A study of piano string power and duration as a function of bridge/soundboard impedance

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1 Impedance in String Instruments

The dynamics of most linear, elastic media which support the propagation of acoustic waves may be accurately modeled by a partial differential equation known as the "wave equation." Solutions to the wave equation, assuming appropriate boundary conditions, describe the spatiotemporal value of the displacement and velocity of the propagation medium, which for musical instruments is often a column of air or a string under tension.

In an air column, waves propagate axially, and are reflected by a high-impedance boundary condition at the open end of the column. At certain frequencies, forward and backward-propagating waves superimpose to create the appearance of a wave "standing still" – a so-called axial standing wave – which may achieve a very large magnitude.

In a string, waves also propagate axially, but the medium (the string itself) displaces perpendicular to the axis of propagation, creating a transverse standing wave. Taken alone, whether a standing wave is transverse or axial is of little importance; however, the boundary conditions seen by the wave medium may be vastly different for a string than for an air column. In fact, one might reasonably argue that the dynamic characteristics of a string’s "free boundary" (the end of the string that is permitted to vibrate) are primarily responsible for making string instruments sound, well, different than wind and percussion instruments!

Of course, most acoustic string instruments use a stiff piece of wood to bridge the free boundary to a sounding board that will transduce the string’s standing wave into an acoustic pressure wave. Thus, to perhaps state the obvious, the central difference between wind and string instruments is that wind instruments generate pressure waves that are directly perceived by our ears, whereas string instruments must employ an intermediary – the sound board – to transform their vibrations into perceptible pressure waves. (Note that percussive instruments like bells and xylophones fall into the "wind" category by this logic.)

So here is the main point: the intermediary necessarily has dynamic characteristics of its own. It will emphasize or attenuate certain frequencies presented to it by the string, and it will most certainly have its own vibrational modes. Sound boards are, in and of themselves, interesting subjects of study! But in the final analysis, a sound board has only two jobs: it must (1) efficiently transduce axial vibrations at its bridge into acoustic pressure waves, and (2) maintain a high enough bridge impedance to support standing waves for a sufficiently long duration. These two objectives are inherently at odds with one another. On the one hand, a sound board of maximum efficiency will drain the string’s vibrational energy in a very short, high intensity burst – much like a drum head. On the other hand, a sound board with too-high bridge impedance will permit long sustains, but with very low power.

What makes the question of optimal bridge/soundboard impedance so complex is that impedance is a frequency-dependent quantity. Any sound board, when presented with an ultra high frequency (ultrasonic, for example) will appear to be an immovable rigid mass. Thus, vibrational energy in a hypothetical ultra-high-frequency string will theoretically remain trapped by an "infinite impedance" bridge. Similarly, a hypothetical super-low-frequency string will see a soundboard far too stiff to comply with such low frequencies and its energy too will remain confined by an infinite impedance bridge. Somewhere in the middle, the combination of board stiffness and mass density is "just right," and the board is compliant enough to accept vibrations and convert them into pressure waves.

While the mass density and stiffness of the board determine the frequency dependence of its impedance, there must also be a certain amount of real damping present. Masses and springs store energy, but do not dissipate it. For a board to execute its primary function, it must dissipate vibrational energy into the air, and in this sense it is a damper. (Other forms of damping may be present, including friction effects in the board and string, and hysteresis.) Thus, in sum total, the sound board may be viewed as a large, interconnected web of masses, springs and dampers – and this is, in point of fact, exactly how vibrating membranes are
numerically approximated in computer code. For the purposes of this investigation, we want to simplify the situation by assuming only one effective mass, stiffness and damping coefficient.

Figure 1: A string of length L. Its transverse displacement is given by $u(t,x)$, as both a function of location $x$ and time $t$, where $x$ varies between 0 and L. One end is fixed, and the other is attached to a mass-spring-damper system with mass $m$, damping constant $b$, and stiffness $k$.

The situation is pictured in figure 1. In piano terminology, the left end of the string starts at the agraffe or capo bar (either of which is assumed to be totally immovable), and the right end terminates at the bridge pin.

It may be reasonably asked what effect string downbearing has. (Downbearing is present in all sting instruments for the purpose of holding the string against the bridge.) Downbearing is a static force that will compress the spring in the model of figure 1, and in reality the very state of being under constant compression will alter the spring’s stiffness because most materials, including wood, exhibit nonlinear elasticity under large strain. However, once a relationship between downbearing and effective board stiffness is known, it is safe to assume that the board responds linearly to the minute displacements induced by the string.

Now we may precisely define what is meant by the "impedance" of the bridge/soundboard system. Impedance is the ratio of the shear force applied to the bridge by the string to the resulting velocity attained by the bridge. As mentioned earlier, impedance is frequency-dependent, so we imagine that we have a sinusoidal force pushing and pulling on the bridge, and we are measuring a sinusoidal resultant bridge velocity. Simply computing the ratio of their peak values gives the magnitude of the impedance. However the magnitude of the impedance is only half the story. Almost always, a sinusoidal applied force will produce a sinusoidal resulting velocity that is out of phase with the force. The relative phase of the impedance is important because it indicates whether the bridge/board system will dissipate energy or not; impedance with zero phase (e.g. the force and velocity vibrate in phase) is 100% dissipative, or "real," while impedance with ±90 degree phase is non-dissipative, or "reactive." (Everything in between is a little of both, and the impedance is said to be "complex." ) This explains why the characteristic impedance of a sound board is so important: it dictates not only the power and sustain of a given note but also its frequency spectrum (or tonal color) just as much as the interplay between string and hammer, or the scaling and sizing of the strings. String fundamentals and partials at frequencies the correspond to largely real impedance will sound with greater initial power but dissipate quickly, whereas frequencies corresponding to highly reactive impedance cannot dissipate through the board as easily.

2 Mathematics

The following work assumes the use of IP units (length in inches, mass in pounds). IP and FP suffer a slight inconsistency, in that units of mass must always equal force divided by acceleration. Since the "pound" is usually accepted as a unit of force, then mass must actually have units of $lb \cdot s^2/in$. This combination of units has never been named, but author Robert Norton makes a good (and humorous) case for the name "blob", 

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Abbreviated bl. It is reminiscent of lb for pound, but if mass is specified in lb, one must always divide by 386 (earth gravitational acceleration in \(in/s^2\)) to get bl before doing any calculations. Long live metric!

Assuming a steel string of length \(L\) that has negligible stiffness, the one-dimensional wave equation is given by

\[
p\ddot{u}(t, x) + Tu''(t, x) = 0, \quad x \in [0, L], \quad t \geq 0,
\]

where \(p := \frac{\partial^2 u}{\partial t^2}\), and \(u'' := \frac{\partial^2 u}{\partial x^2}\). The boundary conditions for (1) are

\[
u(t, 0) = 0, \quad m\ddot{u}(t, L) + \frac{\partial u(t, L)}{\partial x} = 0.
\]

A full derivation of these equations appears in the appendix, but referring back to figure 1, we see that equation (3) balances the string boundary shear force with the internal forces in the "sound board." Equation (2) simply pins down the left side of the string. In this work, we assume initial conditions at rest, so that \(\ddot{u}(0, x) = 0\), and set \(u(0, x)\) equal to an arbitrary function describing the initial string shape with \(u(0, 0) = u(0, L) = 0\).

Some of the parameters above are known if a particular note is selected, say A-440. \(T\) is the string tension, as

\[
\begin{align*}
T &= 490 \Omega^2 \mu \rho, \\
\mu &= 490 \Omega^2 \mu \rho,
\end{align*}
\]

where \(\Omega\) is the string tension, \(\mu\) is the volumetric mass density of steel, and \(d\), the string diameter, as \(\rho = 0.25\pi\mu\). Typically, \(d = 0.039\) in and \(\mu = 490 \Omega^2 \mu \rho^3\) (which is \(7.34 \times 10^{-4}\) bl/in^3). Even though the right-hand boundary of the string is not pinned in this model, for the moment we can assume that it is and use the formula

\[
f_1 = \sqrt{\frac{T}{\rho}}
\]

to calculate the standing wave fundamental frequency. With a 1.55% decrease in \(\rho\), we find \(f_1 = 440\) Hz. (The slight decrease may be attributed to the elongation of the string as it is tensed; this effectively lowers its linear mass density.)

That leaves unknown parameters \(m\) (soundboard effective mass), \(b\) (effective damping) and \(k\) (effective stiffness). Again we stress that these are quantities that must be measured under dynamic (oscillatory) conditions – \(m\) is not simply the static mass of the soundboard and bridge! We leave for further investigation the discovery of suitable numbers here, but take as our charge now to investigate the behavior of the string/bridge model for various combinations of the \(m\), \(b\) and \(k\).

We now return to the question of impedance. The right-hand boundary equation (3) holds the two variables of interest, applied string shear force \(Tu'(t, L)\) and bridge/bridge velocity \(\ddot{u}(t, L)\). Engineers almost always transform these types of expressions from the time domain into the frequency domain in order to express the impedance as a complex number (with a magnitude and phase) that is a function of frequency. This gives the bridge impedance as

\[
\Omega(f) = \left[ms + b + \frac{k}{s^j}ight]_{s=2\pi jf}
\]

where \(j = \sqrt{-1}\). Two things are immediately evident upon calculating the magnitude of \(\Omega\). First, it is monotonically increasing both as \(f \to 0\) from the right and as \(f \to \infty\) from the left,

\[
|\Omega| = \sqrt{\left(2\pi m \frac{k}{2\pi f} \right)^2 + b^2}.
\]

This is by now intuitive: as \(f\) trends toward ultrasonic frequencies, the impedance becomes large and is "mass dominated," \(|\Omega| \simeq 2\pi mf\). As \(f\) decreases toward super low frequencies, the impedance becomes large again but is now "stiffness dominated," \(|\Omega| \simeq k/2\pi f\). Second, there is a frequency \(f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}\) at which mass and stiffness effects cancel each other, and the only contributor to the impedance is damping factor \(b\). At this point, \(|\Omega|\) has achieved a minimum, and vibrational energy at this frequency will be dissipated at the fastest possible rate.

In view of the preceding discussion, even the simple model presented here now begins to yield insight into the interaction between note power, duration and frequency content. String energy at frequencies near a minimum of the board impedance function \(\Omega\) will dissipate quickly; those far from a minimum may ring longer but with less power. It is important to remember that only the damper in the model is able to actually transduce vibrational string energy into audible pressure waves – the mass and spring merely store energy. Thus, at frequencies where the impedance is mass or stiffness dominated, energy will only slowly find its way into the damper, and apparent acoustic power (the rate at which energy is transduced) will be small.
3 Simulations

The string and soundboard PDE and boundary conditions are approximated by a system of ordinary differential equations (ODEs) using the finite element method (FEM). The hammer strike point is located at $x = L/8$, so no more than 8 elements, or 9 nodes including the boundary at $x = 0$, are necessary to approximate the transient response and frequency characteristics of the system. (For illustrative purposes, however, a plucked string was simulated with 256 nodes under conditions of zero boundary impedance, infinite boundary impedance, and optimal boundary impedance. The resulting string dynamics were captured in three videos.) The development of the ODE system is given in the appendix.

Simulations fall into one of two categories: (1) transient responses with real termination impedance, i.e. no mass or stiffness effects, and (2) transient responses with complex termination impedance. Three files convey the results of each simulation. The first two are graphics in JPG and EPS format. An example is shown in figure 2. The third is a WAV sound file, as if the damper in the soundboard model were a lossless transducer converting damping force $b\dot{u}(t, L)$ to acoustic pressure waves. Several "experiments" were undertaken, choosing various values of $m$, $b$, and $k$. The resulting graphic and WAV files are therefore named "mxxxbyyykzzz" where xxx, yyy, and zzz are numbers. These files are all available on the web site, with some comments indicating interesting features of each.

![Bridge/Soundboard Acoustic Impedance](image1)

![Transient Response Normalized Power Spectrum](image2)

![Bridge/Sb Damping Force](image3)

![Dissipated Energy](image4)

Figure 2: Impulse response with a high-stiffness, low-mass soundboard. Note the fundamental (440Hz) and its first six partials in the power spectrum. The impedance minimum (at about the 5th partial) will allow the board to dissipate high partials much faster than lower partials.
Now some comments on what appears in each graphic.

1. Bridge/Soundboard Acoustic Impedance. This is simply a log-log plot of $|\Omega|$ vs frequency. In the various trials, the minimum impedance was intentionally located at different points in the frequency spectrum. Note how very small values of $b$ create a "notch" in the plot – when this happens, the "soundboard" is said to be underdamped and will sympathetically resonate at this frequency. Obviously, all soundboards have some degree of resonance, but they should always be well damped if they are to serve as efficient transducers.

2. Transient Response Normalized Power Spectrum. In each trial, the string is virtually plucked. A Fast Fourier Transform is applied to the resulting transient response $b\dot{u}(t,L)$ and the frequency spectrum then adjusted so the highest peak starts at 0 db. The peaks in each spectral plot indicate predominant frequencies, and in figure 2 one can clearly see the fundamental at 440Hz and the first 6 partials. No more partials appear because the string is plucked at $L/8$, the location of a vibrational node for the 7th partial and above. Partials appear within 10db of each other when no impedance is present. However, in figure 2, one can also see that high partials resonate almost -30db below the fundamental. This is because the impedance function assumes its minimum around 3000Hz, quickly draining energy from high partials while leaving the fundamental and lower partials to vibrate for a longer period.

3. Bridge/Sb damping force. This is a time plot of damping force $b\dot{u}(t,L)$, the force appearing on the "transducer". It represents the waveform for what would be audibly heard. In figure 2 note that high partials start with a relatively large amplitude but decay quickly. Lower frequencies are left, sounding with much lower power but much longer sustain. The waveforms appearing in the lower left plot were recorded in WAV files, but scaled so that all WAV recordings start at the same magnitude (to ensure the recordings are not too soft to hear or so loud that they clip.) WAV recordings give the listener a sense of note duration and frequency content, but to compare initial note power, one must look at the plots.

4. Dissipated energy. The impulse applied to the string imparts an energy of approximately 0.47 in-lbs. This plot shows how much energy has dissipated through the damper as a function of time. This is the plot that reveals power output. The system’s initial output (or transduced) power $P_{initial}$ is the slope of the energy dissipation curve near $t = 0$. Naturally, peak power occurs at $t = 0$. But in many cases with complex impedance, certain frequencies die out quickly (with high power) while others are left ringing at lower power, so it is interesting to compare the slopes of the energy curve at various points in time. The final output power $P_{final}$ occurs at $t = 2$. Both are calculated and displayed.

4 Appendix

4.1 Derivation of dynamical equations

Let real-valued function $u \in [0, \infty) \times C^2[0,L]$ and $t_0, t_f > 0$ be arbitrary real numbers. The kinetic and potential energies of the system from figure 1 are

$$KE = \frac{1}{2} \int_0^L \rho \dot{u}^2 dx + \frac{1}{2} m \dot{u}(t,L)^2,$$

$$PE = \frac{1}{2} \int_0^L T u^2 dx + \frac{1}{2} k u(t,L)^2,$$

where $u$ without arguments implies $u(t,x)$. The virtual work done by the system on the boundary damper is

$$\delta W = \delta u(t,L)b\dot{u}(t,L).$$

Thus, setting the Hamilton integral invariant yields

$$0 = \delta \int_{t_0}^{t_f} \frac{1}{2} \left( \int_0^L \rho \ddot{u}^2 - T u'^2 dx \right) + \frac{1}{2} m \ddot{u}(t,L)^2 - k u(t,L)^2 dt$$

$$- \int_{t_0}^{t_f} \delta u(t,L)b\dot{u}(t,L)dt,$$

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with \( \delta u(t, 0) = 0, \delta u(t_0, x) = 0 \) and \( \delta u(t_f, x) = 0 \) but all other variations arbitrary. Applying the variational operator, integrating by parts and collecting terms gives

\[
0 = \int_{t_0}^{t_f} \left( \int_0^L (\rho \ddot{u} - Tu'' \delta u) \, dx \right) + m \dot{u}(t, L) \delta \dot{u}(t, L) - (ku(t, L) + b_\dot{u}(t, L)) \delta u(t, L) \, dt
\]

\[
= \int_{t_0}^{t_f} \left( \int_0^L (\rho \ddot{u} - Tu'' \delta u) \, dx \right) - (m \dot{u}(t, L) + ku(t, L) + b_\dot{u}(t, L)) \delta u(t, L) \, dt
\]

\[
+ \int_0^L \rho \ddot{u} \big|_{t = t_0}^{t = t_f} \, dx - \int_{t_0}^{t_f} Tu' \delta u \big|_{x = L} \, dt + m \dot{u}(t, L) \delta u(t, L) \big|_{t = t_0}^{t = t_f}
\]

\[
= \int_{t_0}^{t_f} \left( \int_0^L (\rho \ddot{u} + Tu' \delta u) \, dx \right)
\]

\[
- (m \ddot{u}(t, L) + ku(t, L) + b_\dot{u}(t, L) + Tu'(t, L)) \delta u(t, L) \, dt.
\]

All remaining variations are arbitrary, so momentarily assume that \( \delta u(t, x) = 0 \) at \( x = L \) but is arbitrary elsewhere. Then the kernel of the spatial integral term must be zero. Knowing this, let \( \delta u(t, L) \) assume an arbitrary shape. Then the kernel of the time integral must be zero. Together these give

\[
\rho \ddot{u} - Tu'' = 0 \tag{12}
\]

\[
m \ddot{u}(t, L) + ku(t, L) + b_\dot{u}(t, L) + Tu'(t, L) = 0. \tag{13}
\]

The invariant left-hand boundary condition adds \( u(t, 0) = 0 \). Time initial conditions are discussed earlier.

### 4.2 FEM set up

Writing (1) in variational form and applying integration by parts (with appropriate boundary conditions) gives

\[
0 = \int_0^L (\rho \ddot{u} - Tu'') \, dx \tag{14}
\]

\[
= \int_0^L (\rho \ddot{u} + Tu' \delta u) \, dx - Tu'(t, L) \delta u(t, L)
\]

\[
= \int_0^L (\rho \ddot{u} + Tu' \delta u) \, dx + m \ddot{u}(t, L) + ku(t, L) + b_\dot{u}(t, L).
\]

Approximate \( u(t, x) \) as a finite linear combination of independent basis functions \( \phi \in C^1[0, L] \) with time-varying coefficients:

\[
u(t, x) = \sum_{k=0}^n a_k(t) \phi_k(x) = \underline{a}(t)^T \underline{\phi}(x) = \underline{\phi}(x)^T \underline{a}(t), \tag{15}
\]

where underscores denote vectorized variables. Dropping \( x \) and \( t \) arguments where possible, we now have

\[
0 = \int_0^L \rho \delta \underline{a}^T \underline{\phi} \underline{\dot{a}} + T \delta \underline{a} \delta \underline{a}^T dx + \delta \underline{a}^T \left[ \underline{\phi} \underline{\phi}^T \right] \left( m \underline{\dot{a}} + b_\underline{\dot{a}} + k \underline{a} \right)
\]

\[
= \delta \underline{a}^T \left( \rho \int_0^L \underline{\phi} \underline{\phi}^T \, dx + T \int_0^L \underline{\phi}' \underline{\phi}'^T \, dx + \left[ \underline{\phi} \underline{\phi}^T \right] \left( m \underline{\dot{a}} + b_\underline{\dot{a}} + k \underline{a} \right) \right). \tag{16}
\]

Given that \( \delta \underline{a}^T \in \mathbb{R}^{n+1} \) is arbitrary, it follows that

\[
M \underline{\dot{a}} + B \underline{\dot{a}} + K \underline{a} = 0, \tag{17}
\]

where

\[
M := \rho \int_0^L \underline{\phi} \underline{\phi}^T \, dx + m \underline{\phi} \underline{\phi}^T \tag{18}
\]

\[
B := b_\underline{\phi} \underline{\phi} \underline{\phi}^T \tag{19}
\]

\[
K := T \int_0^L \underline{\phi}' \underline{\phi}'^T \, dx + k \underline{\phi} \underline{\phi}^T. \tag{20}
\]
Simulation code is written assuming unit amplitude piecewise linear basis functions centered at node locations $x_k$ where $k = 0...n$:

$$\phi_k(x) := \begin{cases} 
\frac{1}{x_k - x_{k-1}}(x - x_{k-1}) & x \in [x_{k-1}, x_k) \\
\frac{1}{x_{k+1} - x_k}(x_{k+1} - x) & x \in [x_k, x_{k+1}) \\
0 & \text{elsewhere}
\end{cases} \quad (21)$$

This way, $a_k(t) = u(t, x_k)$. Consequently, $a_0(t) = u(t, 0) = 0$ (from the pinned boundary condition), and the dimensionality of the system in (17) is reduced from $n + 1$ to $n$ with a full rank mass matrix.

Next it is convenient to expand (17) into a system of $2n$ first-order ODEs for computational purposes. Let

$$\dot{\bar{z}} = \begin{bmatrix} \dot{a} \\ \dot{x} \end{bmatrix}; \quad A = \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ -M^{-1}K & -M^{-1}B \end{bmatrix} \quad (22)$$

so that

$$\dot{\bar{z}} = Az \quad (23)$$

Integrating gives solutions

$$\bar{z}(t) = \bar{z}(0)e^{At}; \quad t \geq 0, \quad (24)$$

which can be computed to desired accuracy using any standard matrix exponentiation routine (e.g. Padé approximation or truncated Taylor series).