The $z$-Transform and Its Application to the Analysis of LTI Systems

Inversion of the $z$-Transform

Analysis of LTI Systems in the $z$-Domain

Causality and Stability
Inversion of the $z$-Transform

$$H(z) = \frac{Y(z)}{X(z)}, \quad H(z) \rightarrow_{inv z} h(n)$$

Inverse $z$-Transform:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{-n-1} dz$$

where the integral is a (counter-clockwise) contour integral over a closed path $C$ that encloses the origin and lies within the region of convergence of $X(z)$.

Methods of Inverse $z$-Transform

(1) Contour integration

(2) Power series expansion (using long division)

(3) Partial-fraction expansion
Inverse $z$-Transform by Partial-Fraction Expansion

$X(z)$ is rational function.

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

A rational function is proper if $a_N \neq 0$ and $M < N$.

Inverse $z$-Transform by Partial-Fraction Expansion

An improper rational function ($M \geq N$) can always be written as the sum of a polynomial and a proper rational function.

$$X(z) = \frac{B(z)}{A(z)} = c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{B_1(z)}{A(z)}$$

The inverse $z$-transform of the polynomial can easily be found by inspection.

We focus our attention on the inversion of proper rational function.
Inverse \( z \)-Transform by Partial-Fraction Expansion

Let \( X(z) \) be a proper rational function.

\[
X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}
\]

Since \( N > M \),

\[
\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}
\]

is proper.

Inverse \( z \)-Transform by Partial-Fraction Expansion

(1) Distinct poles. Suppose that the poles \( p_1, p_2, \ldots, p_N \) are all different.

\[
\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}
\]

We want to determine the coefficients \( A_1, A_2, \ldots, A_N \).

\[
\frac{(z - p_k)X(z)}{z} = \frac{(z - p_k)A_1}{z - p_1} + \cdots + A_k + \cdots + \frac{(z - p_k)A_N}{z - p_N}
\]

Therefore,

\[
A_k = \left. \frac{(z - p_k)X(z)}{z} \right|_{z=p_k}, \quad k = 1, 2, \ldots, N
\]

(In addition, if \( p_2 = p_1^* \), \( A_2 = A_1^* \).)
Inverse $z$-Transform by Partial-Fraction Expansion

(2) Multiple-order poles. $X(z)$ has a pole of multiplicity $m$, that is, it contains in its denominator the factor $(z - p_k)^m$.

The partial-fraction expansion must contain the terms

$$\frac{A_{1k}}{(z - p_k)} + \frac{A_{2k}}{(z - p_k)^2} + \cdots + \frac{A_{mk}}{(z - p_k)^m}$$

Therefore,

$$A_{mk} = \left. \frac{(z - p_k)^m X(z)}{z} \right|_{z=p_k}$$

$$A_{(m-1)k} = \frac{d}{dz} \left[ \left. \frac{(z - p_k)^m X(z)}{z} \right|_{z=p_k} \right], \cdots$$

$$A_{1k} = \frac{d^{(m-1)}}{dz^{(m-1)}} \left[ \left. \frac{(z - p_k)^m X(z)}{z} \right|_{z=p_k} \right]$$

Inverse $z$-Transform by Partial-Fraction Expansion

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots + \frac{A_N}{1 - p_N z^{-1}}$$

$$\mathcal{Z}^{-1} \left\{ \frac{1}{1 - p_k z^{-1}} \right\} = \begin{cases} (p_k)^n u(n), & \text{ROC: } |z| > |p_k| \text{ (causal)} \\ -(p_k)^n u(-n - 1), & \text{ROC: } |z| < |p_k| \text{ (anticausal)} \end{cases}$$
Inverse $z$-Transform by Partial-Fraction Expansion

In the case of a double pole:

$$X(z) = \frac{A}{(z-p)^2} + \cdots$$

$$X(z) = \frac{A z^{-1}}{(1-p z^{-1})^2} + \cdots$$

$$Z^{-1} \left\{ \frac{p z^{-1}}{(1-p z^{-1})^2} \right\} = \left\{ \begin{array}{l}
np^n u(n), \quad \text{ROC: } |z| > |p| \ (causal) \\
-np^n u(-n-1), \quad \text{ROC: } |z| < |p| \ (anticausal)
\end{array} \right.$$  

Decomposition of Rational $z$-Transform

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M} (1-z_k z^{-1})}{\prod_{k=1}^{N} (1-p_k z^{-1})}$$

With real signals,

$$X(z) = \sum_{k=0}^{M-N} \gamma_k z^{-k} + \sum_{k=1}^{K_1} \frac{\beta_k}{1 + \alpha_k z^{-1}} + \sum_{k=1}^{K_2} \frac{\beta_{0k} + \beta_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}}$$

$$= v_0 \prod_{k=1}^{K_1} \frac{1 + v_k z^{-1}}{1 + u_k z^{-1}} \prod_{k=1}^{K_2} \frac{1 + v_{1k} z^{-1} + v_{2k} z^{-2}}{1 + u_{1k} z^{-1} + u_{2k} z^{-2}}$$

where $K_1 + 2K_2 = N$.

Coefficients $\alpha_k, \beta_k, \gamma_k, u_k, v_k$ are real.
Analysis of LTI Systems in the $z$-Domain

Zero-pole systems represented by linear constant-coefficient difference equations with arbitrary initial conditions.

$$H(z) = \frac{B(z)}{A(z)}$$

Assume that the input signal $x(n)$ has a rational $z$-transform $X(z)$

$$X(z) = \frac{N(z)}{Q(z)}$$

The system is initially relaxed, i.e.,

$$y(-1) = y(-2) = \cdots = y(-N) = 0.$$  

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)}$$

Analysis of LTI Systems in the $z$-Domain

Suppose that the system contains simple poles $p_1, p_2, \ldots, p_N$ and the $z$-transform of the input signal contains poles $q_1, q_2, \ldots, q_L$, where $p_k \neq q_m$ for all $k$ and $m$.

In addition, suppose that there is no pole-zero cancellation.

A partial-fraction expansion of $Y(z)$ yields

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}}$$

Inverse transform of $Y(z)$:

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

natural response  \hspace{1cm} forced response
Transient Response and Steady-State Response

\[ y_{nr}(n) = \sum_{k=1}^{N} A_k(p_k)^n u(n) \]

If \(|p_k| < 1\) for all \(k\), then \(y_{nr}(n)\) decays to zero as \(n\) approaches infinity. The natural response is called the transient response.

\[ y_{fr}(n) = \sum_{k=1}^{L} Q_k(q_k)^n u(n) \]

If the poles fall on the unit circle and consequently, the forced response persists for all \(n > 0\). The forced response is called the steady-state response of the system.

Causality

Causal LTI system: \(h(n) = 0, \ n < 0\).

(The ROC of the z-transform of a causal sequence is the exterior of a circle. )

A LTI system is causal iff the ROC of the system function is the exterior of a circle of radius \(r < \infty\), including the point \(z = \infty\).
Stability

BIBO stable LTI system: $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$$

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}|$$

When evaluated on the unit circle, i.e., $|z| = 1$,

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)| < \infty \Rightarrow \text{The ROC includes the unit circle.}$$

Causality and Stability

A causal and stable LTI system must have a system function converges for $|z| > r$, where $r < 1$.

A causal LTI system is BIBO stable $iff$ all the poles of $H(z)$ are inside the unit circle.

cf. A causal LTI system with a rational transfer function $H(s)$ is stable $iff$ all poles of $H(s)$ are in the left half of the $s$-plane, i.e., the real parts of all poles are negative.
A LTI system is characterized by the system function

\[ H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}} \]

\[ = \frac{1}{1 - 0.5z^{-1}} + \frac{2}{1 - 3z^{-1}} \]

Specify the ROC of \( H(z) \) and determine \( h(n) \) for the following conditions:

(1) The system is stable.

(2) The system is causal.

(3) The system is anticausal.

Solution. The system has poles at \( z = 0.5 \) and \( z = 3 \).

(1) Since the system is stable, its ROC must include the unit circle and hence it is \( 0.5 < |z| < 3 \).

\[ h(n) = (0.5)^n u(n) - 2(3)^n u(-n - 1) \Rightarrow \text{noncausal} \]

(2) Since the system is causal, its ROC is \( |z| > 3 \).

\[ h(n) = (0.5)^n u(n) + 2(3)^n u(n) \Rightarrow \text{unstable} \]

(3) Since the system is anticausal, its ROC is \( |z| < 0.5 \).

\[ h(n) = -(0.5)^n u(-n - 1) - 2(3)^n u(-n - 1) \Rightarrow \text{unstable} \]
Pole-zero cancellations can occur either in the system function itself or in the product of the system function $H(z)$ with the $z$-transform of the input signal $X(z)$. 