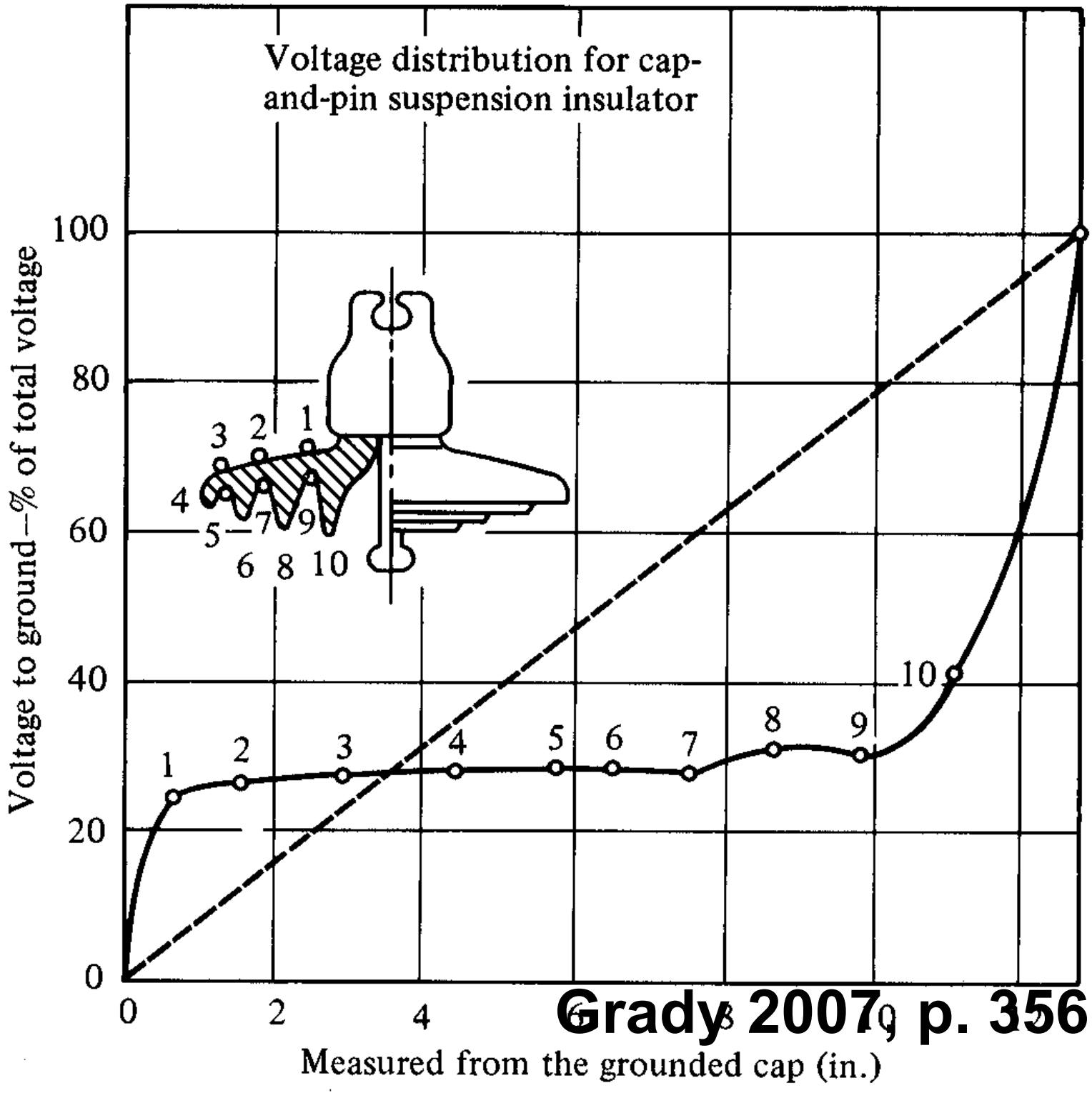

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Figure 6.34 Voltage distribution along the surface of a single clean cap-and-pin insulator unit. [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

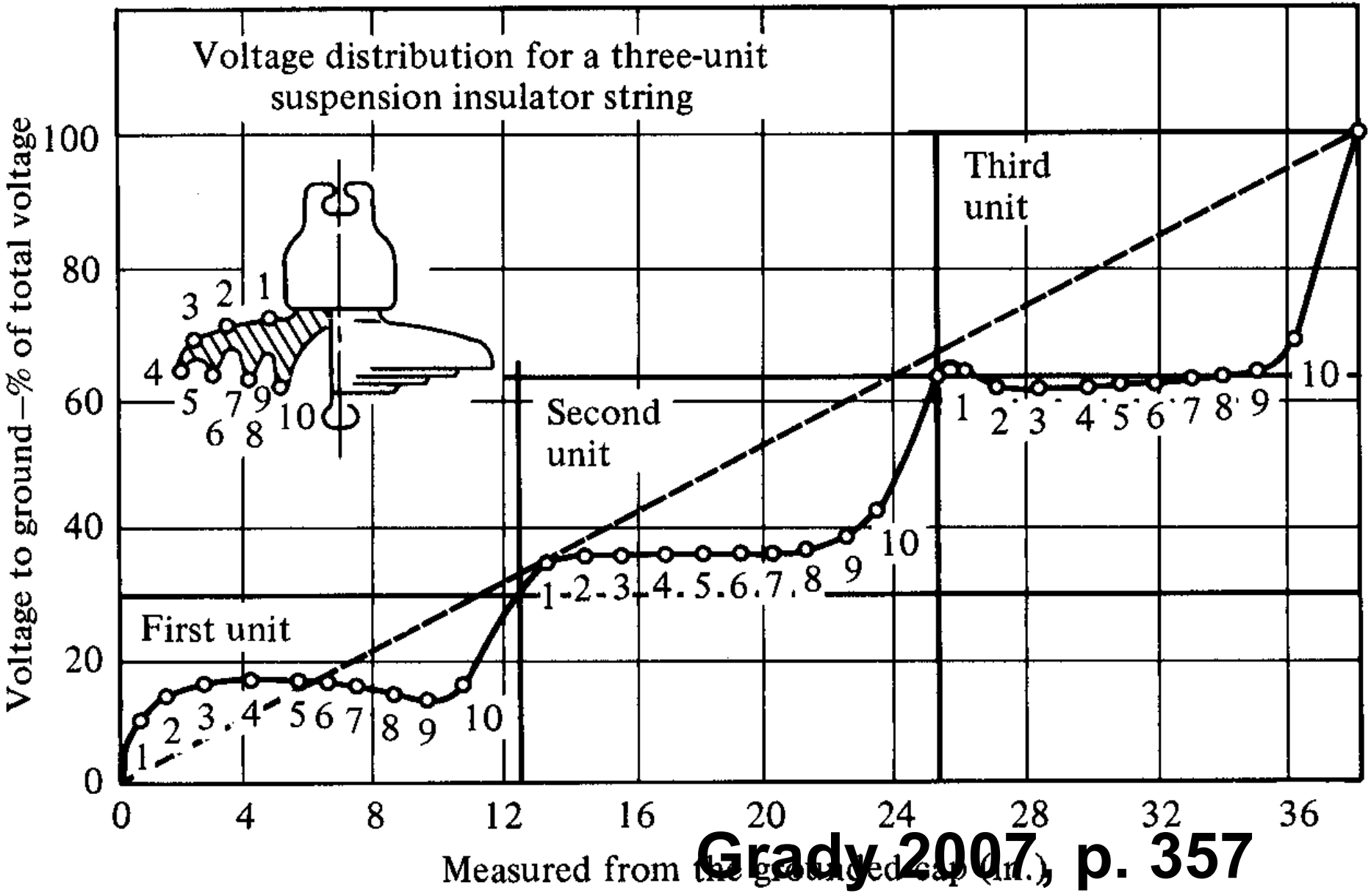

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Figure 6.35 Typical voltage distribution on the surface of three clean cap-and-pin insulator units in series. [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

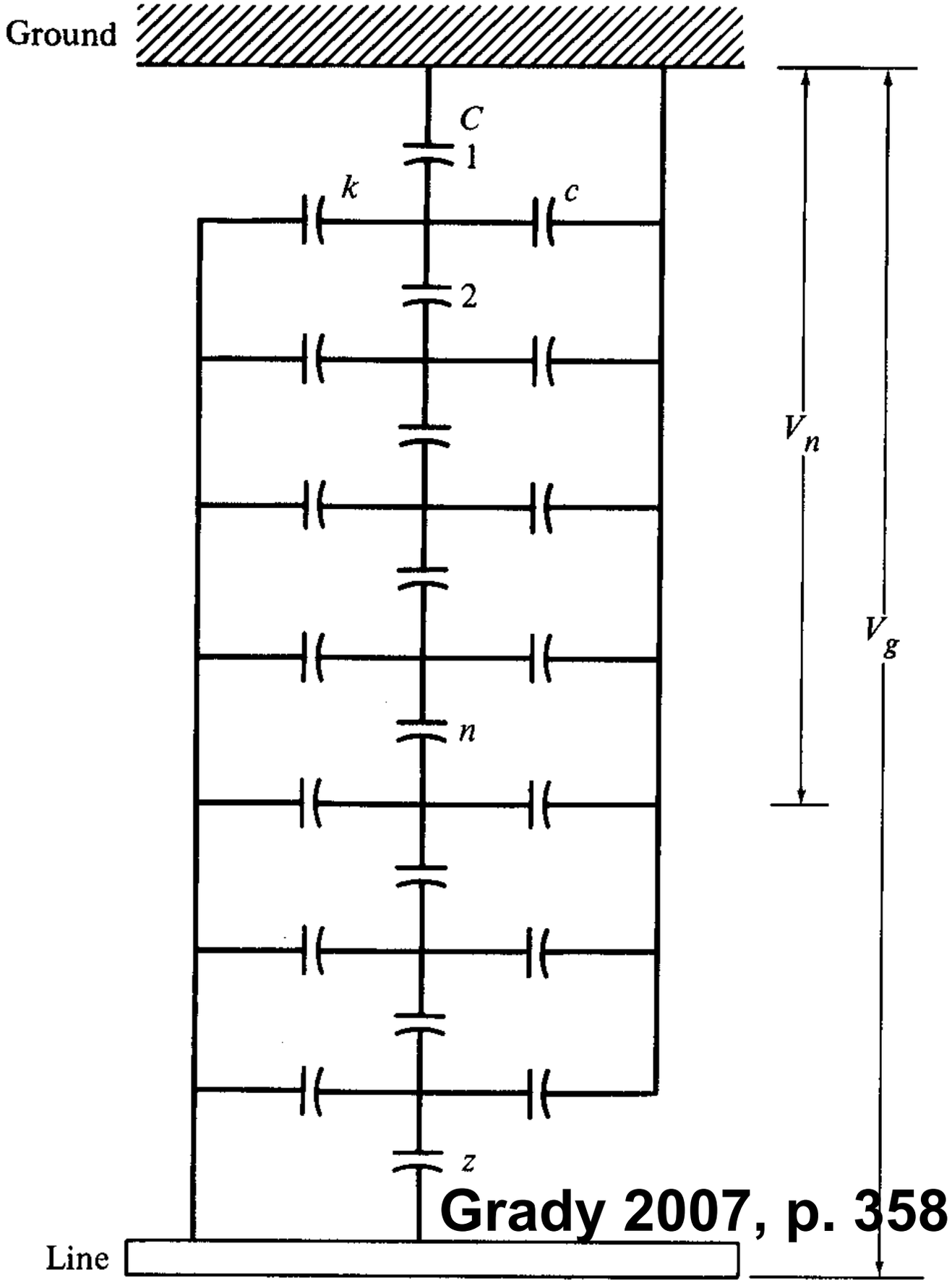

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Figure 6.36 Equivalent circuit for the voltage distribution along a clean eight-unit insulator string.

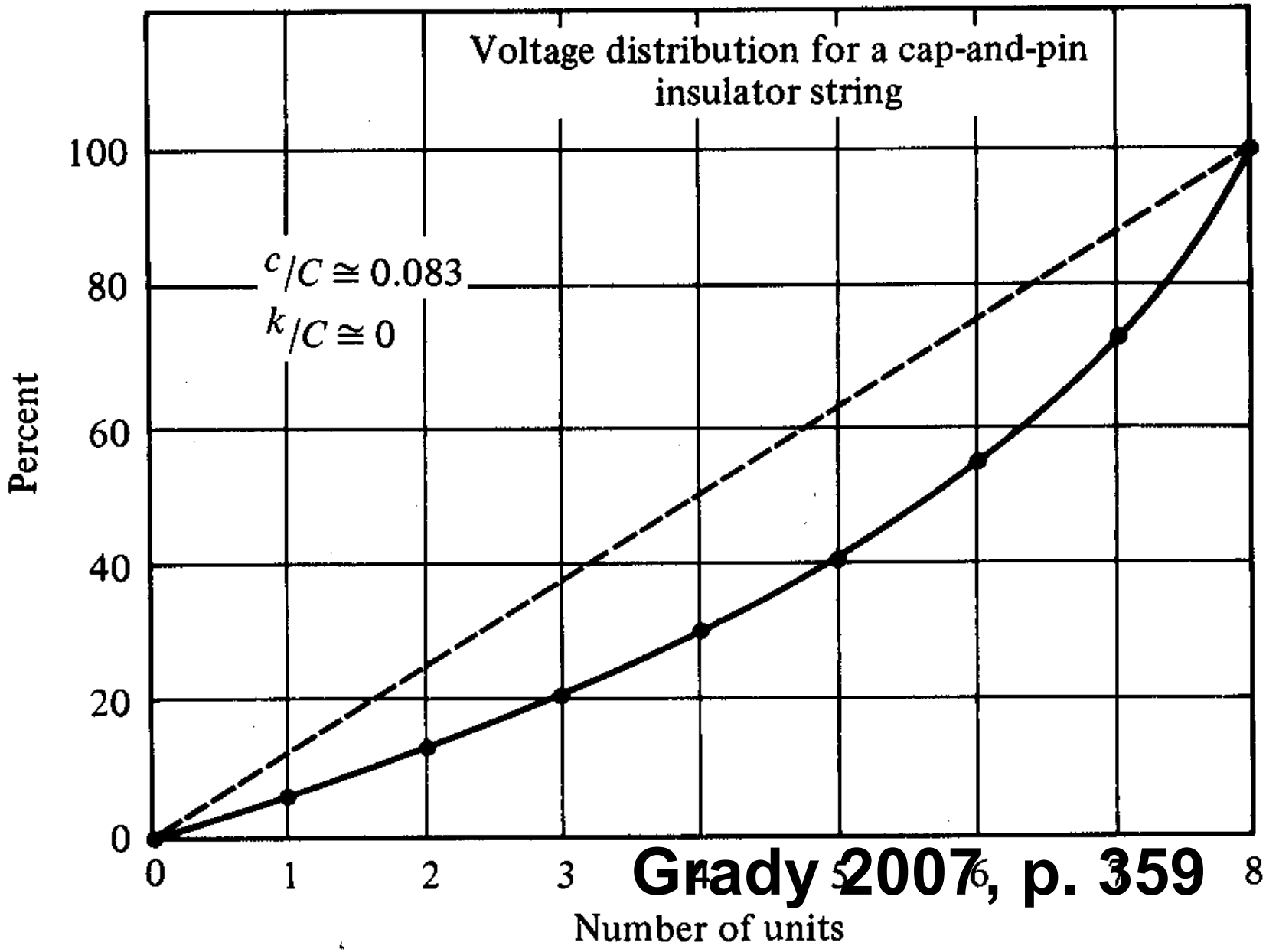

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Figure 6.37 Voltage distribution along a clean eight-unit cap-and-pin insulator string. [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

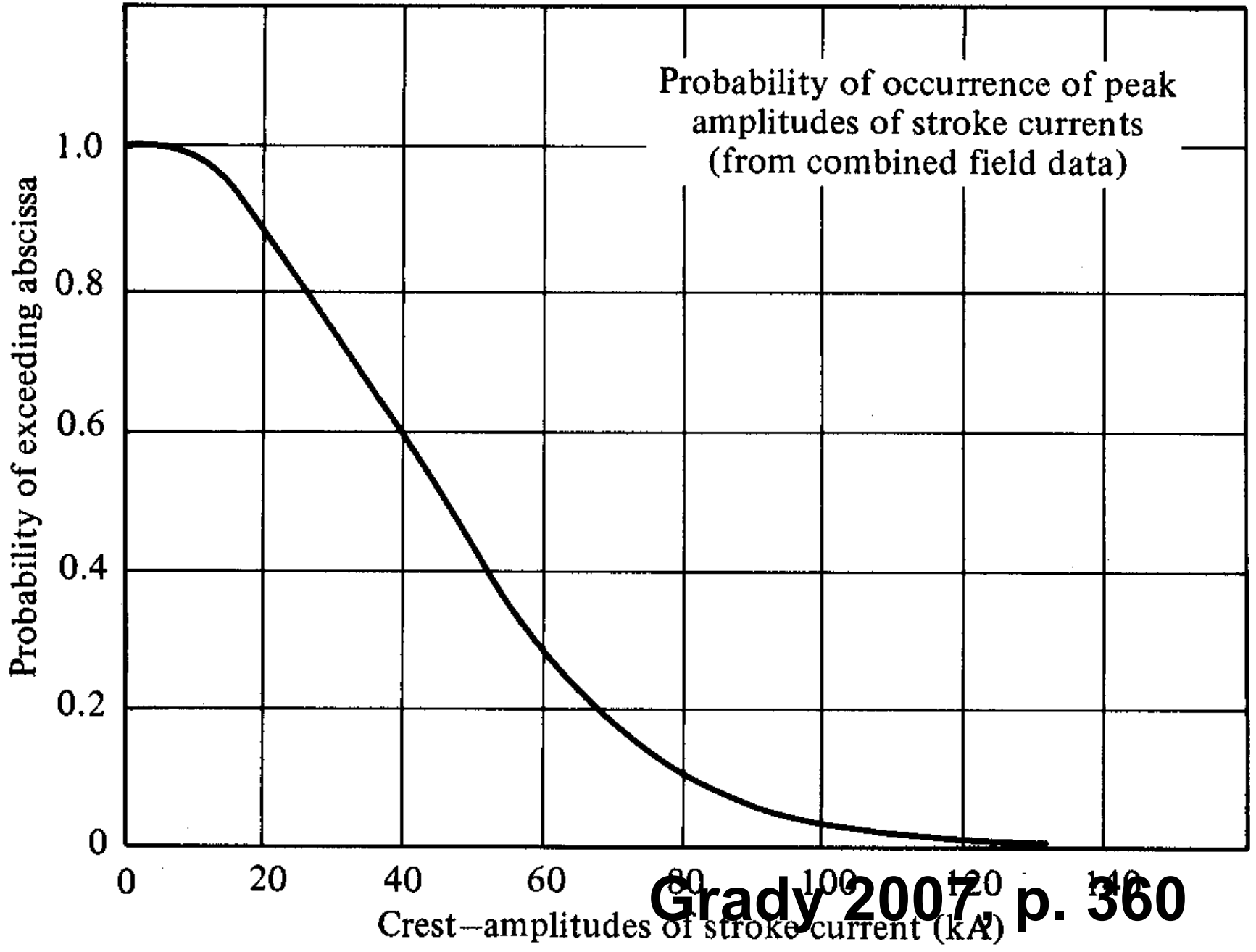

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Figure 6.38 Probability of occurrence of peak amplitudes of stroke currents (from combined field data). [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

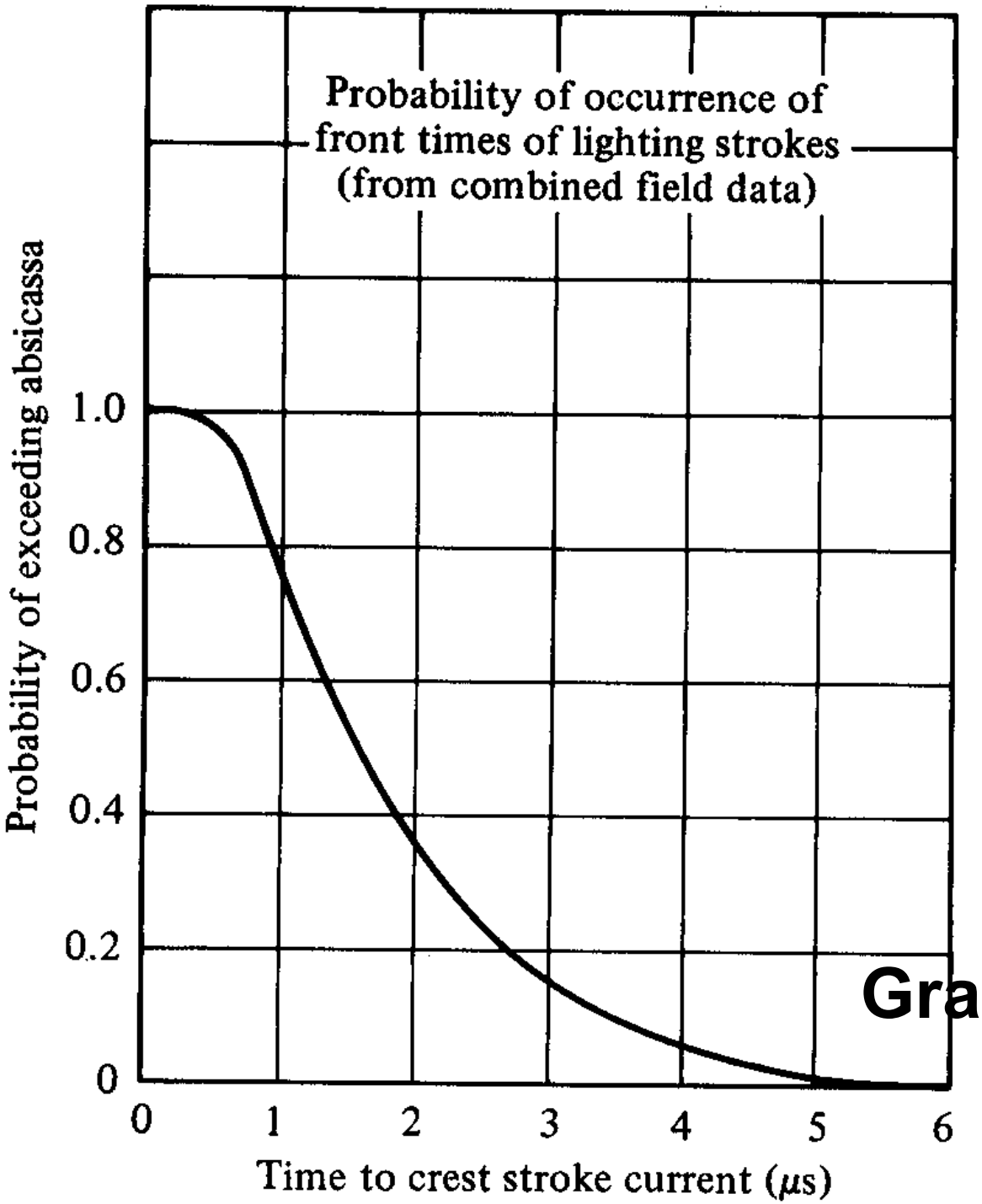

Figure 6.39 Probability of occurrence of front times of lightning strokes (from combined data). [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

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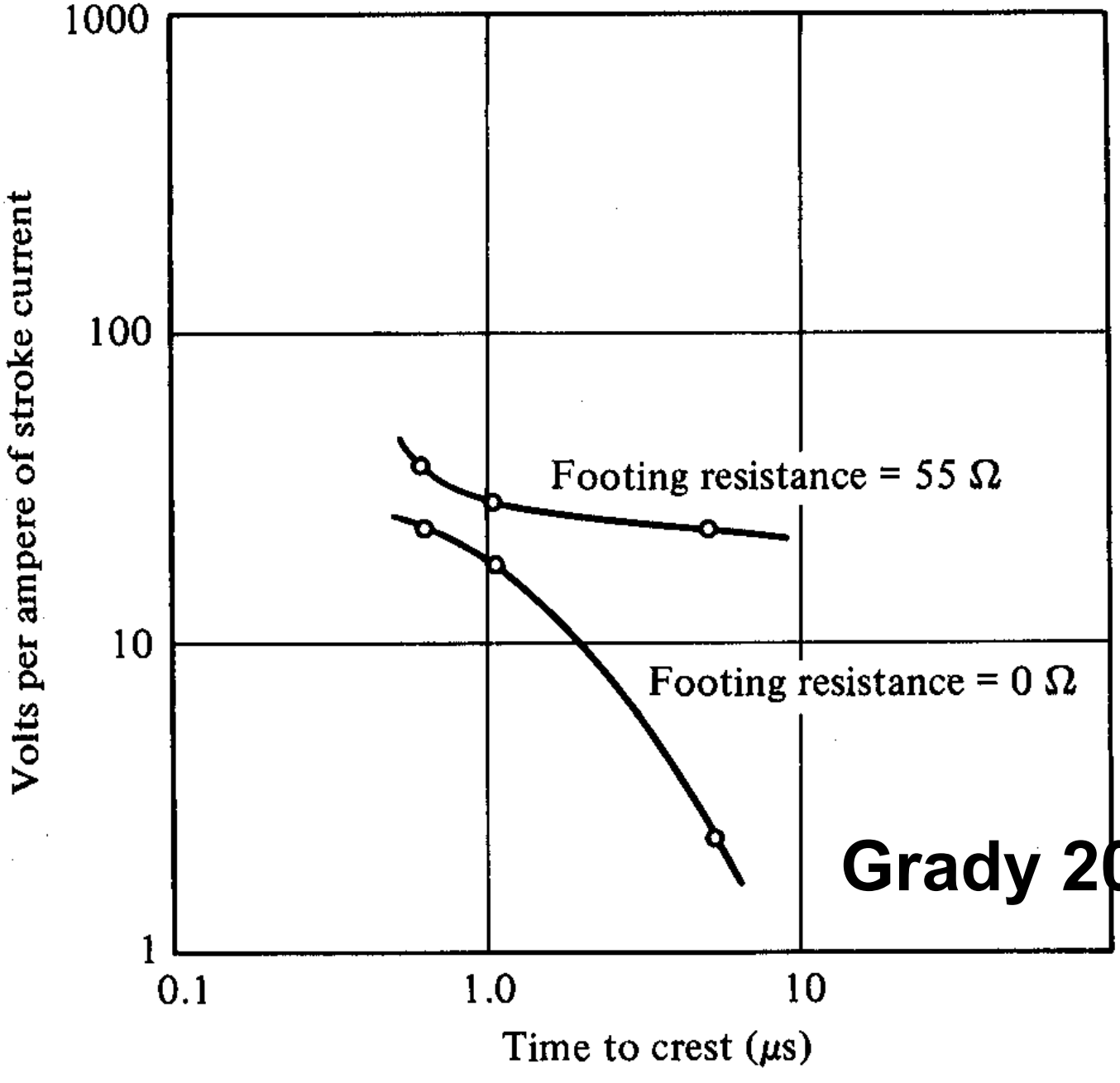

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Figure 6.40 Representative measured values of the peak voltage generated across an insulator string by a stroke. [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

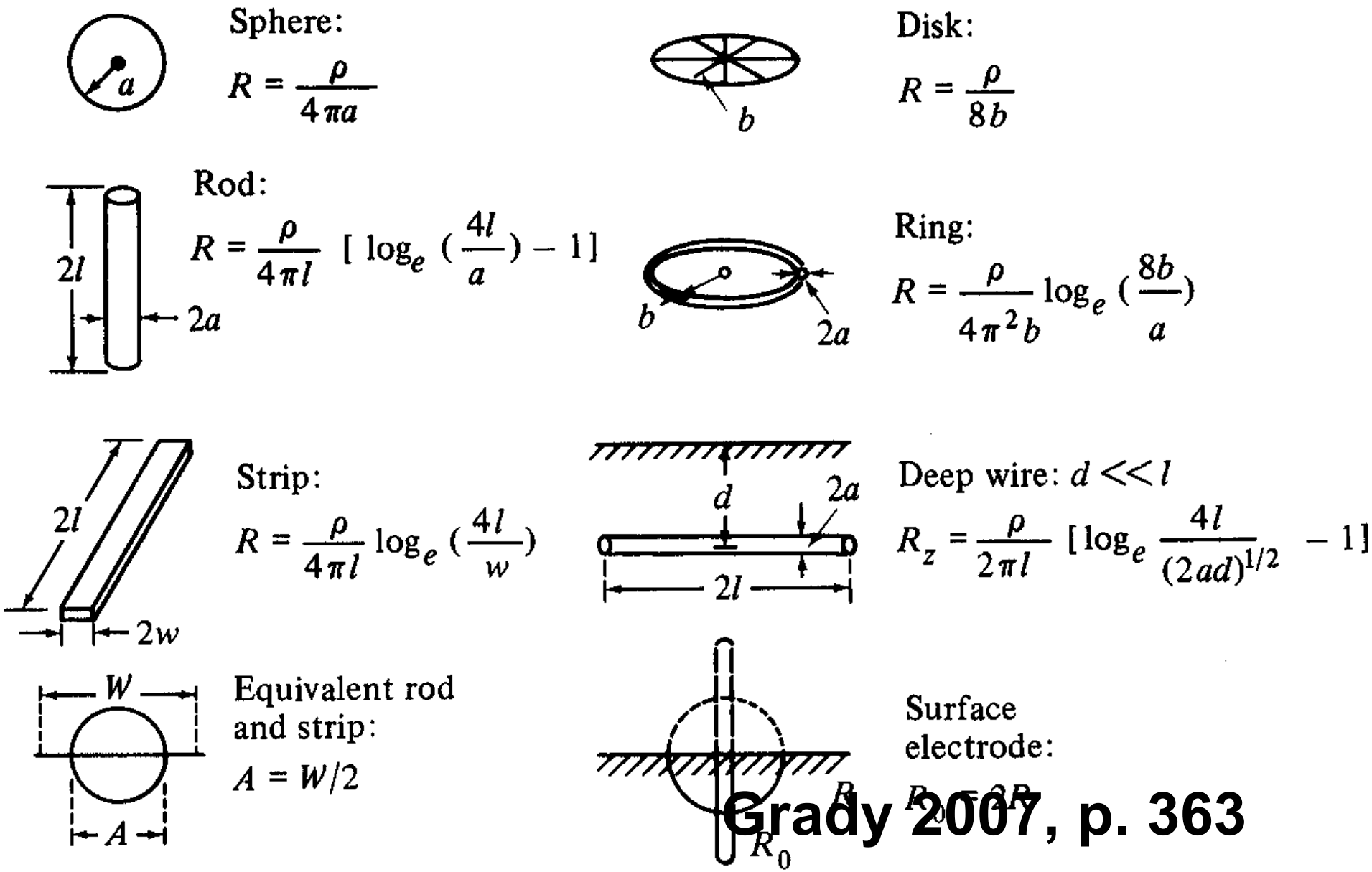

Figure 6.41 Ground resistance of elementary electrodes, buried deeply in earth. [Source: *EHV Transmission Line Reference Book* (Edison Electric Inst.).]

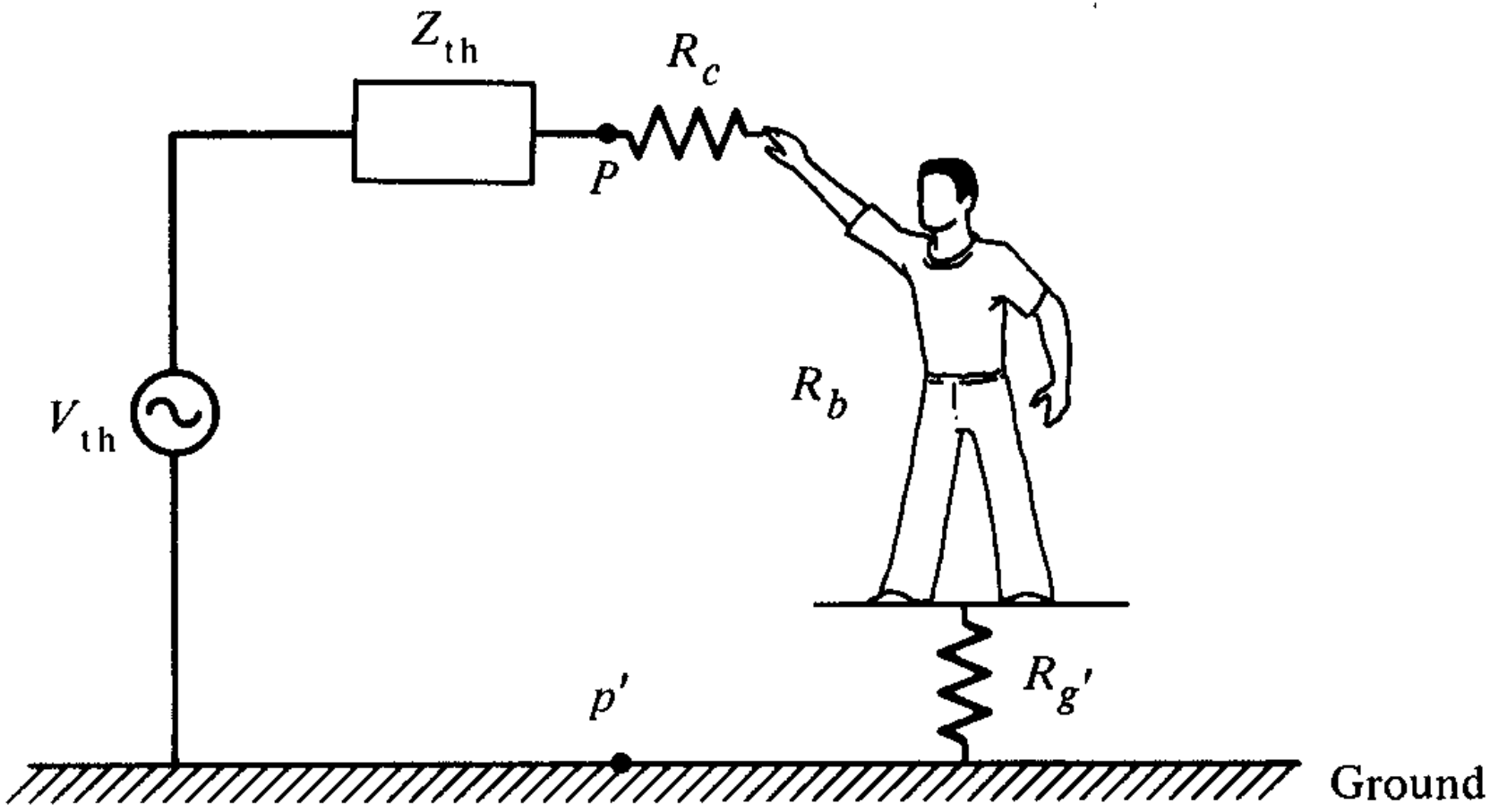

Figure 6.25 Equivalent circuit for estimating the current in a body contacting a grounded "incidental" line.

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$V_{th} = V_{oc}$ at $P_{p'}$, $Z_{th} = R_g + Z_0 \tanh \gamma d$, $Z_m = R_{contact} + R_{body} + R_{ground'}$

$R_c = R_{contact} \approx 500\ \Omega$, $R_b = R_{body} \approx 1000\ \Omega$, $R_{g'} = R_{ground} \approx 600\ \Omega$

6.7.3 Hazards to People

According to Dalziel's[4] electrocution formula, the 60-Hz body current leading to a possible fatality through ventricular fibrillation is

$$I = 0.165/\sqrt{t} \tag{6.9}$$

where t is in seconds in the range from about 8.3 ms to 5 s. The current that will flow in the body depends on the body resistance, the effective grounding resistance of the body, the open circuit voltage at the point on the conductor which contacts the individual, and the impedance seen looking back into the system at the point of contact. The latter two factors will be recognized as the essentials in the Thévenin equivalent circuit for the contact points (ground and the conductor). The current can then be calculated with the aid of this equivalent circuit,

$$I = \frac{V_{\text{th}}}{Z_{\text{th}} + Z_m}$$

where Z_m is the total impedance in the "human" part of the circuit as described above. The idea is indicated in the equivalent in Fig. 6.36. Typical values of resistances are shown in the table, but these are subject to wide variations. A typical person to ground capacitance when on a 1-cm-thick insulator is about 200 pF.

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TABLE 6.1 Typical Electric Shock Energy Effects

	Millijoules
1. Minimum primary shock energy	50,000
2. More conservative value (IEEE)	25,000
3. "Unpleasant" shock energy	250
4. Minimum secondary shock energy	0.5–1.5
5. Perception threshold	0.12

The current calculated in this way may be used in Eq. 6.9 to determine the time to a possible fatality, for if the current exceeds that given by Eq. (6.9) for a stated time, a fatality is likely. Moreover, currents in the range from 5 to 20 mA, if sustained, are likely to interfere with muscular control so that the individual cannot "let go" and free him or herself from the circuits. If currents in excess of the "let go" levels for an individual are continued, the result is likely to be a collapse, unconsciousness, asphyxia, and death. Tables 6.1 and 6.2 show effects of current on people.

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TABLE 6.2 Typical Effects of Currents on People[a]

Effect	Current (mA)			
	Direct Current		60 Hz rms	
	Men	Women	Men	Women
1. No sensation on hand	1	0.6	0.4	0.3
2. Slight tingling. Perception threshold	5.2	3.5	1.1	0.7
3. Shock—not painful but muscular control not lost	9	6	1.8	1.2
4. Painful shock—painful but muscular control not lost	62	41	9	6
5. Painful shock—let go threshold	76	51	16.0	10.5
6. Painful and severe shock, muscular contractions, breathing difficult	90	60	23	15
7. Possible ventricular fibrillation from short shocks:				
(a) shock duration 0.03 s	1300	1300	1000	1000
(b) Shock duration 3.0 s	500	500	100	100
(c) Almost certain ventricular fibrillation (if shock duration is over one heart beat interval)	1375	1375	275	275

[a] Threshold for 50% of the males and females tested.

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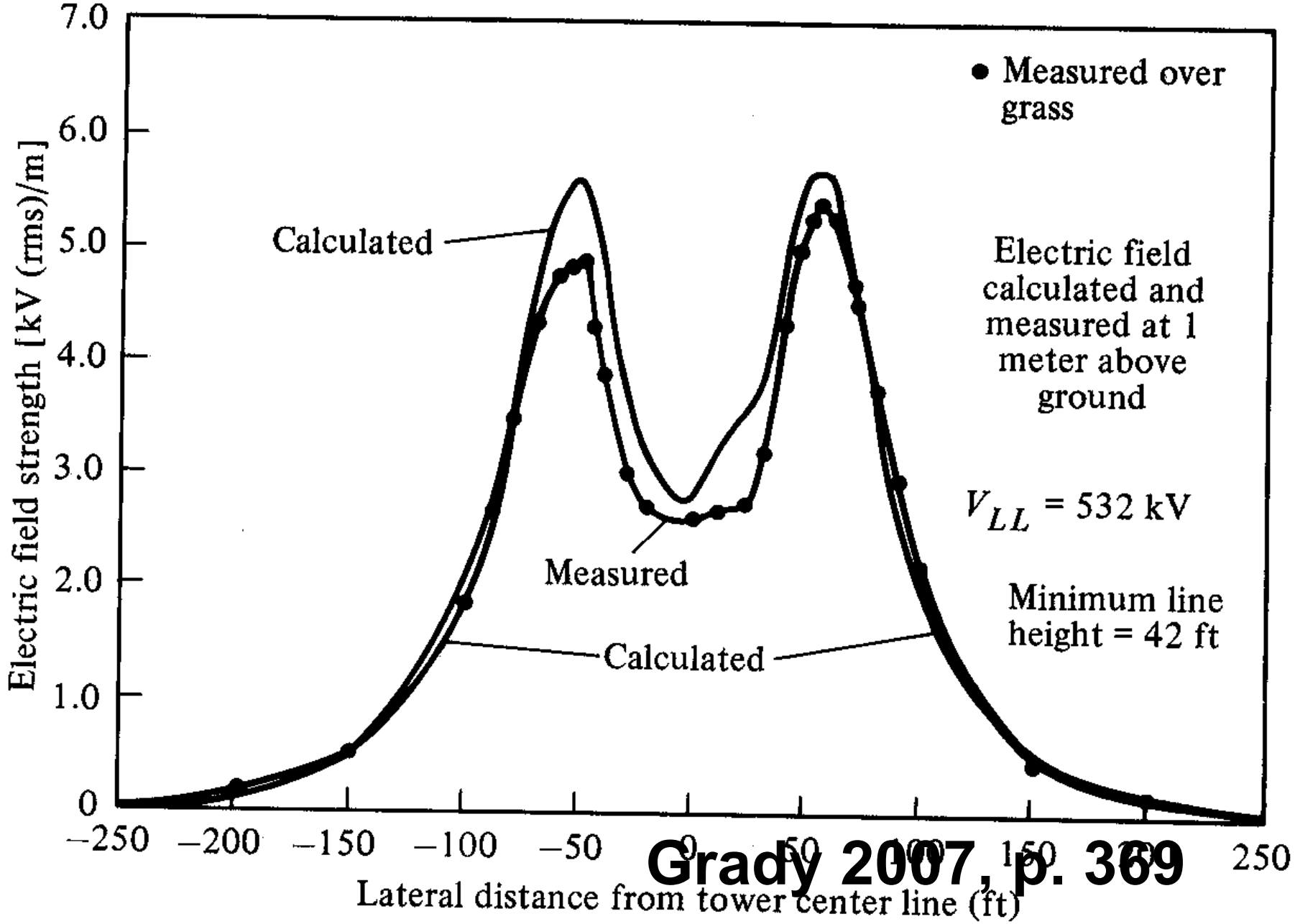
If the field strength levels are high enough, people in the region begin to sense the field in one way or another even though there are no significant currents to ground. The threshold of perception is in the range of 10–15 kV/m, where the sensations are body hair erection and tingling between body and clothes. Such activity is apparently not harmful, but at higher fields, perhaps as low as 15–20 kV/m, sparks may form, and contacts to grass, weeds, or the handling of conducting objects are unpleasant, at least. At higher fields, corona may form at some places with small radii such as ears. This phenomenon is also disturbing to animals.

Grady 2007, p. 368



Grady 2007, p. 369

Figure 6.24 Comparison of measured and computer calculated electric field 500-kV line.

14.4 A lightning stroke which reaches a peak current of 35 kA in 1 μs strikes a 20 m tower on a 345 kV transmission line. The line has a ground wire joining the tops of the towers; its surge impedance is 520 Ω. The tower surge impedance is 90 Ω and the ground footing resistance is 40 Ω.

Determine whether the line insulators will flashover as a consequence of the surge, assuming that their impulse flashover strength is 1050 kV. A coupling factor 0.3 with the phase conductors can be assumed; the impedance of the stroke channel can be ignored; a wave velocity on the tower of 2.98×10^8 m/s can be assumed.

Grady 2007, p. 370

14.6

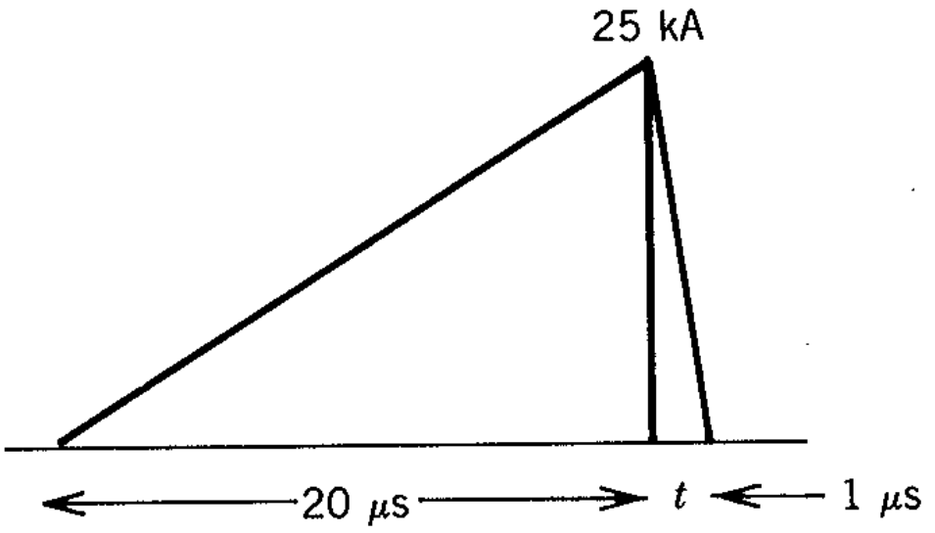


Fig. 14P.3.

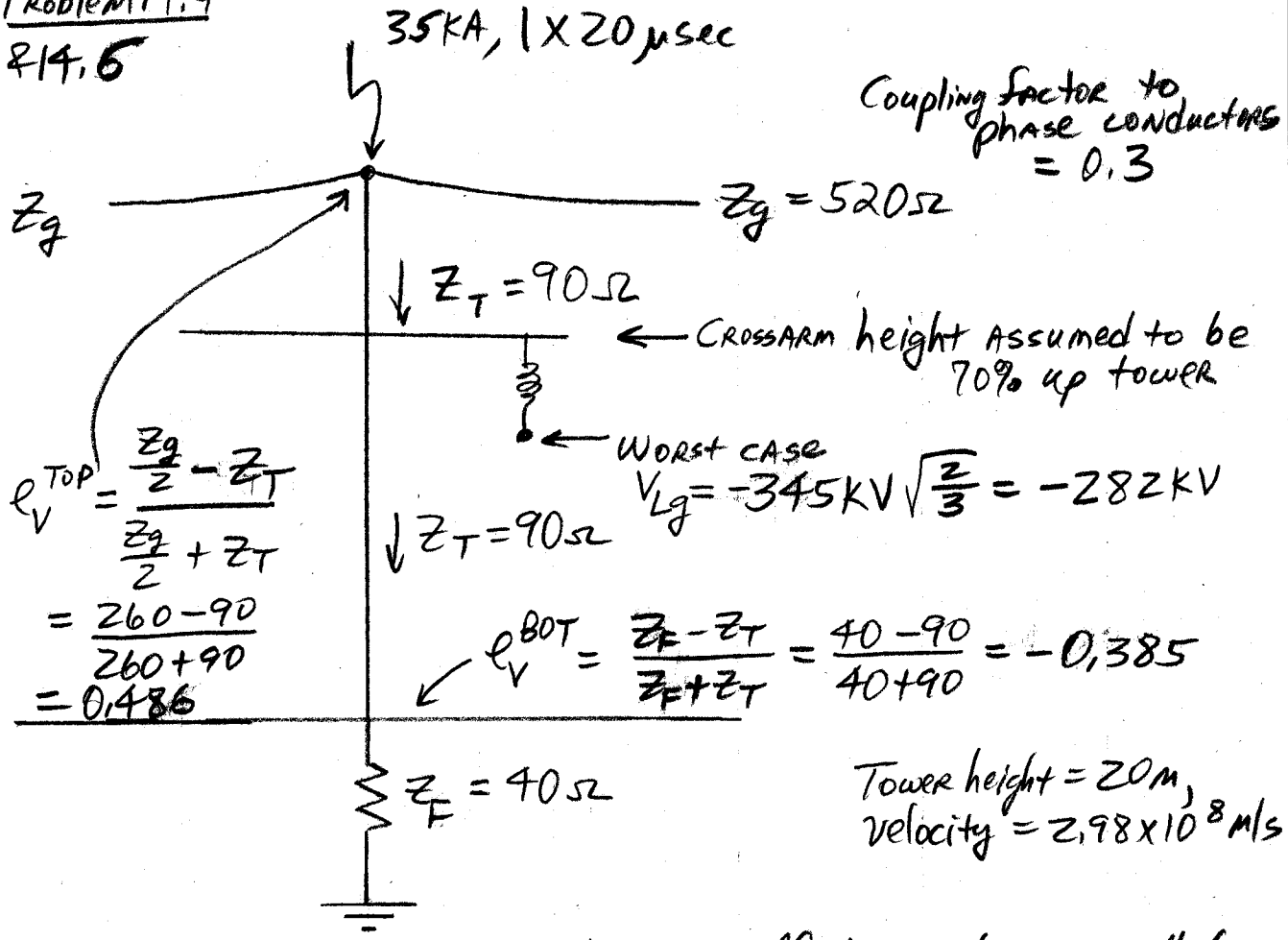
A lightning current surge with the waveshape of Fig. 14P.3 strikes a tower which has a single ground wire in both directions. The characteristics are as follows:

surge impedance of lightning channel	$= 1500\ \Omega$
surge impedance of tower	$= 150\ \Omega$
surge impedance of ground wire	$= 340\ \Omega$
velocity of wave propatation on lines	$= 298\ \mathrm{m}/\mu\mathrm{s}$
velocity of wave propagation on tower	$= 240\ \mathrm{m}/\mu\mathrm{s}$
coupling factor of phase conductors	$= 0.25$
height of tower	$= 30\ \mathrm{m}$
effective tower footing resistance	$= 40\ \Omega$

Grady 2007, p. 371

Determine the maximum tower top potential. Plot the surge voltage distribution on a phase conductor 0.4 μs after the tower has been struck.

Problem 14.4
&14.5



$$\rho_V^{TOP} = \frac{\frac{Z_g}{2} - Z_T}{\frac{Z_g}{2} + Z_T} = \frac{260-90}{260+90} = 0.486$$

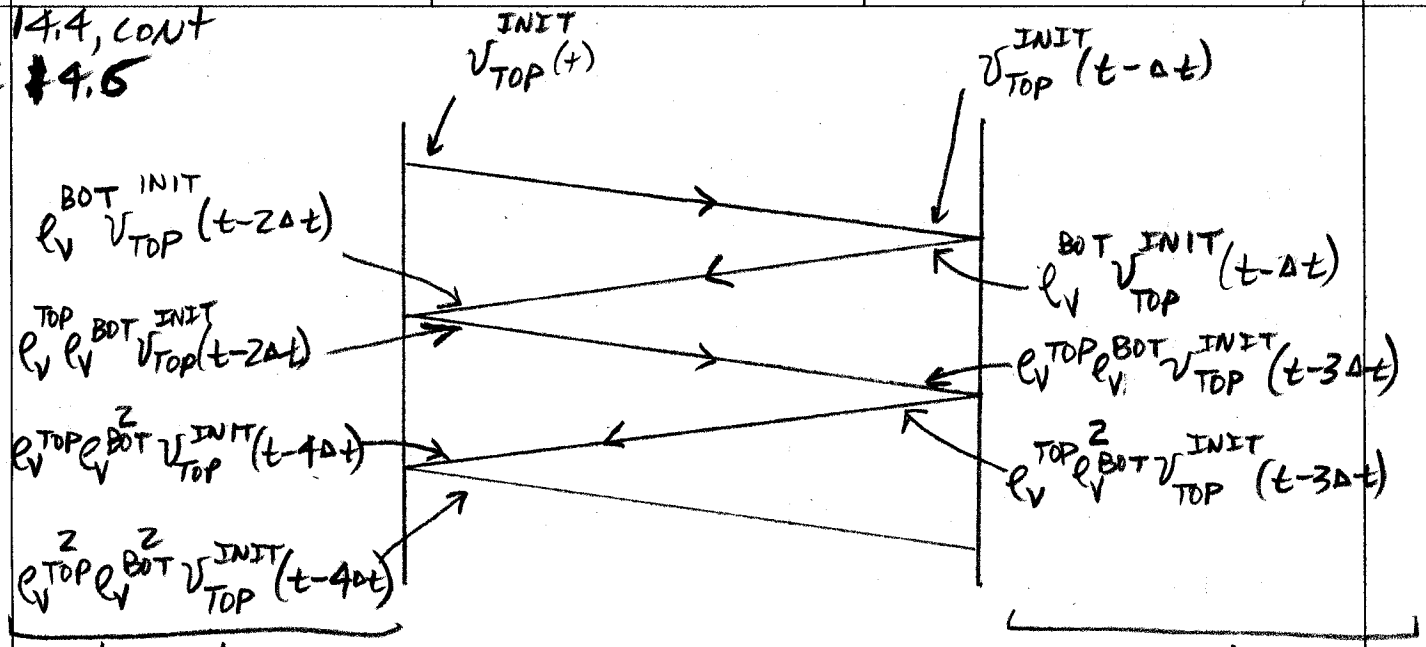
$$\rho_V^{BOT} = \frac{Z_F - Z_T}{Z_F + Z_T} = \frac{40-90}{40+90} = -0.385$$

The initial tower top voltage will be the parallel combination $Z_g \| Z_g \| Z_T = \frac{Z_g}{2} \| Z_T = \frac{(260)(90)}{260+90} = 66.9\Omega$ times the lightning current, so $v_{TOP}^{INIT} = 66.9\,i$. If the tower was infinitely tall, v_{TOP} would crest at $(66.9)(35KA) = 2342KV$. (see V_{TOP-1} plot on Excel sheet). Bounces from the tower base reduce this crest value considerably.

- Tower top & bottom. A lattice diagram is useful in developing the expression for tower top & bottom.

Grady 2007, p. 372

14.4, cont & 14.5



Add all terms that have "hit" by any time to get voltage at tower top.

Add all terms that have "hit" by any time to get voltage at bottom of tower

The reflected terms get progressivly smaller, as seen in the table below

Δt	Bottom Bounce	$\rho_V^{BOT} =$	-0.385	5 top bounces are sufficient to model
$2\Delta t$	Top Bounce	$\rho_V^{TOP}\rho_V^{BOT} =$	-0.187	
$3\Delta t$	Bottom Bounce	$\rho_V^{TOP}\rho_V^{2BOT} =$	0.072	
$4\Delta t$	Top Bounce	$\rho_V^{2TOP}\rho_V^{2BOT} =$	0.035	
			And so on	

- Voltage at Cross Arm

  Approximate using $v_{CROSS\,ARM}(t) = v_{BOTTOM}(t) + 0.70\left[v_{TOP}(t) - v_{BOTTOM}(t)\right]$

- Induced voltage on phase conductors

  $\Delta v_{Induced\ on\ phase}(t) = 0.3\, v_{TOP}(t)$

Grady 2007, p. 373

14-4, cont & 14.5

- Voltage across insulator string,
  (worst case, pick the phase with negative V_{peak})

$$V_{string}(t) = V_{\substack{CROSS\\ARM}}(t) - \left[\Delta V_{\substack{INDUCED\\ON\,phase}} - V_{peak}\right]$$

$$V_{string}(t) = V_{\substack{CROSS\\ARM}} - \Delta V_{\substack{INDUCED\\ON\,phase}} + V_{peak}$$

Results from Excel

Peak tower top voltage $\simeq$ 1332 KV

Peak voltage across string $\simeq$ 1164 KV
(will flashover 1050KV rated)

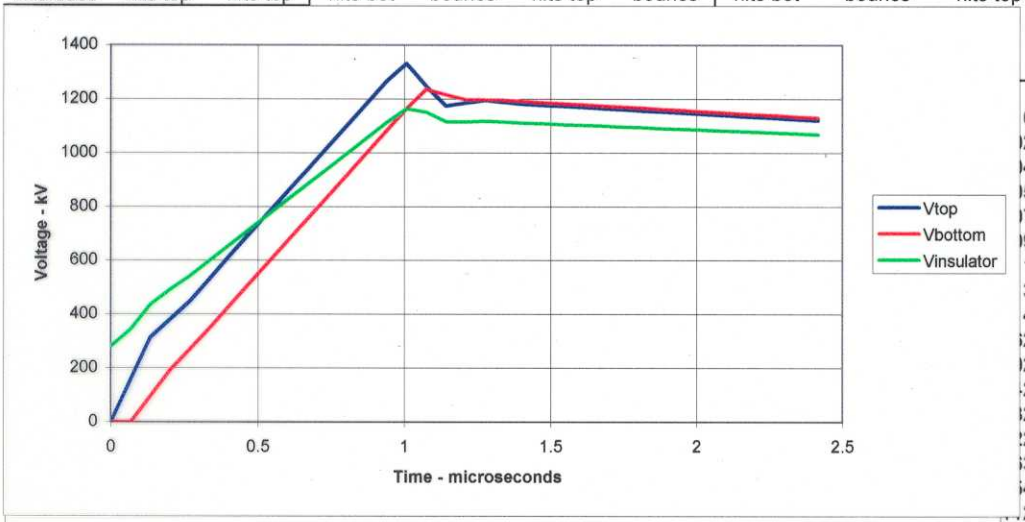
If the footing resistance is lowered to 10Ω,
Peak tower top voltage $\simeq$ 512 KV
Peak voltage across string = 585KV (No Flash)

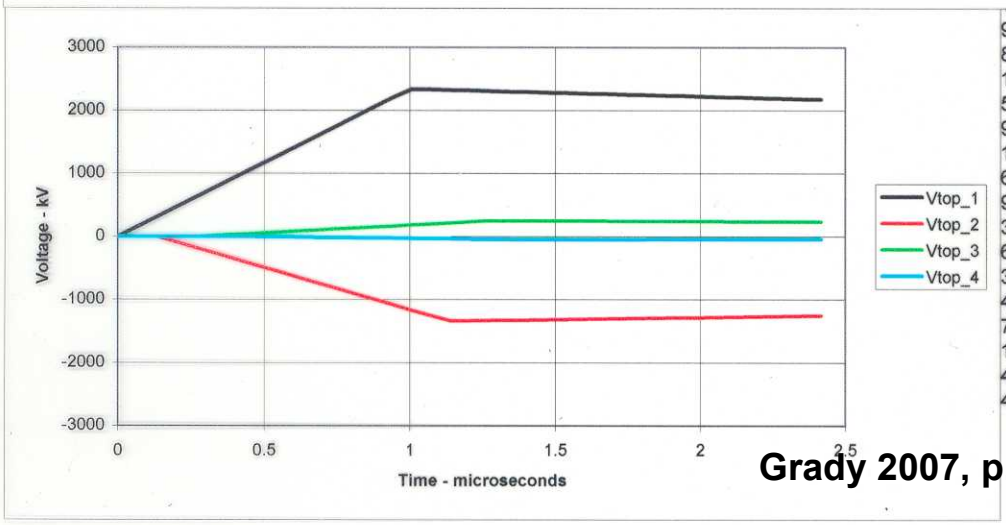
Problem 4-5. Same Excel program used, lightning channel assumed ∞ Ω.

Important observations for both problems.

- Low footing resistance is the <u>most</u> important factor in reducing V_{TOP} & induced voltage.
- The tower quickly disappears from the analysis, leaving only the two ground wires & the footing resistance. For 4.4, this is 520||520||40 = 34.7Ω. (35KA)(34.7Ω) = 1215KV. This is the pos Grady 2007, p.374 in the graph.

ZG	ZTOWER	ZFOOT	Tower Height	Tower velocity	Phase Coupling Factor	Crossarm Height Ratio	Peak kA	Rise Time microsec	Fall Time microsec	Line Voltage kVrms(l-l)	rhov
520	90	40	20	2.98E+08	0.3	0.7	35	1	20	345	0.4
t microsec	v(t)-kv hits top	(non-neg) v(t)-kv hits top	v(t)-kv hits bot	v(t)-kv bot bounce	v(t)-kv hits top	v(t)-kv top bounce	v(t)-kv hits bot	v(t)-kv bot bounce	v(t)-kv hits top	v(t)-kv top bounce	v(hits



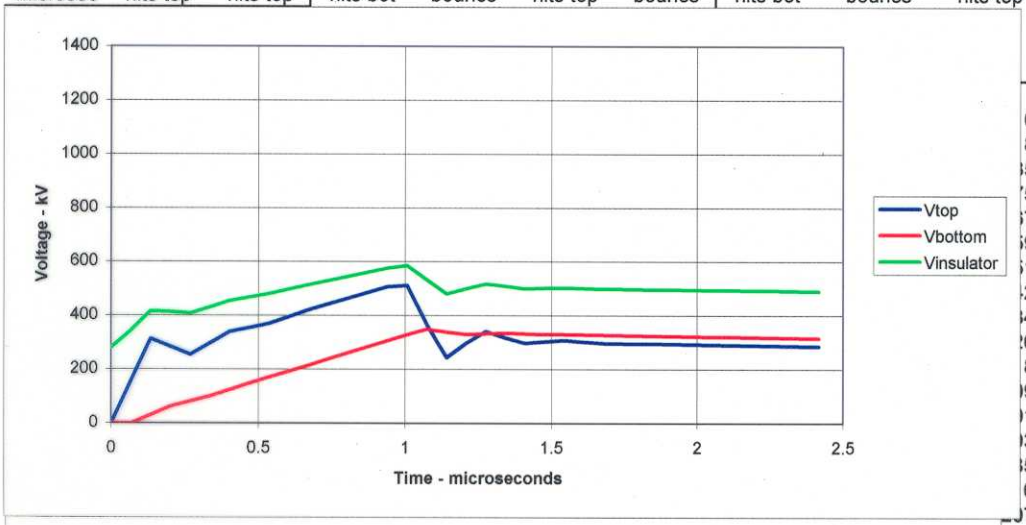


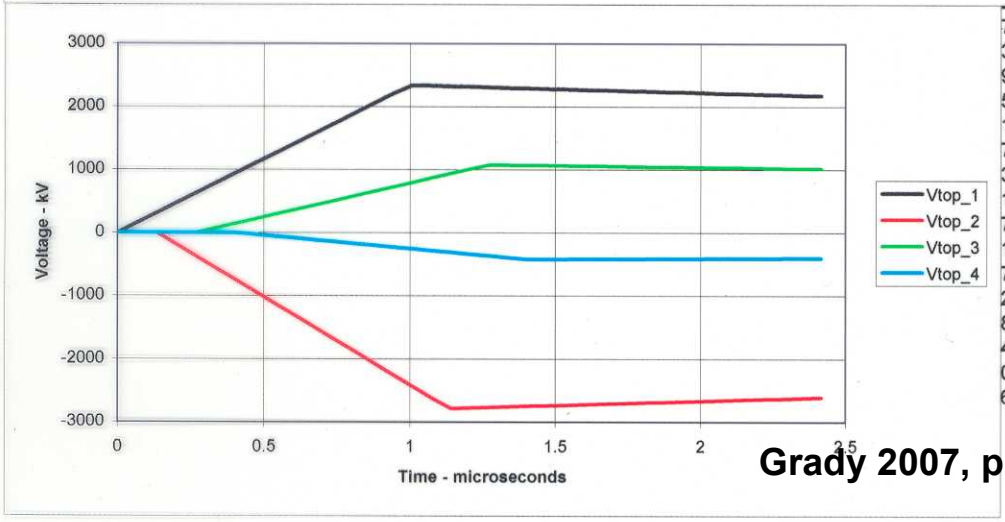
										0	0
										2	5.480809
										4	10.96162
										5	16.44243
										7	21.92323
										9	27.40404
										1	32.88485
										3	38.36566
										4	43.84647
										2	49.32728
										2	54.80809
										2	60.2889
										2	65.7697
										2	71.25051
										3	76.73132
										4	81.63665
										2	81.36261
										947	81.08857
										828	80.81452
										186	80.54048
										544	80.26644
										902	79.9924
										126	79.71836
										618	79.44432
										976	79.17028
										334	78.89624
										692	78.6222
										305	78.34816
										408	78.07412
										766	77.80008
										124	77.52604
										482	77.252
										484	76.97796

Grady 2007, p. 375

ZG	ZTOWER	ZFOOT	Tower Height	Tower velocity	Phase Coupling Factor	Crossarm Height Ratio	Peak kA	Rise Time microsec	Fall Time microsec	Line Voltage kVrms(l-l)	rhov
520	90	10	20	2.98E+08	0.3	0.7	35	1	20	345	0.4

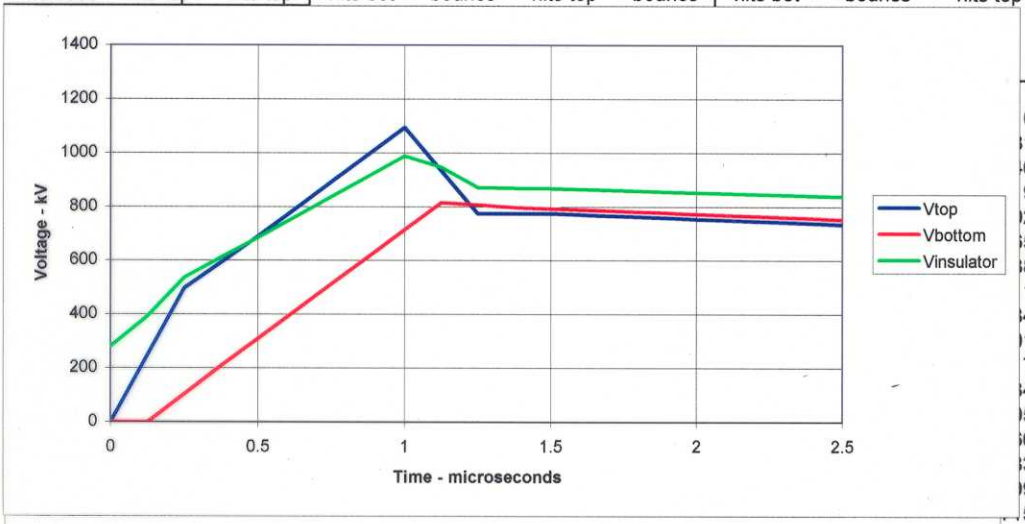
t microsec	v(t)-kv hits top	(non-neg) v(t)-kv hits top	v(t)-kv hits bot	v(t)-kv bot bounce	v(t)-kv hits top	v(t)-kv top bounce	v(t)-kv hits bot	v(t)-kv bot bounce	v(t)-kv hits top	v(t)-kv top bounce	v(hit
0										0	
8										23.71217	
5										47.42434	23.
5										71.13651	47.
7										94.84868	71.
9										118.5609	94.
1										142.273	118
2										165.9852	14
4										189.6974	165
6										213.4095	189
8										237.1217	213
9										260.8339	237
1										284.5461	260
3										308.2582	284
5										331.9704	308
6										353.1928	331
7										352.0072	353
797										350.8216	352
387										349.636	350
978										348.4504	34
568										347.2647	348
159										346.0791	347
749										344.8935	346
339										343.7079	344
193										342.5223	343
752										341.3367	342
111										340.1511	341
701										338.9655	340
292										337.7799	338
882										336.5943	337
472										335.4087	336
063										334.223	335
653										333.0374	33

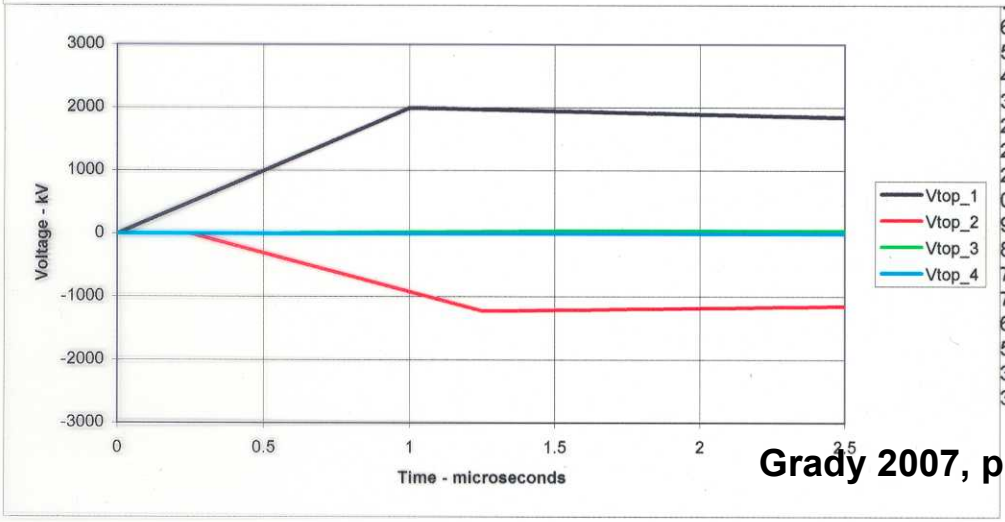




Grady 2007, p. 376

ZG	ZTOWER	ZFOOT	Tower Height	Tower velocity	Phase Coupling Factor	Crossarm Height Ratio	Peak kA	Rise Time microsec	Fall Time microsec	Line Voltage kVrms(l-l)	rhov
340	150	40	30	2.40E+08	0.25	0.7	25	1	20	345	0
t microsec	v(t)-kv hits top	(non-neg) v(t)-kv hits top	v(t)-kv hits bot	v(t)-kv bot bounce	v(t)-kv hits top	v(t)-kv top bounce	v(t)-kv hits bot	v(t)-kv bot bounce	v(t)-kv hits top	v(t)-kv top bounce	v(hits
										0	0
										1	0.326046
										6	0.652091
										9	0.978137
										2	1.304183
										5	1.630228
										8	1.956274
										1	2.28232
										4	2.608365
										1	2.592063
										7	2.575761
										4	2.559458
										5	2.543156
										6	2.526854
										3	2.510552
										9	2.494249
										5	2.477947
										632	2.461645
										548	2.445342
										464	2.42904
										381	2.412738
										297	2.396436
										213	2.380133
										213	2.363831
										046	2.347529
										962	2.331226
										879	2.314924
										795	2.298622
										711	2.28232
										628	2.266017
										544	2.249715
										346	2.233413
										377	2.217111





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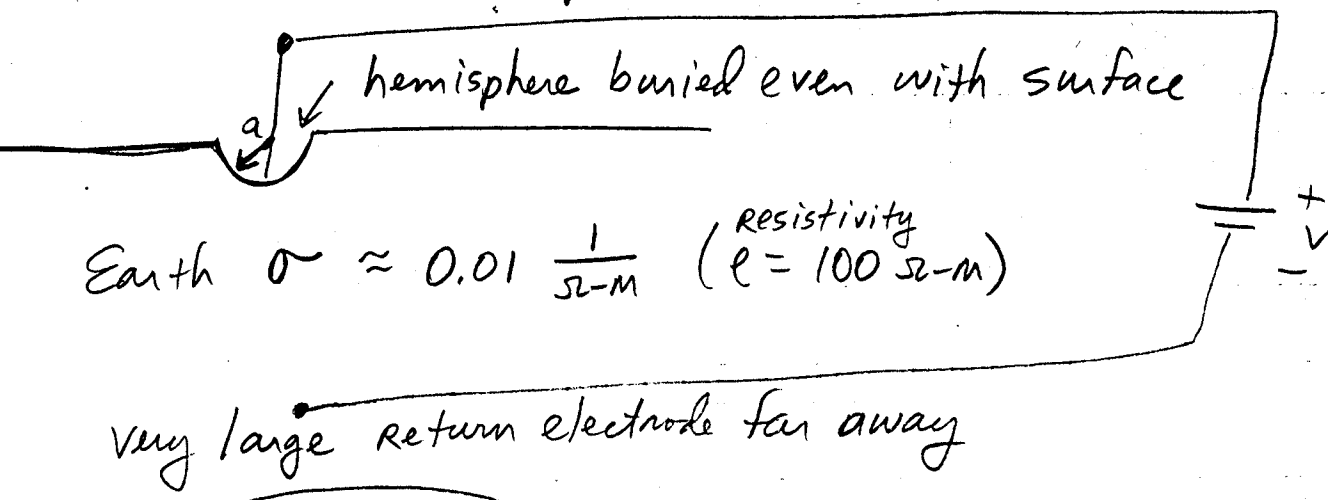
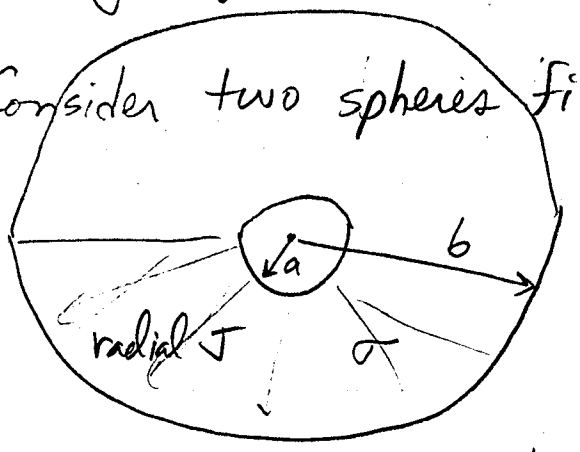
Resistance of hemisphere in Earth



Earth $\sigma \approx 0.01 \frac{1}{\Omega\text{-m}}$ ($\rho = 100\ \Omega\text{-m}$) resistivity

Consider two spheres filled with dirt. Treat as concentric shells in series



Using $R = \frac{L}{\sigma A}$, $dR = \frac{dr}{\sigma(4\pi r^2)}$ for each shell

(dR: resistance; dr: radius)

R adds in series, so $R = \int_{r=a}^{b} dR = \frac{1}{4\pi\sigma}\int_{r=a}^{b}\frac{dr}{r^2}$

$$R = \frac{1}{4\pi\sigma}\left[-\frac{1}{r}\right]\Big|_{r=a}^{b} = \frac{1}{4\pi\sigma}\left[\frac{1}{a} - \frac{1}{b}\right].$$

as $b \to \infty$, $R = \frac{1}{4\pi\sigma a}$.

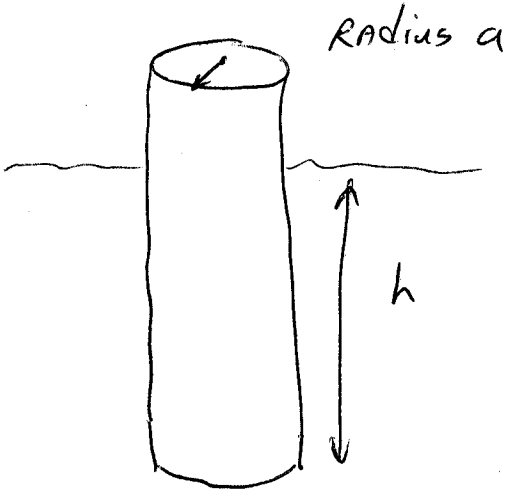
Check $RC = \frac{\epsilon_0}{\sigma}$, and $C = 4\pi\epsilon_0 a$

$$RC = \frac{4\pi\epsilon_0 a}{4\pi\sigma a} = \frac{\epsilon_0}{\sigma} \quad \text{OK}$$

So a hemisphere has twice this, $R = \frac{1}{2\pi\sigma a}$

Grady 2007, p. 378

Ground Rods



$$R = \frac{1}{2\pi\sigma h}\left(\ln\frac{4h}{a} - 1\right)$$

dirt

$$\sigma \simeq 0.01 \frac{1}{\Omega m}$$

Let $a = 1/4$ inch, $h = 8$ feet,

$$R = \frac{1}{2\pi(0.01)(2.44m)}\left(\ln\left[\frac{4(8)(12)}{1/4}\right] - 1\right)$$

$$= 6.53(6.34) = 41\,\Omega$$

Work hard to get $20\,\Omega$. At each tower foot.

- Perhaps not the best contact between dirt & rod
- Dirt dries out & backs away

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Electrocution Caused by prolonged shock, OR sudden energy

Prolonged Shock
- Dalziel's formula (IEEE Spectrum Feb 1969, Feb 1972)

$$I_{FATAL} = \frac{0.165}{\sqrt{t}},$$

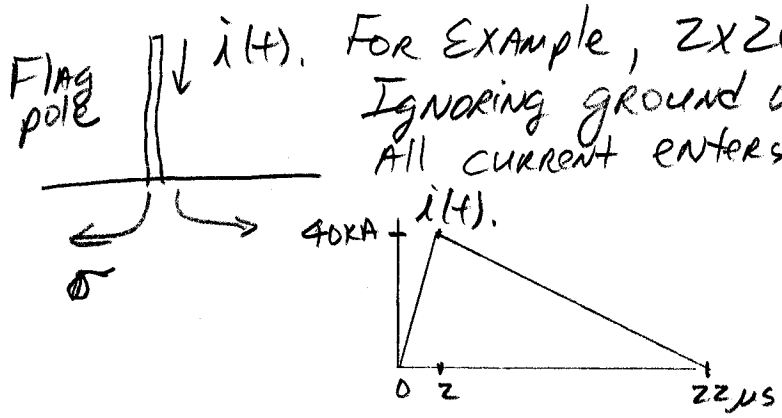
for $8.3\,ms \le t \le 5\,s$

↑ 1/2 cycle

- Weeks Table 6.2
  0.1A for 3 sec

Sudden Energy (Weeks doesn't say lethal?)
- Weeks Table 6.1
  Primary Energy — 25 to 50 J

Shock Energy in Lightning Strike



For Example, 2x20, 40KA.
Ignoring ground wires — Assuming All current enters the ground.
Assume bouncing is over quickly so i(t) goes into the ground relatively unaffected

In Earth

$$J(r) = \underbrace{\frac{i(t)}{2\pi r^2}}_{\text{Hemisphere}}, \quad E(r) = \frac{J(r)}{\sigma} = \frac{i(t)}{2\pi\sigma r^2}$$

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Shock Energy, cont

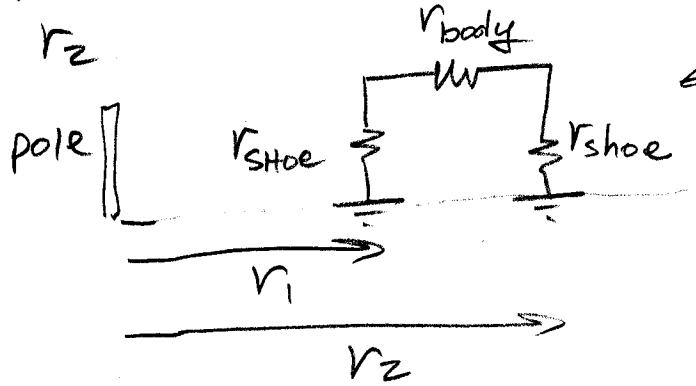
$$\Delta V_{1,2} = \int_{r_1}^{r_2} E(r)\cdot dr = \frac{i(t)}{2\pi\sigma}\int_{r_1}^{r_2}\frac{dr}{r^2} = \frac{i(t)}{2\pi\sigma}\int_{r_1}^{r_2} r^{-2}\,dr$$

r_1, r_2 from pole

$$= \frac{-i(t)}{2\pi\sigma} r^{-1}\Big|_{r_1}^{r_2} = \frac{-i(t)}{2\pi\sigma}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$$

$$= \frac{i(t)}{2\pi\sigma}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) = \frac{i(t)}{2\pi\sigma}\left(\frac{r_2-r_1}{r_1 r_2}\right)$$

A person with one foot at r_1, the other foot at r_2



← each r about 1 kΩ, so person has little effect on potential at ground surface

$$\Delta V_{Body} = \Delta V_{1,2}\left(\frac{r_{body}}{2r_{shoe}+r_{body}}\right), \text{ voltage divider}$$

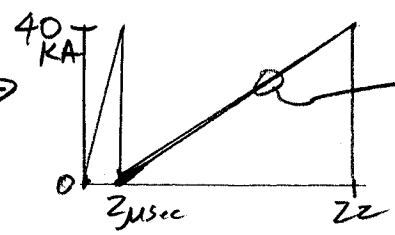
$$\text{Energy to } r_{body} = \int \frac{v_{body}^2}{r_{body}}\,dt,$$

$$v_{body} = \left[\underbrace{\frac{i(t)}{2\pi\sigma}\left(\frac{r_2-r_1}{r_1 r_2}\right)}_{\Delta V_{1,2}}\underbrace{\left(\frac{r_{body}}{2r_{shoe}+r_{body}}\right)}_{\text{voltage divider}}\right]$$

← has shape of 2x20 triangles

Now, examine the integral of the squared triangle shape

Symmetry demands that the integral will be the same as for this →



doesn't matter if slope down or slope up in time

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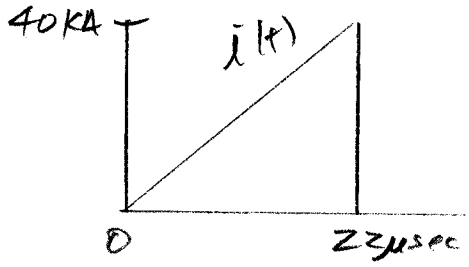
Symmetry demands that the contribution to the integral for the first 2 μsec is $\frac{2}{20}$ times the contribution of the 2–22 μsec. So, one can conclude that integrating the current (or squared) triangle below gives you the desired result



General form, peak value A, duration t

$$\int_0^T i^2(t)\,dt = \int_0^T \left(\frac{A}{T}t\right)^2 dt$$

$$= \frac{A^2}{T^2}\int_0^T t^2\,dt = \frac{A^2}{3T^2}t^3\Big|_0^T = \frac{A^2T^3}{3T^2}$$

$$\int_0^T i^2(t)\,dt = \frac{A^2T}{3}$$

So, E_{body} (Joules) $= \frac{1}{r_{body}}\int_0^T v_{body}^2\,dt$ becomes

$$= \left(\frac{1}{r_{body}}\right)\left[\frac{1}{2\pi\sigma}\left(\frac{r_2-r_1}{r_1 r_2}\right)\left(\frac{r_{body}}{2r_{shoe}+r_{body}}\right)\right]^2 \frac{I_{peak}^2 T}{3}$$

$$= \frac{r_{body}}{4\pi^2\sigma^2}\left(\frac{r_2-r_1}{r_1 r_2}\right)^2\left(\frac{1}{2r_{shoe}+r_{body}}\right)^2 \frac{I_{peak}^2 T}{3}$$

$$E_{Body}\ (\text{Joules}) = \left(\frac{1}{4\pi^2\sigma^2}\right)\left(\frac{r_2-r_1}{r_1 r_2}\right)^2 \underbrace{\left[\frac{r_{body}}{(2r_{shoe}+r_{body})^2}\right]}_{\equiv R_{FACTOR}}\left[\frac{I_{peak}^2 T}{3}\right]$$

E_{BODY} (JOULES) $\propto I_{peak}^2$, $\propto T$, $\propto \left(\frac{1}{\sigma}\right)^2 = \rho^2$ (Resistivity)

$(r_2 - r_1)$ often considered to be 1 m

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sigma	Ipeak - kA	Duration - microsec	Rshoe	Rbody	(R2 - R1) - m	Rfactor
0.01	40.00	22.00	1000	1000	1	0.000111111

R1 - m	Body Peak Voltage	Body Joules
1	106103.30	82.56
1.1	91864.33	61.89
1.2	80381.28	47.38
1.3	70972.10	36.94
1.4	63156.72	29.25
1.5	56588.42	23.48
1.6	51011.20	19.08
1.7	46232.37	15.67
1.8	42104.48	13.00
1.9	38512.99	10.88
2	35367.77	9.17
2.1	32597.02	7.79
2.2	30142.98	6.66
2.3	27958.71	5.73
2.4	26005.71	4.96
2.5	24252.18	4.31
2.6	22671.64	3.77
2.7	21241.90	3.31
2.8	19944.23	2.92
2.9	18762.74	2.58
3	17683.88	2.29

sigma	Ipeak - kA	Duration - microsec	Rshoe	Rbody	(R2 - R1) - m	Rfactor
0.001	40.00	22.00	1000	1000	1	0.000111111

R1 - m	Body Peak Voltage	Body Joules
1	1061032.95	8255.80
2	353677.65	917.31
3	176838.83	229.33
4	106103.30	82.56
5	70735.53	36.69
6	50525.38	18.72
7	37894.03	10.53
8	29473.14	6.37
9	23578.51	4.08
10	19291.51	2.73
11	16076.26	1.90
12	13602.99	1.36
13	11659.70	1.00
14	10105.08	0.75
15	8841.94	0.57
16	7801.71	0.45
17	6934.86	0.35
18	6204.87	0.28
19	5584.38	0.23
20	5052.54	0.19
21	4593.22	0.15

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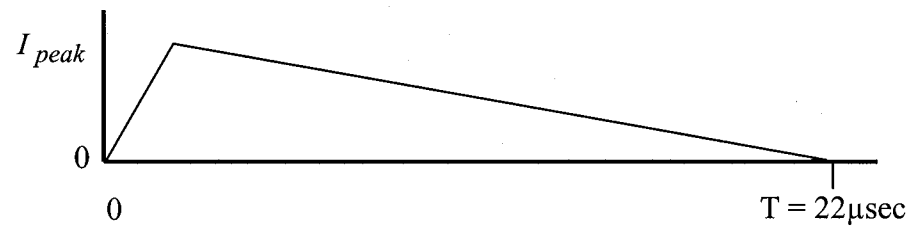
Six equally-weighted problems. To receive proper credit, you must show all your work.

Problem 6. Lightning, with the waveform shown below, hits a tree.



A person stands with one foot $R_1 = 3$ meters from the base of the tree, and the other foot R_2 meters from the base of the tree. Each shoe has resistance 1kΩ, and the body resistance is 1kΩ. Assuming that

- the Earth has uniform conductivity, so that the current density decreases according to $\frac{1}{R^2}$,
- the person does not significantly affect the voltages at the Earth's surface when a lightning strike occurs,
- $(R_2 - R_1) = 1\text{m}$,
- Lethal shock energy is 50J,

determine the minimum lethal I_{peak} for dry soil conditions ($\sigma = 0.001$).

Hint – the formula developed in class for this analysis is

$$E_{bodyjoules} = \left(\frac{1}{4\pi^2\sigma^2}\right)\left(\frac{R_2 - R_1}{R_1 R_2}\right)^2\left(\frac{R_{body}}{(2R_{shoe} + R_{body})^2}\right)\left(\frac{I_{peak}^2 T}{3}\right).$$

$$I_{PEAK\,MIN}^2 = \frac{E_{body} \leftarrow 50}{\underbrace{\left(\frac{1}{4\pi^2\sigma^2}\right)}_{25{,}330}\underbrace{\left(\frac{R_2-R_1}{R_1R_2}\right)^2}_{0.00694}\underbrace{\left[\frac{R_{body}}{(2R_{shoe}+R_{body})^2}\right]}_{111.1\times10^{-6}}\underbrace{\left(\frac{T}{3}\right)}_{7.33\times10^{-6}}}$$

$$\boxed{I_{PEAK\,MIN} = 18.69\,\text{kA}}$$

Problem 1 – Harmonic Filter Design.
A 12.5kV distribution feeder is experiencing a problem with high 5th harmonic voltages caused by large six-pulse adjustable-speed motor drives. Your task is to design a "generic" 5th harmonic, grounded-wye, series-tuned filter. **Each phase** of the filter will employ a 300kVAr (@ 12.5kV/√3) capacitor and a series inductor. Ignore inductor resistance. (Note - the number of these generic filters needed to solve the harmonic problem is not addressed here)

a. Specify the inductance (mH) of each inductor.
b. If the feeder voltage is 12.5kV/√3, determine the 60Hz rms current in each inductor.

Now, assuming that the 5th harmonic current in the filter has twice the magnitude of the fundamental,

c. Specify the rms ampere rating of each inductor.
d. Determine the rms 5th harmonic voltage across the capacitor.

Problem 2 – Estimating the Filter kVArs Needed to Correct a Seriously Distorted Distribution System.
Highly-distorted distribution systems, such as ski areas with large six-pulse adjustable-speed motor drives, require significant filtering. Conventional shunt capacitors must be avoided – they must be used only as filters. A good "starting point" for developing a filter plan is to:

a. power factor correct the distribution system DPF to 0.95 with harmonic filters,
b. for every 4 units of kVArs committed for 5th harmonic filters, commit 2 units for 7th harmonic filters, 1 unit for 11th harmonic filters, and 1 unit for 13th harmonic filters,
c. assume that contributions of harmonics above the 13th will not be a significant.

If a distribution system has 15MW, 10MVAr of conventional load, and 5MW of six-pulse drives (assume drive DPF = 0.85), determine a "starting point" estimate for the net kVArs needed for 5th, for 7th, for 11th, and for 13th harmonic filters.

Problem 3 – Partial Harmonic Current Cancellation Due to Variations in DPF.
Two six-pulse phase adjustable-speed motor drives operate side by side. Each draws 200kW from a 480V bus. One drive has DPF = 0.90, and the other has DPF = 0.70. Assume that

- both drives have the standard six-pulse waveform, except that one lags the other in time according to their DPFs.
- the 5th harmonic current for each drive is 20% of fundamental

What is the net 5th harmonic current drawn by the pair, in rms amperes?

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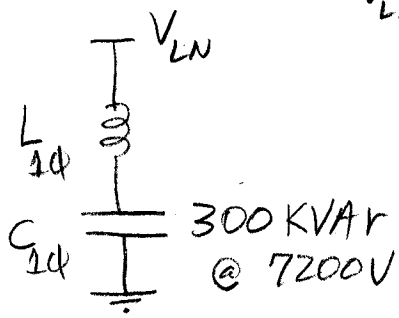
Problem 1



$$V_{LN}^2 \omega C = Q_{1\phi}, \quad C_{1\phi} = \frac{Q_{1\phi}}{V_{LN}^2 \omega} = \frac{300 \times 10^3}{\left(\frac{12500}{\sqrt{3}}\right)^2 (120\pi)}$$

$$\boxed{C_{1\phi} = 15.28\,\mu F}$$

$$\frac{1}{\sqrt{LC}} = \omega_{Res}, \quad \frac{1}{\sqrt{LC}} = 2\pi f_{Res},$$

$$\frac{1}{LC} = (2\pi f_{Res})^2, \quad L = \frac{1}{(2\pi f_{Res}^2)C}$$

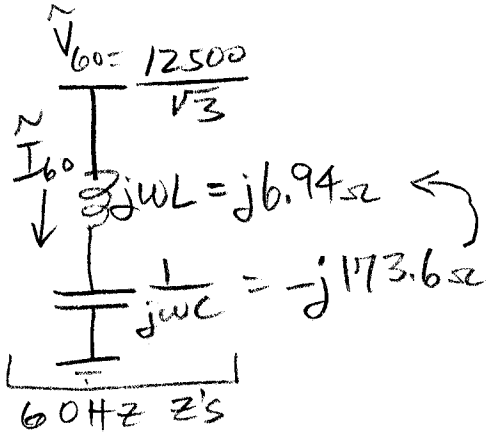
Check $\frac{1}{\sqrt{LC}}$

$= 1885$ R/s

$= 300$ Hz (OK)

$$L = \frac{1}{(2\pi \cdot 60 \cdot 5)^2 (15.28 \times 10^{-6})}$$

(a) $\boxed{L_{1\phi}\ 18.42\ mH}$



Checks as 5^2 times bigger

(b)

$$\tilde{I}_{60} = \frac{\frac{12500}{\sqrt{3}}\angle 0}{j(6.94 - 173.6)} = \boxed{43.3\angle +90^\circ\ A_{RMS}}$$

(c) If $|\tilde{I}_{300}| = 2|\tilde{I}_{60}| = 86.6\ A_{RMS}$

$$|\tilde{I}_{L1}|^2 = |\tilde{I}_{60}|^2 + |\tilde{I}_{300}|^2 = |I_{60}|^2(1+2^2) = 5|I_{60}|^2$$

$$\boxed{|I_{L1}| = 96.8\,A \text{ (the rating)}}$$

(d) $|V_{300}| = |I_{300}||Z_{Cap\,300}| = (86.6)\left(\frac{173.6}{5}\right) = \boxed{3000\,V}$

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Problem 2

Linear Load 15MW, 10MVAr

Drive Load 5MW, $Q = P\sqrt{\left(\frac{1}{DPF}\right)^2 - 1} = 5\sqrt{\left(\frac{1}{0.85}\right)^2 - 1} = 3.10\,MVAr$

$P_{TOT} = 15 + 5 = 20\,MW$

$Q_{TOT} = 10 + 3.10 = 13.10\,MVAr$

We want $Q_{NEW} = P\sqrt{\left(\frac{1}{0.95}\right)^2 - 1} = 20\sqrt{\left(\frac{1}{0.95}\right)^2 - 1} = 6.57\,MVAr$

Using $Q_{NEW} = Q_{TOT} + Q_{CAP}$, $Q_{CAP} = 6.57 - 13.10 = -6.53\,MVAr$

Lets use ⊕ notation for cap Q

$$Q_{CAP} = \underbrace{4Q}_{5^{th}} + \underbrace{2Q}_{7^{th}} + \underbrace{1Q}_{11^{th}} + \underbrace{1Q}_{13^{th}}$$

$$Q_{CAP} = 8Q, \quad Q = \frac{6.53}{8} = 816\,KVAr$$

$$\boxed{\begin{aligned} Q_5 &= 3.26\,MVAr \\ Q_7 &= 1.63\,MVAr \\ Q_{11} &= 0.82\,MVAr \\ Q_{13} &= 0.82\,MVAr \end{aligned}}$$

TOT 6.53

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Problem 3

Drive 1. $S_{1\phi} = V_{LN} I^*$, $I = \dfrac{S_{1\phi}^*}{V_{LN}^*}$

$$S_{1\phi} = \frac{P_{1\phi}}{DPF} \angle \cos^{-1}(DPF)$$

$$I = \left[\frac{P_{1\phi} \angle \cos^{-1}(DPF)}{(DPF)\, V_{LN}}\right]^* = \left[\frac{\frac{200000}{3} \angle \cos^{-1}(0.90)}{(0.90)(480/\sqrt{3})}\right]^*$$

$$\boxed{\tilde{I}_{D1} = 267 \angle -25.8^\circ \text{ A}}$$

Drive 2. $\tilde{I}_{D2} = \left[\dfrac{\frac{200000}{3} \angle \cos^{-1}(0.70)}{(0.70)(480/\sqrt{3})}\right] = 344 \angle -45.6^\circ \text{ A}$

The above are fundamental freq. currents.
The fundamental angle of D2 lags the fund. Angle of D1 by $(45.6 - 25.8) = 19.8^\circ$.
Thus, maintaining the waveshape appearance, the 5th harmonic current of D2 lags the 5th of D1 by $5(19.8) = 99.0^\circ$

$$\frac{267}{5} \angle \theta_5 + \frac{344}{5} \angle \theta_5 - 99.0^\circ$$

The value of θ_5 doesn't matter when determining the mag of the phasor sum. Let $\theta_5 = 0$

$$\frac{1}{5}\left[267 + 344 \angle -99^\circ\right] = \frac{267 - 54 - j340}{5} = \frac{213 - j340}{5}$$

$$= 42.6 - j68$$

Mag = 80.2A

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