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Nonregressivity in switched linear circuits and mechanical systems☆

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Abstract

We analyze several examples of switched linear circuits and a switched spring–mass system to illustrate the physical manifestations of regressivity and nonregressivity for discrete and continuous time systems as well as hybrid discrete/continuous systems from a time scales perspective. These examples highlight the role that nonregressivity plays in modeling and applications, and they point out a fascinating dichotomy between purely continuous systems and discrete, continuous, or hybrid systems. We conclude with a physically realizable null space criterion for inducing nonregressivity.

Keywords: Nonregressivity; Switched linear system; Switched circuit; Time scale

1. Introduction

Discrete and continuous dynamical systems are ubiquitous in engineering applications and the interaction between discrete and continuous (so called *hybrid*) systems is itself a rapidly emerging field of research [1,2]. The introduction of mixed discrete/continuous domains leads to many intriguing mathematical questions. In this paper, we investigate these types of systems from the unified perspective of time scales and point out how this theory illuminates various interesting phenomena in applications of switched circuits and mechanical systems. By doing so, we present several interesting consequences of the formerly abstract concept of regressivity and nonregressivity of dynamic systems.

Time scales have proven effective in various applications such as high-gain adaptive control [3] and improved bandwidth allocation for real-time communication networks [4]. However, these applications have relied heavily on new results in stability theory [5–7] for time scales. This highlights the connection between progress in the theoretical arena and breakthrough applications.

Both mathematicians and engineers have shown interest in switched systems of various complexities, evidenced by vast number of papers in the area; [8–19] should serve as a representative sample of references. It is our hope that time scales can be helpful here, too.

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Various methods have been brought to bear on questions involving the stability of switched systems such as Lie algebraic techniques [8,13], stability preserving mappings [17], and an average dwell time approach with piecewise Lyapunov functions [18]. Switched systems provide a natural context for applying classical Floquet theory for ordinary differential equations in the continuous case [10] and hence a generalized Floquet theory for the time scale case [6]. Several of the aforementioned papers achieve asymptotic stability for a switched system whose subsystems are unstable [15] as well as instability for a system whose subsystems are stable [8]. This illuminates the subtleties of this type of analysis even in the continuous case. The general time scale case is even more delicate as our examples will indicate.

A complete tutorial on the theory of time scales is beyond the scope of this paper; however, we refer the reader to the excellent texts by Bohner and Peterson [20,21] for a thorough treatment.

The paper is organized as follows. In Section 2, we compare and contrast the concept of nonregressivity in scalar versus multidimensional (vector) problems. In Sections 3–6, we look at three examples of nonregressivity in switched linear circuits on three different time scales. In Section 7, we do a similar analysis of nonregressivity in a switched spring–mass system. In Section 8, we give a simple null space criterion for nonregressivity for multidimensional systems, and in Section 9 we verify that our models tend to the familiar continuous models in the limit. We present our conclusions in Section 10.

2. Regressivity and nonregressivity

The scalar dynamic equation on a time scale $\mathbb{T}$

$$x^{\Delta}(t) = a(t)x(t) \tag{1}$$

is said to be *regressive* if $a(t) \in \mathcal{R}$, the regressive group, given by

$$\mathcal{R} := \{f : \mathbb{T} \to \mathbb{R} \mid 1 + f(t)\mu(t) \neq 0 \ \forall t \in \mathbb{T}^{\kappa} \text{ and } f \in C_{rd}\}. \tag{2}$$

A system is *nonregressive* if it is not regressive. To the best of our knowledge, regressivity was first introduced in the seminal works of Stefan Hilger [22,23] and plays a crucial role in developing the fundamental theory of linear dynamic equations. In particular, any results that appeal to the generalized time scale exponential function require regressivity since $e_a(t, t_0)$ is defined only for $a \in \mathcal{R}$. See Chapter 2 of [21].

This topic might be unfamiliar to those who study ordinary differential equations (ODEs) exclusively. This is because purely continuous[1] dynamical systems (e.g., ODEs) are *always* regressive since the underlying time scale, $\mathbb{R}$, has $\mu(t) \equiv 0$ and hence (2) holds for all $t \in \mathbb{R}$. However, nonregressivity is always a possibility in purely discrete dynamical systems (difference/recursive equations with uniform or nonuniform step size) or dynamical systems where the underlying domain consists of a mixture of discrete and continuous parts (e.g., hybrid systems). In fact, if there is even one point in $\mathbb{T}$ with nonzero graininess, then nonregressivity is possible.

Having identified when nonregressivity is ruled in or out by the domain of the system, what are the consequences of nonregressivity? An almost immediate observation is the following: *assuming a nontrivial initial condition*, (1) *is nonregressive at* t^* *if and only if* $x(t) \equiv 0$ *for all* $t \geq t^*$. The physical ramifications of this are interesting: a system modeled by (1) is nonregressive if and only if there exists some time t^* such that the right combination of graininess $\mu(t^*)$ and system parameter $a(t^*)$ will force the state variable to be zero and stay zero thereafter. This may prove useful in applications as we will see later.

Regressivity in multidimensional (vector) systems such as

$$\vec{x}^{\Delta}(t) = \mathbf{A}(t)\vec{x}(t), \quad \mathbf{A} \in \mathbb{R}^{n\times n}, \quad \vec{x} \in \mathbb{R}^{n\times 1}, t \in \mathbb{T}, \tag{3}$$

is very different from the scalar case. In [21], Eq. (3) is said to be *regressive* if and only if

$$\det(\mathbf{I} + \mu(t)\mathbf{A}(t)) \neq 0, \quad \forall t \in \mathbb{T}^{\kappa}. \tag{4}$$

It turns out that this is equivalent to having all of the (scalar) eigenvalues $\lambda_i(t)$, $i = 1, \ldots, n$, of $\mathbf{A}(t)$ regressive in the sense of (2).

[1] By this, we mean dynamical systems whose underlying domain is continuous, such as a closed interval.

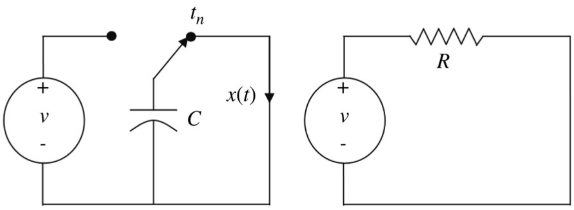

Fig. 1. Left: A switched capacitor circuit. Right: The resistor circuit approximated by the switched capacitor circuit.

A natural question to ask at this juncture is whether the "hit zero, stay zero" phenomenon apparent in nonregressive scalar systems carries over to multidimensional systems. We will show that the answer is no, but we will provide nonregressivity conditions sufficient to force (3) to behave this way. This leads to some interesting revelations about regressive/nonregressive dynamic equations in the context of switched circuits and switched spring–mass systems.

3. Switched capacitor circuits

A commonly used[2] switched circuit is the *switched capacitor circuit* [24,25] shown on the left side of Fig. 1. A switch connects the capacitor, C, to the voltage source with voltage v on the left. The switch then connects the right hand short circuit at times $\{t_n\}$. The back and forth process continues. The time scale consists of these discrete points, i.e. $\mathbb{T} = \{t_n\}$. The variable under consideration is the cumulative charge, x, delivered from the capacitor to the short circuit. The entire charge of the capacitor, Cv, is discharged instantaneously through the short circuit at time t_n. Thus, the cumulative charge is

$$x(t_{n+1}) = x(t_n) + vC,$$

which we can recast as the dynamic equation

$$x^{\Delta}(t) = Cv/\mu(t), \quad t \in \mathbb{T}.$$

The Hilger derivative of the charge is an approximation for the temporal derivative of charge: the current emanating from the voltage source. The voltage is in turn proportional to the Hilger derivative of the current, and the proportionality constant is the resistance, R. When the graininess is constant, say $\mu(t) \equiv \mu$, then resistance[3] is given by $R = \mu/C$. Insofar as the Hilger derivative of the charge approximates the continuous time derivative of charge [21], the switched capacitor circuit on the left in Fig. 1 approximates the resistor circuit on the right.

4. A switched *LC* circuit

Another simple switched circuit is shown in Fig. 2. When the switch is to the left at time t_{2n}, the inductor with inductance L and the capacitor with capacitance C form an oscillator with voltage satisfying $\ddot{x} = -\omega^2 x$ and frequency $\omega = 1/\sqrt{LC}$. When the switch is to the right, between times t_{2n} and t_{2n+1}, the value of $x(t)$ is a constant, i.e. $\dot{x} = 0$; otherwise, there is sinusoidal oscillation. An illustration of the dynamics[4] is shown in Fig. 2.

Due to the physics of the circuit,[5] immediately after the switch connects the inductor, $\dot{x}^+_{2n+1} = 0$, where we have adopted the notation $z^+_k = z(t^+_k)$. Thus, the solution $x(t)$ is of the form

$$x(t) = \begin{cases} x^+_{2n}, & t \in [t_{2n}, t_{2n+1}], \\ x^+_{2n+1} \cos(\omega(t - t^+_{2n+1})), & t \in [t_{2n+1}, t_{2n+2}]. \end{cases} \tag{5}$$

[2] One reason is that in VLSI applications, resistance requires much chip area, but capacitance does not. So resistors are emulated by using switched capacitor circuits.

[3] The resistance in switched capacitor circuits is commonly written as $R = (fC)^{-1}$ where $f = 1/\mu$ is the switching frequency.

[4] To discuss the dynamics at switching point t_m, let t^-_m denote instances immediately before the switching occurs and t^+_m immediately after.

[5] Let y denote the current entering the top of the capacitor. The same current, passing through the inductor L, is related to the voltage x as $x = -L\dot{y}$. Thus, for x to be bounded at t^+_{2n+1}, y must be continuous. The current through the inductor immediately prior to the switching is zero. Hence, $y = 0$ both immediately before and immediately after the switching occurs at t_{2n+1}. For the capacitor, $y = C\dot{x}$. Since $y = 0$ at t^+_{2n+1}, so too does $\dot{x} = 0$.

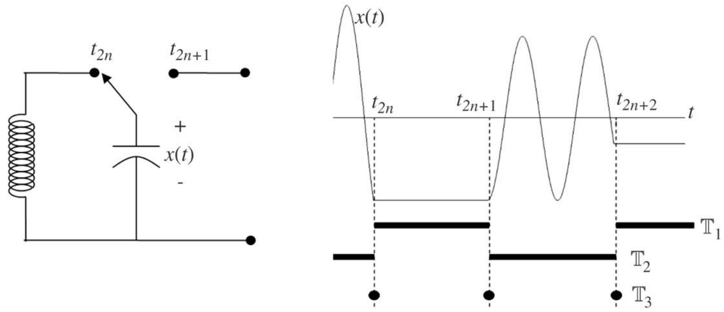

Fig. 2. A simple switched *LC* circuit.

There are three time scales we consider for this switched circuit, all illustrated in Fig. 2:

$$\mathbb{T}_1 = \{t \in [t_{2n}, t_{2n+1}], n \in \mathbb{N}_0\},$$
$$\mathbb{T}_2 = \{t \in [t_{2n+1}, t_{2n+2}], n \in \mathbb{N}_0\},$$
$$\mathbb{T}_3 = \{t_n, n \in \mathbb{N}_0\}.$$

For $\mathbb{T}_1$, from (5),

$$x^+_{2n+2} = x^-_{2n+2} = x(t^+_{2n+1}) \cos(\omega\mu(t_{2n+1})).$$

Hence, on $\mathbb{T}_1$, we have a scalar problem of the form (1) with

$$a(t) = \begin{cases} 0, & t \in [t_{2n}, t_{2n+1}), \\ \dfrac{\cos(\omega\mu(t)) - 1}{\mu(t)}, & t = t_{2n+1}. \end{cases}$$

Nonregressivity of the system, when modeled on $\mathbb{T}_1$, can occur only at $t = t_{2n+1}$. From (2), nonregressivity occurs when $1 + \mu(t)a(t) = \cos(\omega\mu(t)) = 0$, $t = t_{2n+1}$, that is, when $\mu(t) = \frac{\pi(2p-1)}{2\omega}$, $t = t_{2n+1}$, $p \in \mathbb{N}$. If switching occurs at the point where the sinusoid hits zero, the system is nonregressive. Thus, $x(t)$ becomes zero and remains zero thereafter.

Although nonregressivity occurs on $\mathbb{T}_1$ for the circuitry in Fig. 2, analysis with time scale $\mathbb{T}_2$ shows that the system is *always regressive on* $\mathbb{T}_2$. This is true, despite the clear demonstration in the previous example that $x(t)$ can hit zero and stay there. Since $x(t_{2n}) = x(t_{2n+1})$, then (3) takes the form

$$\begin{bmatrix} \dot{x}^\Delta(t) \\ x^\Delta(t) \end{bmatrix} = \mathbf{A}(t) \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix},$$

where

$$\mathbf{A}(t) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, & t \in [t_{2n+1}, t_{2n+2}), \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & t = t_{2n+2}. \end{cases}$$

Since the determinant condition in (4) always holds, the system is always regressive on time scale $\mathbb{T}_2$. This points out how the same physical phenomenon—when modeled on different time scales—can lead to different regressivity/nonregressivity results.

For time scale $\mathbb{T}_3$, we are again dealing with a scalar problem of the form (1) with

$$a(t) = \begin{cases} \dfrac{\cos(\omega\mu(t)) - 1}{\mu(t)}, & t = t_{2n+1}, \\ 0, & t = t_{2n+2}. \end{cases}$$

Nonregressivity via (2) on $\mathbb{T}_3$ occurs under the same condition as for $\mathbb{T}_1$.

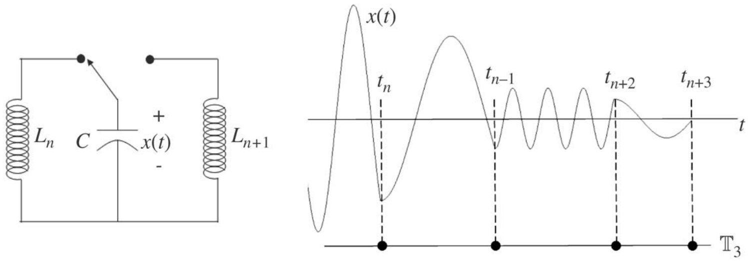

Fig. 3. A second order switched *LCL* circuit capable of nonregressivity for appropriate choices of graininess.

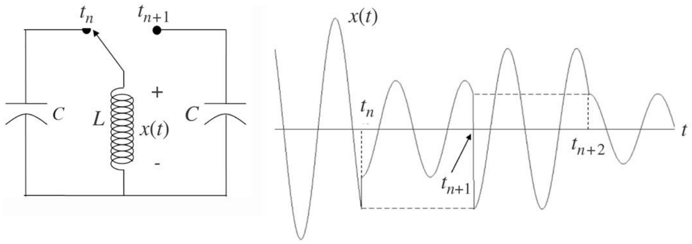

Fig. 4. A second order switched *CLC* circuit capable of nonregressivity for appropriate choices of graininess.

5. A switched *LCL* circuit

A second switched oscillator is shown in Fig. 3. The switch to the left connects the capacitor, C, to the inductor L_n. At time t_{n+1} the switch goes to the right to inductor L_{n+1}. The inductor L_n is replaced by L_{n+2} and the process continues. The oscillation over the interval $[t_n, t_{n+1}]$ occurs at frequency $\omega_n = 1/\sqrt{L_n C}$. Sample dynamics are shown in Fig. 3.

We consider the time scale $\mathbb{T}_3$ as illustrated in Fig. 3. Analogously to the oscillator case in Fig. 2 (and for the same physical reasons), $\dot{x}_n^+ = 0$. Thus

$$x(t) = x_n^+ \cos(\omega_n (t - t_n)), \quad t \in [t_n, t_{n+1}],$$

and so

$$x_{n+1}^+ = x_{n+1}^- = x_n^+ \cos(\omega_n \mu(t_n)),$$

which leads to

$$x^\Delta(t) = \frac{\cos(\omega_n \mu(t)) - 1}{\mu(t)} x(t), \quad t = t_n. \tag{6}$$

Similarly to the case for the first oscillator, nonregressivity occurs when $\mu_n = \frac{\pi(2p-1)}{2\omega}$ for any $p \in \mathbb{N}$. For $p = 2$, nonregressivity in Fig. 3 occurs at t_{n+2} corresponding to the clamping of $x(t)$ to zero at t_{n+3}.

6. A switched *CLC* circuit

A third oscillator is illustrated in Fig. 4. Note that the two capacitors are equal here. As in the previous examples, $\dot{x} = 0$ immediately after a switch is closed. An illustration of the response of this circuit is also shown in Fig. 4. The dynamics of the right hand circuit are

$$x(t) = x_n^+ \cos(\omega(t - t_n)), \quad t \in [t_n, t_{n+1}).$$

Thus $x_{n+1}^- = x_n^+ c_n$, where $c_n := \cos(\omega_n \mu_n)$. Since $x_{n+2}^+ = x_{n+1}^-$, we have

$$x_{n+1}^\Delta = (x_{n+2}^+ - x_{n+1}^+)/\mu(t_{n+1}) = (c_n x_n^+ - x_{n+1}^+),$$

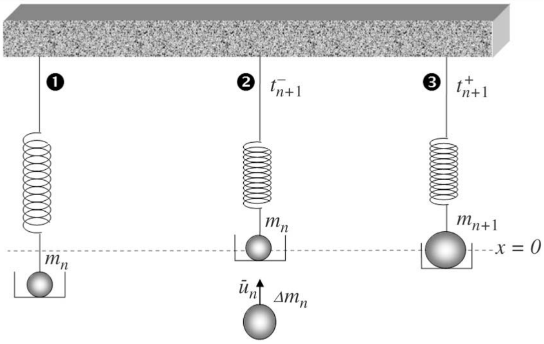

Fig. 5. A switched spring–mass system.

and it follows that

$$\begin{bmatrix} x_{n+1}^{\Delta} \\ x_n^{\Delta} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{\mu_{n+1}} & \dfrac{c_n}{\mu_{n+1}} \\ \dfrac{1}{\mu_n} & \dfrac{-1}{\mu_n} \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}. \tag{7}$$

Thus

$$\mathbf{I} + \mu(t_n)\mathbf{A}(t) = \begin{bmatrix} 1 - \dfrac{\mu_n}{\mu_{n+1}} & \dfrac{\mu_n}{\mu_{n+1}} c_n \\ 1 & 0 \end{bmatrix},$$

and hence $\det(\mathbf{I} + \mu(t)\mathbf{A}(t)) = 0$ when $\mu_n = 0$ or $c_n = 0$, but the former is physically unrealizable. Unlike the case for the previous two oscillators, the total response does not go identically to zero when this occurs. For $c_k = 0$, the response goes to zero over every other interval, i.e. the intervals $t \in [t_{k+2p+1}, t_{k+2p+2}]$, $p \in \mathbb{N}_0$. If, in addition, $c_{k+1} = 0$, the entire solution clamps to zero.

7. A switched mechanical system

A mechanical equivalent of the switched circuit in Fig. 3 is shown in Fig. 5. During time $[t_n, t_{n+1})$, the spring supports mass m_n and oscillates with frequency $\omega_n = \sqrt{k/m_n}$ where k is the Hooke's constant of the spring. Since we cannot (in general) specify that $\dot{x}(t_n) = 0$, the general solution of the oscillation is

$$x(t) = x_n^+ \cos(\omega_n(t - t_n)) + \frac{\dot{x}_n^+}{\omega_n} \sin(\omega_n(t - t_n)), \quad t \in [t_n, t_{n+1}).$$

Thus

$$\dot{x}(t) = -\omega_n x_n^+ \sin(\omega_n(t - t_n)) + \dot{x}_n^+ \cos(\omega_n(t - t_n)), \quad t \in [t_n, t_{n+1}).$$

Evaluating the above two expressions at $t = t_{n+1}^-$ gives

$$\begin{bmatrix} \dot{x}_{n+1}^- \\ x_{n+1}^- \end{bmatrix} = \begin{bmatrix} c_n & \omega_n s_n \\ s_n/\omega_n & c_n \end{bmatrix} \begin{bmatrix} \dot{x}_n^+ \\ x_n^+ \end{bmatrix}, \tag{8}$$

where $s_n := \sin(\omega_n \mu_n)$.

To make the transition from $\dot{x}_{n+1}^-$ to $\dot{x}_{n+1}^+$, assume inelastic momentum conservation of the mass, m_n, moving at velocity $\dot{x}_{n+1}^-$, with a mass Δm_n traveling with velocity $\dot{u}_{n+1}$. This is illustrated in Fig. 5. Then

$$m_{n+1}\dot{x}_{n+1}^+ = m_n \dot{x}_{n+1}^- + \Delta m_n \dot{u}_{n+1}.$$

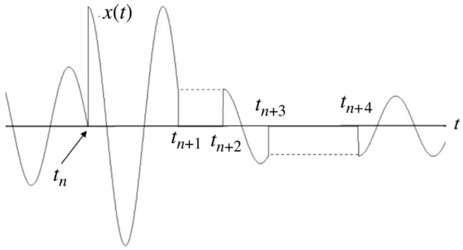

Fig. 6. An example of the circuit in Fig. 4 in a nonregressive state.

Equivalently, since $\Delta m_n = m_{n+1} - m_n$,

$$\dot{x}^+_{n+1} = \alpha_n \dot{x}^-_{n+1} + (1-\alpha_n)\dot{u}_{n+1},$$

where $\alpha_n := m_{n+1}/m_n$. Express the velocity of Δm_n as proportional to $\dot{x}^-_{n+1}$,

$$\dot{u}_{n+1} = \lambda_n \dot{x}^-_{n+1},$$

so that $\dot{x}^-_{n+1} = \dot{x}^+_{n+1}$. By the continuity of position, we obtain

$$\begin{bmatrix}\dot{x}^+_{n+1}\\ x^+_{n+1}\end{bmatrix} = \begin{bmatrix}\beta_n & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\dot{x}^-_{n+1}\\ x^-_{n+1}\end{bmatrix},$$

where $\beta_n := \alpha_n + \lambda_n(1-\alpha_n)$. Applying (8), we see

$$\begin{bmatrix}\dot{x}^+_{n+1}\\ x^+_{n+1}\end{bmatrix} = \begin{bmatrix}\beta_n c_n & -\beta_n\omega_n s_n\\ s_n/\omega_n & c_n\end{bmatrix}\begin{bmatrix}\dot{x}^+_{n}\\ x^+_{n}\end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix}\dot{x}^\Delta(t)\\ x^\Delta(t)\end{bmatrix} = \mathbf{A}(t)\begin{bmatrix}\dot{x}(t)\\ x(t)\end{bmatrix} = \frac{1}{\mu(t)}\begin{bmatrix}\beta_n c_n - 1 & -\beta_n\omega_n s_n\\ s_n/\omega_n & c_n - 1\end{bmatrix}\begin{bmatrix}\dot{x}(t)\\ x(t)\end{bmatrix}, \quad t = t_n. \tag{9}$$

Setting $\det(\mathbf{I} + \mu(t)\mathbf{A}(t)) = 0$ yields $\beta_n = 0$. Thus, for the system to be nonregressive on $\mathbb{T}_3$ (Fig. 6), we require

$$\lambda_n = \frac{-\alpha_n}{1-\alpha_n} = \frac{-m_n}{\Delta m_n}. \tag{10}$$

To illustrate, if $m_n = \Delta m_n$, then $\lambda_n = -1$. Two equal masses collide going at the same velocity but in opposite directions. If the collision occurs at the equilibrium point of the spring ($x = 0$), the system will be at rest thereafter. The nonregressivity condition in (10) indicates the equal and opposite velocities, but does not require the collision to occur at the system's point of equilibrium. If nonregressive collision occurs at a nonequilibrium point, the composite mass will be motionless momentarily before resuming its oscillations (see Fig. 7).

8. A null space criterion for nonregressivity

If we rewrite (3) in its recursive form, we obtain

$$\vec{x}(t_{n+1}) = (\mathbf{I} + \mu(t)\mathbf{A}(t))\vec{x}(t),$$

which leads to the following null space criterion for nonregressivity: *If there exists a* $t^* \in \mathbb{T}$ *such that* $\vec{x}(t^*) \in \ker(\mathbf{I}+\mu(t)\mathbf{A}(t))$, *then* $\vec{x}(t) \equiv 0$ *for all* $t \geq t^*$. This is consistent with the observation in the spring–mass systems that both position and velocity need to be zero at the switching instance in order to have the "hit zero, stay zero" behavior.

Since (3) is nonregressive if and only if all of the (scalar) eigenvalues $\lambda_i(t)$, $i = 1, \ldots, n$, of $\mathbf{A}(t)$ are regressive (in the sense of (1)), one might conjecture that it is possible to induce this "hit zero, stay zero" phenomenon—which characterizes nonregressivity in scalar equations—in vector systems via a sequence of times t_i, $i = 1, \ldots, n$,

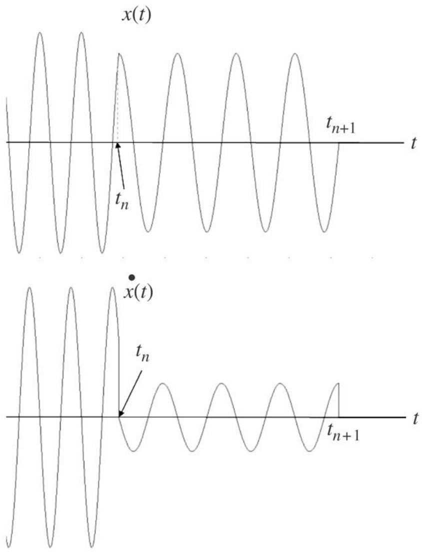

Fig. 7. Illustration of nonregressivity in the mechanical system in Fig. 5. At both times t_n and t_{n+1}, the nonregressivity condition in (10) holds for $\lambda = -1$. In both cases, the oscillating mass on the spring is hit by a second mass going at the same velocity in the opposite direction. Doing so always makes the instantaneous velocity of the mass after the impact equal to zero. At time t_n, the spring continues to oscillate. At time t_{n+1}, it does not. This is because the impact occurs when the mass position is at equilibrium.

such that $\lambda_i(t_i) = 0$. However, this is not the case; counterexamples can be easily constructed which demonstrate, for example in the switched spring–mass example, that even though one state variable becomes zero through this "sequential nonregressivity" process, it does *not* have to stay zero at future times. Of course, this does not occur in scalar problems, and we see yet another interesting dichotomy between scalar and vector problems.

9. Limiting dynamic equations

To evaluate the limiting case of the time scale dynamics, let $\mathbb{T}_h = ht_n$ and $h \to 0$. Then

$$\begin{aligned}
&\mu(t) \to \lim_{h\to 0} h\mu(t) = \psi(t)\,\mathrm{d}t,\\
&\omega_n \to \omega(t),\\
&c_n = \cos(\mu(t_n)\omega_n) \to 1,\\
&x^{\Delta} \to \dot{x},\\
&\dot{x}^{\Delta} \to \ddot{x}.
\end{aligned}$$

9.1. The second oscillator circuit

For the time scale dynamics in (6) on $\mathbb{T}_3$, the limiting equation is $\dot{x} = 0$. Physically, the voltage across the capacitor is not permitted to change due to the current inertia of the inductors.

9.2. The mechanical system

For the spring–mass problem, the following additional limits are needed:

$$\begin{aligned}
&\Delta m_n \to \mathrm{d}m := \mathrm{d}m(t),\\
&m_n \to m := m(t),\\
&\lambda_n \to \lambda := \lambda(t)\\
&s_n = \sin(\mu(t_n)\omega_n) \to \psi(t)\omega(t)\mathrm{d}t,
\end{aligned}$$

$$\alpha_n \to \left(1 + \frac{\mathrm{d}m}{m}\right)^{-1},$$
$$\beta_n \to \frac{m + \lambda \mathrm{d}m}{m + \mathrm{d}m}.$$

Then, as $h \to 0$, (9) becomes

$$\begin{bmatrix} \dot{x}_n^{\Delta} \\ x_n^{\Delta} \end{bmatrix} \to \begin{bmatrix} \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dfrac{1-\lambda}{m\psi(t)} \dfrac{\mathrm{d}m}{\mathrm{d}t} & -\omega^2(t) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix}. \tag{11}$$

Of special interest is the case where mass is being added or subtracted at the same velocity of the oscillating mass. Then $\lambda = 1$, and (11) becomes the standard second order system

$$\begin{bmatrix} \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & -\omega^2(t) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix},$$

corresponding to the harmonic oscillator $\ddot{x} + \omega^2 x = 0$.

10. Consequences and conclusions

Our goal with this short paper was to give some insight into the unexplored concept of regressivity and nonregressivity and see what role they play in physical systems. These are not purely abstract mathematical conditions one might impose on a system; indeed, there are physical ramifications of doing so. Maybe this "hit zero, stay zero" phenomenon and its connection to nonregressivity can be exploited in discrete problems such as sampled data systems or adaptive control problems with nonuniform, discrete sampling as well systems where the domain is a mixture of discrete and continuous parts such as hybrid systems.

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