5. Newton-Raphson Method

- Iterative Solutions for Nonlinear Equations

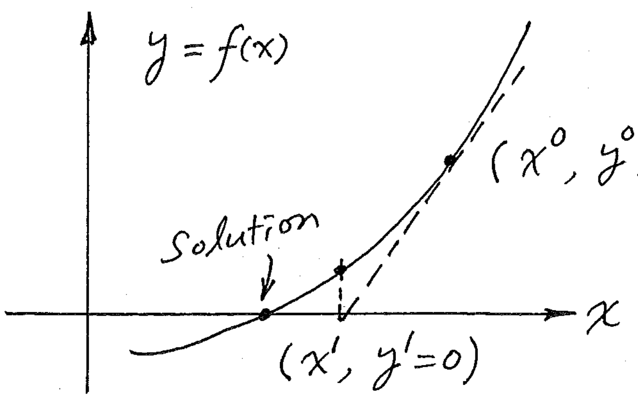
Consider a scalar equation:

$$y = f(x)$$

Taylor series expansion of $f(x)$ about an operating point x_0 yields

$$y = f(x_0) + \left.\frac{df}{dx}\right|_{x_0}(x - x_0) + \frac{d^2f}{dx^2}\frac{(x-x_0)^2}{2!} + \cdots$$

or $$y = f(x + \Delta x) = f(x) + \frac{df}{dx}\Delta x + \cdots$$



$$f(x^1) = f(x^0) + f'(x^0)(x^1 - x^0)$$

$$y^1 = f(x^1) \neq 0$$

In approximation,

$$0 = f(x^0) + f'(x^0)(x^1 - x^0)$$

$$\Rightarrow \quad x^1 = x^0 - \frac{f(x^0)}{f'(x^0)}$$

In iteration,

$$x^k = x^{k-1} - \frac{f(x^{k-1})}{f'(x^{k-1})}$$

Repeating the iteration, the true solution for

$$0 = f(x)$$

can be found as shown in Fig.

For vector equations

$$y_1 = f_1(x_1, x_2, \cdots, x_n)$$
$$\vdots$$
$$y_n = f_n(x_1, x_2, \cdots, x_n)$$

Taylor series expansion about $(x_1^o, \cdots, x_n^o)$ yields

$$y_1 = f_1(x_1^o, \cdots, x_n^o) + \left.\frac{\partial f_1}{\partial x_1}\right|_0 \Delta x_1^o + \cdots + \left.\frac{\partial f_1}{\partial x_n}\right|_0 \Delta x_n^o$$
$$\vdots$$
$$y_n = f_n(x_1^o, \cdots, x_n^o) + \left.\frac{\partial f_n}{\partial x_1}\right|_0 \Delta x_1^o + \cdots + \left.\frac{\partial f_n}{\partial x_n}\right|_0 \Delta x_n^o$$

$\Rightarrow$

$$\begin{bmatrix} y_1 - f_1^o \\ \\ y_n - f_n^o \end{bmatrix} = \underbrace{\begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_0 & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_0 \\ \vdots & & \\ \left.\frac{\partial f_n}{\partial x_1}\right|_0 & \cdots & \left.\frac{\partial f_n}{\partial x_n}\right|_0 \end{bmatrix}}_{\text{Jacobian Matrix}} \begin{bmatrix} \Delta x_1^o \\ \vdots \\ \Delta x_n^o \end{bmatrix}$$

In matrix form,

$$\Delta y = J \, \Delta x$$

which is a linear (matrix) equation that we can solve using methods such as gaus elimination procedure.

Ex.
$$f_1(x_1,x_2) = 2x_1^3 + 3x_1^2 x_2 - x_2^2 - 2 = 0$$
$$f_2(x_1,x_2) = x_1 x_2^2 + 2x_1^2 - 3x_2^2 + 16 = 0$$

$$\frac{\partial f_1}{\partial x_1} = 6x_1^2 + 6x_1 x_2 \qquad \frac{\partial f_1}{\partial x_2} = 3x_1^2 - 2x_2$$

$$\frac{\partial f_2}{\partial x_1} = x_2^2 + 4x_1 \qquad \frac{\partial f_2}{\partial x_2} = 2x_1 x_2 - 6x_2$$

Let $(x_1^0, x_2^0) = (2, 2)$, initial guess

$$f_1^0 = 16 + 24 - 4 - 2 = 34$$
$$f_2^0 = 8 + 8 - 12 + 16 = 20$$
$$\left.\frac{\partial f_1}{\partial x_1}\right|_0 = 24 + 24 = 48 \qquad \left.\frac{\partial f_1}{\partial x_2}\right|_0 = 12 - 4 = 8$$
$$\left.\frac{\partial f_2}{\partial x_1}\right|_0 = 4 + 8 = 12 \qquad \left.\frac{\partial f_2}{\partial x_2}\right|_0 = 8 - 12 = -4$$

$$y_1 - f_1^0 = \left.\frac{\partial f_1}{\partial x_1}\right|_0 \Delta x_1^0 + \left.\frac{\partial f_1}{\partial x_2}\right|_0 \Delta x_2^0$$
$$y_2 - f_2^0 = \left.\frac{\partial f_2}{\partial x_1}\right|_0 \Delta x_1^0 + \left.\frac{\partial f_2}{\partial x_2}\right|_0 \Delta x_2^0$$

$\Rightarrow$
$$0 - 34 = 48\,\Delta x_1^0 + 8\,\Delta x_2^0 \quad \cdots ①$$
$$0 - 20 = 12\,\Delta x_1^0 - 4\,\Delta x_2^0 \quad \cdots ②$$

$$① + 2 \times ②: \quad -74 = 72\,\Delta x_1^0 \quad \rightarrow \quad \Delta x_1^0 = -1.03$$
$$①: \quad \Delta x_2^0 = 1.9$$

$$\Rightarrow \quad x_1^1 = x_1^0 + \Delta x_1^0 = 2 - 1.03 = 0.97$$
$$x_2^1 = x_2^0 + \Delta x_2^0 = 2 + 1.9 = 3.97$$

2nd iteration:

$$4.3 = 28.4\,\Delta x_1^1 - 5.0\,\Delta x_2^1$$
$$13.0 = 19.1\,\Delta x_1^1 - 15.8\,\Delta x_2^1$$

$$\Rightarrow \quad \Delta x_1^1 = 0.0085, \quad \Delta x_2^1 = -0.812$$

$$\Rightarrow \quad x_1^2 = 0.97 + 0.0085 = 0.9785$$
$$x_2^2 = 3.97 - 0.812 = 3.09$$

In few iterations the solution converges to the true solution, $(1, 3)$.

6. Power-Flow Solution by Newton-Raphson

Power equations:

$$P_k = V_k \sum_{n=1}^{N} Y_{kn} V_n \cos(\delta_k - \delta_n - \theta_{kn})$$

$$Q_k = V_k \sum_{n=1}^{N} Y_{kn} V_n \sin(\delta_k - \delta_n - \theta_{kn})$$

$$\Rightarrow \quad y = f(x), \quad \text{nonlinear}: 2N \times 1$$

where

$$y = \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} P_1 \\ \vdots \\ P_N \\ \hdashline Q_1 \\ \vdots \\ Q_N \end{bmatrix} \quad x = \begin{bmatrix} \delta \\ V \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_N \\ \hdashline V_1 \\ \vdots \\ V_N \end{bmatrix} \quad f(x) = \begin{bmatrix} P(x) \\ Q(x) \end{bmatrix} = \begin{bmatrix} P_1(x) \\ \vdots \\ P_N(x) \\ \hdashline Q_1(x) \\ \vdots \\ Q_N(x) \end{bmatrix}$$

$\Rightarrow$

$$y_1 = f_1(x_1, x_2, \cdots, x_n)$$
$$\vdots$$
$$y_n = f_n(x_1, x_2, \cdots, x_n)$$

Taylor series expansion around $(x_1^o, \cdots, x_n^o)$:

$$y_1 = \underbrace{f_1(x_1^o, \cdots, x_n^o)}_{y_1^o} + \left.\frac{\partial f_1}{\partial x_1}\right|_0 \Delta x_1^o + \cdots + \left.\frac{\partial f_1}{\partial x_n}\right|_0 \Delta x_n^o$$
$$\vdots$$
$$y_n = \underbrace{f_n(x_1^o, \cdots, x_n^o)}_{y_n^o} + \left.\frac{\partial f_n}{\partial x_1}\right|_0 \Delta x_1^o + \cdots + \left.\frac{\partial f_n}{\partial x_n}\right|_0 \Delta x_n^o$$

where

$$\Delta y_1^o \triangleq y_1 - y_1^o, \quad \Delta x_1^o \triangleq x_1 - x_1^o, \quad \left.\frac{\partial f_1}{\partial x_1}\right|_0 = \frac{\partial f_1}{\partial x_1}(x_1^o, \cdots, x_n^o)$$
$$\Delta y_n^o \triangleq y_n - y_n^o, \quad \Delta x_n^o \triangleq x_n - x_n^o, \quad \left.\frac{\partial f_n}{\partial x_n}\right|_0 = \frac{\partial f_n}{\partial x_n}(x_1^o, \cdots, x_n^o)$$

$\Rightarrow$

$$\Delta y_1^o = \left.\frac{\partial f_1}{\partial x_1}\right|_0 \Delta x_1^o + \cdots + \left.\frac{\partial f_1}{\partial x_n}\right|_0 \Delta x_n^o$$
$$\vdots$$
$$\Delta y_n^o = \left.\frac{\partial f_n}{\partial x_1}\right|_0 \Delta x_1^o + \cdots + \left.\frac{\partial f_n}{\partial x_n}\right|_0 \Delta x_n^o$$

In matrix form,

$$\begin{bmatrix} \Delta y_1^i \\ \vdots \\ \Delta y_n^i \end{bmatrix} = \underbrace{\begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_i & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_i \\ \vdots & & \\ \left.\frac{\partial f_n}{\partial x_1}\right|_i & \cdots & \left.\frac{\partial f_n}{\partial x_n}\right|_i \end{bmatrix}}_{J,\ \text{Jacobian matrix}} \begin{bmatrix} \Delta x_1^i \\ \vdots \\ \Delta x_n^i \end{bmatrix}$$

For Power Flow Problem:

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} \frac{\partial P}{\partial \delta} & \frac{\partial P}{\partial V} \\ \frac{\partial Q}{\partial \delta} & \frac{\partial Q}{\partial V} \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta V \end{bmatrix}$$

Jacobian Matrix:

$$J = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}, \qquad J_1 = \begin{bmatrix} \frac{\partial P_1}{\partial \delta_1} & \cdots & \frac{\partial P_1}{\partial \delta_N} \\ \vdots & & \vdots \\ \frac{\partial P_N}{\partial \delta_1} & \cdots & \frac{\partial P_N}{\partial \delta_N} \end{bmatrix}$$

J_1: For $n \neq k$, off-diagonals

$$\frac{\partial P_k}{\partial \delta_n} = V_k Y_{kn} V_n \sin(\delta_k - \delta_n - \theta_{kn})$$

For $n = k$, diagonals

$$\frac{\partial P_k}{\partial \delta_k} = -V_k \sum_{\substack{n=1 \\ n \neq k}}^{N} Y_{kn} V_n \sin(\delta_k - \delta_n - \theta_{kn})$$

J_2: For $n \neq k$,

$$\frac{\partial P_k}{\partial V_n} = V_k Y_{kn} \cos(\delta_k - \delta_n - \theta_{kn})$$

For $n = k$,

$$\frac{\partial P_k}{\partial V_n} = \sum_{\substack{n=1 \\ n \neq k}}^{N} Y_{kn} V_n \cos(\delta_k - \delta_n - \theta_{kn}) + 2 V_k Y_{kk} \cos(\delta_k - \delta_k - \theta_{kk})$$

$$= V_k Y_{kk} \cos\theta_{kk} + \sum_{n=1}^{N} Y_{kn} V_n \cos(\delta_k - \delta_n - \theta_{kn})$$

J_3: For $n \neq k$,

$$\frac{\partial Q_k}{\partial \delta_n} = - V_k Y_{kn} V_n \cos(\delta_k - \delta_n - \theta_{kn})$$

For $n = k$,

$$\frac{\partial Q_k}{\partial \delta_k} = V_k \sum_{n=1}^{N} Y_{kn} V_n \cos(\delta_k - \delta_n - \theta_{kn})$$

J_4: For $n \neq k$,

$$\frac{\partial Q_k}{\partial V_n} = V_k Y_{kn} \sin(\delta_k - \delta_n - \theta_{kn})$$

For $n = k$,

$$\frac{\partial Q_k}{\partial V_k} = \sum_{\substack{n=1 \\ n \neq k}}^{N} Y_{kn} V_n \sin(\delta_k - \delta_n - \theta_{kn}) + 2 V_k Y_{kk} \sin(\delta_k - \delta_k - \theta_{kk})$$

$$= V_k Y_{kk} \sin\theta_{kk} + \sum_{n=1}^{N} Y_{kn} V_n \sin(\delta_k - \delta_n - \theta_{kn})$$

<u>Solution Procedure</u>: Given $x^i = \begin{bmatrix} \delta^i \\ V^i \end{bmatrix}$ at the i-th iteration.

Step 1: Compute mismatch,

$$\Delta y^i = \begin{bmatrix} \Delta P^i \\ \Delta Q^i \end{bmatrix} = \begin{bmatrix} P - P(x^i) \\ \underbrace{Q}_{\text{specified}} - \underbrace{Q(x^i)}_{\text{calculated}} \end{bmatrix}$$

Step 2: Compute Jacobian matrix J^i

– updated at x^i

Step 3: Solve

$$J^i \Delta x^i = \Delta y^i$$

for $\Delta x^i = \begin{bmatrix} \Delta \delta^i \\ \Delta V^i \end{bmatrix}$.

Step 4: Update

$$x^{i+1} = x^i + \Delta x^i$$

Repeat iterations until convergence is obtained, or the # of iterations exceeds a specified maximum.

Convergence criteria:

$$|\Delta y^i|_{max} \le \varepsilon, \quad \text{power mismatch}$$

<u>Voltage-Controlled Bus:</u>

V_k is specified.

$\Rightarrow$ Equation for Q_k is not needed.

$\Rightarrow$ Omit V_k & Q_k in x & y vectors.

Omit columns & rows corresponding to $\frac{\partial Q_k}{\partial V_k}$

Compute Q_k

Check $Q_{Gk} = Q_k + Q_{Lk}$ for its limits.

If it exceeds a limit, set it to the limit and

$$Q_k = Q_{Gk\,\text{max or min}} - Q_{Lk}$$

Change the bus to a <u>load</u> bus (P-Q)
Add corresponding rows & columns

<u>Swing Bus:</u>

V_1 & δ_1 : given

$\Rightarrow$ Omit these variables and P_1 & Q_1

$\Rightarrow$ $k = 2, \cdots, N$

Compute P_1 & Q_1 after convergence.