Proof. Refer to FIGURE 3. We find that there are twelve pairs of triangles, each pair of which has a common side or a common angle (or two congruent angles one from each triangle of the pair). Applying P1 and P2 on these triangles we have

$$\begin{aligned}
\frac{AM}{IM}\cdot\frac{IN}{CN} &= \frac{\mathcal{A}(AEG)}{\mathcal{A}(IEG)}\cdot\frac{\mathcal{A}(IFH)}{\mathcal{A}(CHF)} && \text{(P1)}\\
&= \frac{\mathcal{A}(IFH)}{\mathcal{A}(IEG)}\cdot\frac{\mathcal{A}(CBD)}{\mathcal{A}(CHF)}\cdot\frac{\mathcal{A}(ABD)}{\mathcal{A}(CBD)}\cdot\frac{\mathcal{A}(AEG)}{\mathcal{A}(ABD)}\\
&= \frac{IF\cdot IH}{IE\cdot IG}\cdot\frac{CD\cdot CB}{CF\cdot CH}\cdot\frac{IA}{IC}\cdot\frac{AE\cdot AG}{AB\cdot AD} && \text{(P1, P2)}\\
&= \frac{\mathcal{A}(FAC)}{\mathcal{A}(EAC)}\cdot\frac{\mathcal{A}(HAC)}{\mathcal{A}(GAC)}\cdot\frac{\mathcal{A}(DAC)}{\mathcal{A}(FAC)}\cdot\frac{\mathcal{A}(BAC)}{\mathcal{A}(HAC)}\cdot\frac{IA}{IC}\cdot\frac{\mathcal{A}(EAC)}{\mathcal{A}(BAC)}\cdot\frac{\mathcal{A}(GAC)}{\mathcal{A}(DAC)}\\
&= \frac{IA}{IC}. && \text{(P1)} \quad \blacksquare
\end{aligned}$$

Thus, if we let $IA = a$, $IC = c$, $IM = m$, $IN = n$, we get

$$\frac{a-m}{m}\cdot\frac{n}{c-n} = \frac{a}{c},$$

which simplifies to

$$\frac{1}{m}-\frac{1}{n}=\frac{1}{a}-\frac{1}{c}.$$

Hence a butterfly inscribed in a quadrilateral satisfies the same relation (1) as a butterfly inscribed in a circle. Equivalently, the conclusion of the theorem indicates that the ratio of the ratios, $(AM/IM)/(CN/IN)$, is the same as the ratio IA/IC, or that the harmonic mean of IC and IM equals the harmonic mean of IA and IN. In either case, if $IC = IA$, we have $IM = IN$ thereby the analog of the usual butterfly theorem for quadrilaterals.

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Row Rank Equals Column Rank

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Dedicated to George Mackiw, a good friend and an excellent mathematical expositor.

The purpose of this note is to present a short (perhaps shortest?) proof that the row rank of a matrix is equal to its column rank. The proof is elementary and accessible to students in a beginning linear algebra course. It requires only the definition of matrix multiplication and the fact that a minimal spanning set is a basis. It differs in approach from proofs given in textbooks as well as from some interesting proofs in MAA journals [**1, 2**]. And, unlike the latter, this proof is valid over any field of scalars.

But first, recall that if the $m \times n$ matrix $A = BC$ is a product of the $m \times r$ matrix B and the $r \times n$ matrix C, then it follows from the definition of matrix multiplication that the ith row of A is a linear combination of the r rows of C with coefficients from the ith row of B, and the jth column of A is a linear combination of the r columns of B with coefficients from the jth column of C. (If you have trouble understanding this or the next paragraph, you should construct several examples of small matrix products, say, a 3×2 times a 2×3 matrix, etc., with small integer as well as symbolic entries.)

On the other hand, if any collection of r row vectors $\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_r$, spans the row space of A, an $r \times n$ matrix C can be formed by taking these vectors as its rows. Then the ith row of A is a linear combination of the rows of C, say $\overline{a}_i = b_{i1}\overline{c}_1 + b_{i2}\overline{c}_2 + \cdots + b_{ir}\overline{c}_r$. This means $A = BC$, where $B = (b_{ij})$ is the $m \times r$ matrix whose ith row, $\overline{b} = (b_{i1}, b_{i2}, \ldots, b_{ir})$, is formed from the coefficients giving the ith row of A as a linear combination of the r rows of C.

Similarly, if any r column vectors span the column space of A, and B is the $m \times r$ matrix formed by these columns, then the $r \times n$ matrix C formed from the appropriate coefficients satisfies $A = BC$. Now the four sentence proof.

THEOREM. *If A is an $m \times n$ matrix, then the row rank of A is equal to the column rank of A.*

Proof. If $A = 0$, then the row and column rank of A are both 0; otherwise, let r be the smallest positive integer such that there is an $m \times r$ matrix B and an $r \times n$ matrix C satisfying $A = BC$. Thus the r rows of C form a minimal spanning set of the row space of A and the r columns of B form a minimal spanning set of the column space of A. Hence, row and column ranks are both r. ∎

Several other properties of the rank of a matrix over a field are also very easy to obtain. The factorizations $A = I_m A = A I_n$ show that $r \leq m$ and $r \leq n$, which proves that the rank of A is less than or equal to the number of rows and the number of columns of A. Since $A = BC$ implies $A^T = C^T B^T$, the transpose clearly has the same rank as the original matrix. Since $A = BC$ and $D = EF$ implies $AD = B(CD) = (AE)F$, the rank of AD must be less than or equal to the rank of A and to the rank of D.

These concepts suggest the following definition [**5**, p. 123]:

DEFINITION. Let R be a commutative ring with identity and let A be an $m \times n$ matrix over R. Then the *spanning rank of* A is 0 if $A = 0$ and otherwise is the smallest positive integer r such that there is an $m \times r$ matrix B and an $r \times n$ matrix C satisfying $A = BC$.

This definition is one way of extending the notion of rank to matrices over commutative rings. Even if the ring has no identity, it can be embedded in a ring with identity so that the definition can be used. Care must be taken in considering rank over commutative rings, because several different extensions of the definitions over a field can give different results over rings, even though they all give the standard concept of rank over a field. Nonetheless, if the above definition is used, matrices over rings *automatically* have row rank equal to column rank, have rank less than or equal to the number of rows and the number of columns, the rank of the transpose is equal to the rank of the matrix, and the rank of a product is less than or equal to the rank of either factor.

Another application of the spanning rank, first used by the author in a problem [**3**] and later a Note [**5**] in the MAGAZINE, is the proof that a matrix over a commutative ring with spanning rank r satisfies a polynomial equation of degree at most $r + 1$. For if $A = BC$ is an $n \times n$ matrix of spanning rank r, then $D = CB$ is an $r \times r$ matrix with characteristic polynomial $f_D(x) = \det(xI - D)$ of degree r and $f_D(D) = 0$ follows from the Cayley-Hamilton Theorem. (The author has shown [**4**] that the Cayley-Hamilton Theorem holds for matrices over commutative rings.) Thus there is a polynomial $m(x)$ of smallest positive degree such that $m(D) = 0$. Then $p(x) = xm(x)$ is a polynomial of degree $\leq r + 1$ such that $p(A) = Am(A) = BCm(BC) = Bm(CB)C = Bm(D)C = 0$.

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A Modern Approach to a Medieval Problem

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The following problem from *Lilavati* [**1**], a mathematical treatise written by Bhaskaracharya, a 12th-century Indian mathematician and astronomer, deserves a modern approach:

> A snake's hole is at the foot of pillar, nine cubits high, and a peacock is perched on its summit. Seeing a snake at the distance of thrice the pillar gliding towards his hole, he pounces obliquely upon him. Say quickly at how many cubits from the snake's hole they meet, both proceeding an equal distance.

Since both proceed an equal distance, it is reasonable to assume that their speeds are equal. Readers are invited to solve this problem before proceeding.

Assuming that the peacock flies along the hypotenuse of a right-angled triangle and knows the Pythagorean Theorem, it will grab the snake at a distance of 12 cubits from the pillar. Practically, however, such a thing does not happen. Why should a peacock know—*a priori*—to fly along the hypotenuse of a right-angled triangle having a base of 12 cubits? A more peacock-like behavior would be to keep an eagle eye on the snake and change its direction at every instant, always aiming toward the snake.

This type of pursuit problem has a history of over five hundred years. However, this particular problem is a bit different from most. Instead of the prey running away from the predator, here prey and the predator are moving toward each other. Even so, the results are startling.

Although the reader may have seen similar problems, I offer a general analysis. We assume that the snake and the peacock move at different, but constant, speeds: the snake in a straight line toward its hole and the peacock along a curve, changing its direction at every instant so as to be flying directly toward the snake.