1. Basic Principles

Phasors: In AC circuits, voltage and current are sinusoidal, or alternating, from which the word AC (alternating current) came about.

The instantaneous value of a voltage is

$$v(t) = V_{max} \cos(\omega t + \delta)$$

where $\omega = 2\pi f$, f frequency in Hz

V_{max} : peak value

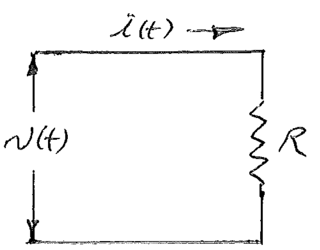
δ : phase angle

Since the frequency is normally fixed (to 50 or 60 Hz), the voltage is uniquely determined by V_{max} and δ (as a cosine or sine function, of course).

In DC (direct current) circuits, however, voltage and current are constant. Then one might ask when AC and DC become "equivalent". We can measure the equivalency by observing the work done, or energy dissipated in a registor when the two types of voltages are applied.

The power loss in a registor (heat loss) is

$$p(t) = R\, i(t)^2 = \frac{1}{R} v(t)^2$$



The average energy loss for a period of one cycle is then

$$\frac{1}{T}\int_0^T p(t)\, dt = \frac{1}{T}\int_0^T v(t)^2\, dt$$

for one ohm registor, where T is the period.

Equating this with the energy loss for the DC voltage,

$$\frac{1}{T} V^2 T = \frac{1}{T}\int_0^T v(t)^2\, dt$$

where V is the DC voltage. Therefore,

$$V = \sqrt{\frac{1}{T}\int_0^T v(t)^2\, dt}$$

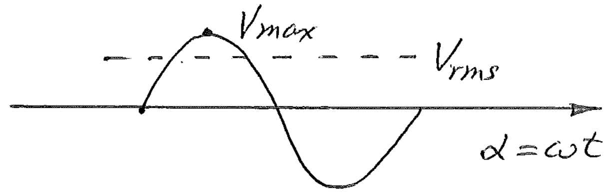
which gives the definition of root-mean-squre (rms) or effective value for AC voltage or current.

Thus, for the voltage

$$v(t) = V_{max}\cos(\omega t + \delta)$$

the rms or effective value is (after the algebra)

$$V_{rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v(\alpha)^2\, d\alpha} = \frac{V_{max}}{\sqrt{2}} \simeq 0.707\, V_{max}$$



The rms value only account for the magnitude of the voltage. In order to account for the phase angle, we introduce the complex variable. Specifically, recall the <u>Euler's identity</u>: $e^{j\phi} = \cos\phi + j\sin\phi$.

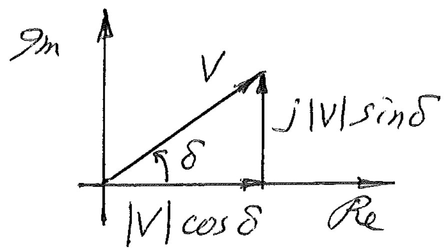
This suggests that the cosine term is the "real part" of the complex variable $e^{j\phi}$.

Recall our voltage is a cosine function and it can be represented as a real part of a complex variable (with an appropriate scale):

$$\begin{aligned} v(t) &= Re\left[V_{max}\, e^{j(\omega t + \delta)}\right] \\ &= Re\left[V_{max}\, e^{j\delta}\, e^{j\omega t}\right] \\ &= Re\left[\sqrt{2}\left(V_{rms} e^{j\delta}\right) e^{j\omega t}\right] \end{aligned}$$

Now, we define the "phasor" as

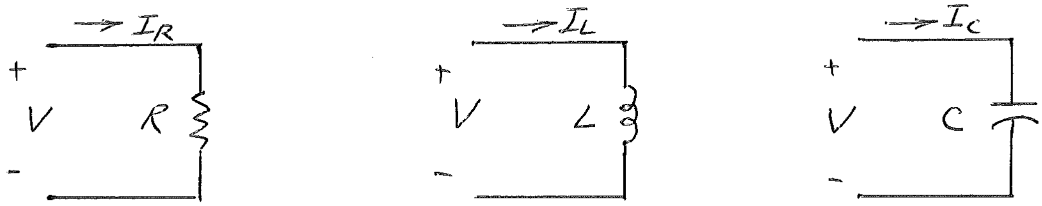
$$V = V_{rms}\, e^{j\delta} = \underbrace{V_{rms}\angle\delta}_{\text{polar form}} = \underbrace{V_{rms}\cos\delta + jV_{rms}\sin\delta}_{\text{rectangular form}}$$



Phasor is a vector in the complex plane, which has magnitude V_{rms} and phase angle δ.

For convenience, we drop the subscript "rms", and use V as rms value or phasor, depending on the context.

<u>Impedances</u>: passive elements

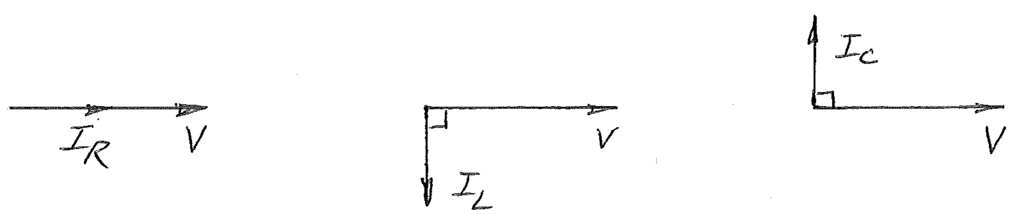


Impedances of resistance, inductance, and capacitance are:

$$Z_R = R, \quad Z_L = j\omega L = jX_L, \quad Z_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C} = -jX_C$$

Currents are, by Ohm's law: $I = \frac{V}{Z}$

$$I_R = \frac{V}{Z_R} = \frac{V}{R}, \quad I_L = \frac{V}{jX_L} = -j\frac{V}{X_L}, \quad I_C = \frac{V}{-jX_C} = j\frac{V}{X_C}$$



Current is: in-phase, lagging by 90°, leading by 90°.

In general, we have a combination of R, L, C

$$Z = |Z|\angle\theta = R + jX$$

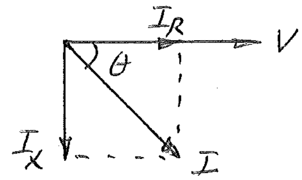
$\theta > 0, \quad X > 0$: inductive reactance

$\theta < 0, \quad X < 0$: capacitive reactance

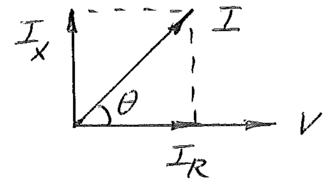
$$I = \frac{V}{Z} = \frac{V}{|Z|}\angle -\theta$$

$\theta > 0$, inductive : lagging current

$\theta < 0$, capacitive : leading current



lagging current
inductive



leading current
capacitive

<u>Powers</u>: Given instantaneous voltage and current:

$$v(t) = V_{max} \cos(\omega t + \delta) \quad [V]$$
$$i(t) = I_{max} \cos(\omega t + \beta) \quad [A]$$

The "instantaneous" power is defined as the product:

$$p(t) = v(t)\, i(t) = V_{max} I_{max} \cos(\underbrace{\omega t + \delta}_{A}) \cos(\underbrace{\omega t + \beta}_{B})$$

Using the identity

$$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$$
$$= \cos(\delta - \beta) + \cos[\underbrace{2(\omega t + \delta)}_{C} - (\underbrace{\delta - \beta}_{D})]$$

Again,

$$\cos(C - D) = \cos C \cos D + \sin C \sin D$$
$$= \cos[2(\omega t + \delta)] \cos(\delta - \beta)$$
$$+ \sin[2(\omega t + \delta)] \sin(\delta - \beta)$$

Thus,

$$p(t) = \frac{V_{max} I_{max}}{2} \cos(\delta - \beta)[1 + \cos[2(\omega t + \delta)]]$$
$$+ \frac{V_{max} I_{max}}{2} \sin(\delta - \beta) \sin[2(\omega t + \delta)]$$
$$= V \underbrace{I \cos(\delta - \beta)} \{1 + \cos[2(\omega t + \delta)]\}$$
$$+ V \underbrace{I \sin(\delta - \beta)} \sin[2(\omega t + \delta)]$$
$$= \underbrace{V I_R \{1 + \cos[2(\omega t + \delta)]\}}_{p_R} + \underbrace{V I_X \sin[2(\omega t + \delta)]}_{p_X}$$

p_R: instantaneous real power, always ≥ 0

p_X: instantaneous reactive power, alternates: +, 0, −

Real Power: Average power, active power

$$P = VI_R = VI \underbrace{\cos(\delta - \beta)}_{\text{power factor}} \qquad [\text{Watts}]$$

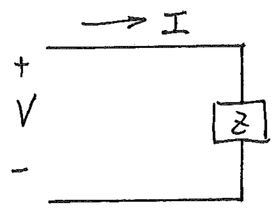
$\theta = \delta - \beta$: power factor angle

Reactive Power: Defined as the amplitude of instantaneous reactive power p_X :

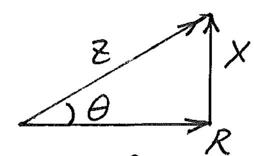
$$Q = VI_X = VI \sin(\delta - \beta) \qquad [\text{Vars}]$$

Var : Volt-Ampere-Reactive

Impedance & Power:

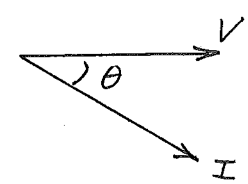


$Z = R + jX = |Z| \angle \theta$



$V = IZ$, $\quad V = |V| \angle 0^\circ$, reference, $\delta = 0^\circ$

$$\Rightarrow I = \frac{V}{Z} = \frac{|V| \angle 0^\circ}{|Z| \angle \theta} = \frac{V}{|Z|} \angle -\theta$$



$$P = |V||I| \cos\theta = (|I||Z|)\,|I| \cos\theta$$
$$= |I|^2 \underbrace{|Z| \cos\theta} = |I|^2 R$$

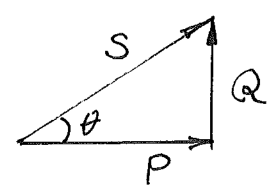
$$Q = |V||I| \sin\theta = (|I|\,|Z|)\,(I| \sin\theta$$
$$= |I|^2 |Z| \sin\theta = |I|^2 X$$

Note: $\sqrt{P^2 + Q^2} = \sqrt{(|V||I|)^2 \cos^2\theta + (|V||I|)^2 \sin^2\theta}$

$= |V|\,|I| \triangleq |S|$, "apparent power"

Power factor & Power factor angle:

$$\frac{Q}{P} = \frac{|V||I| \sin\theta}{|V||I| \cos\theta} = \tan\theta$$



$$pf = \cos\theta = \frac{P}{\sqrt{P^2 + Q^2}} = \frac{P}{|S|} = \frac{P}{|V||I|}$$

Note: The impedance angle θ is preserved in the power triangle!

Complex Power: Let $V = |V| \angle \delta$

$I = |I| \angle \beta$

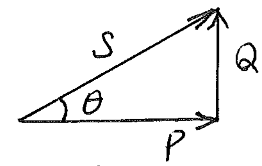
Define

$$S = V I^* = (|V| \angle \delta)(|I| \angle -\beta) = |V||I| \angle \delta - \beta = |S| \angle \delta - \beta$$

$$= \underbrace{|V||I| \cos(\delta - \beta)}_{P} + j \underbrace{|V||I| \sin(\delta - \beta)}_{Q}$$

$|S| = |V||I|$: apparent power [VA]

$\theta = \delta - \beta$: power factor angle



$\delta > \beta$: $\theta > 0$, $Q > 0$, I lagging, inductive load

$\delta < \beta$: $\theta < 0$, $Q < 0$, I leading, capacitive load

Impedance & Complex power:

$$S = V I^* = (I Z)(I^*) = |I|^2 Z$$

$$= |I|^2 |Z| \angle \theta = |S| \angle \theta$$

$$= |I|^2 (R + jX) = \underbrace{|I|^2 R}_{P} + j \underbrace{|I|^2 X}_{Q}$$

Alternatively,

$$S = V I^* = V \left(\frac{V}{Z}\right)^* = \frac{|V|^2}{Z^*} = |V|^2 Y^*$$

$$= \frac{|V|^2}{|Z|} \angle \theta = |S| \angle \theta$$

$$= |V|^2 |Y| \angle \theta = |V|^2 (G + jB)$$

$$= \underbrace{|V|^2 G}_{P} + j \underbrace{|V|^2 B}_{Q}$$

Here, $Y = \frac{1}{Z}$, admittance

$= G - jB$

Complex Power Balance:

The total complex power supplied by the source is equal to the sum of the complex power delivered to the load.