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Time Scale Discrete Fourier Transforms

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***Abstract*—The discrete and continuous Fourier transforms are applicable to discrete and continuous time signals respectively. Time scales allows generalization to to any closed set of points on the real line. Discrete and continuous time are special cases. Using the Hilger exponential from time scale calculus, the discrete Fourier transform (DFT) is extended to signals on a set of points with arbitrary spacing. A time scale $\mathbb{D}_N$ consisting of N points in time is shown to impose a time scale (more appropriately dubbed a *frequency scale*), $\mathbb{U}_N$, in the Fourier domain The time scale DFT's (TS-DFT's) are shown to share familiar properties of the DFT, including the derivative theorem and the power theorem. Shifting on a time scale is accomplished through a boxminus and boxplus operators. The shifting allows formulation of time scale convolution and correlation which, as is the case with the DFT, correspond to multiplication in the frequency domain.**

I. Introduction

A time scale is any collection of closed points on the real line. Continuous time, $\mathbb{R}$, and discrete time $\mathbb{Z}$, are special cases. The calculus of time scales was introduced by Hilger [11]. Time scales have found utility in describing the behavior of dynamic systems [1], [13] and have been applied to control theory [3], [4], [5], [7], [10].

On $\mathbb{R}$ and $\mathbb{Z}$, respectively, the continuous time and discrete time Fourier transforms are well studied [16]. Properties of the Laplace and Fourier transforms on time scales have been extended to time scales with unbounded domains [1], [6], [8], [9], [12], [14], [16].

The conventional *discrete Fourier transform* (DFT) is defined over a finite number of uniformly spaced points. This paper extends the DFT to a finite number of discrete time points that are not uniformly spaced.[1] The time scale of a finite number of N discrete points, $\mathbb{D}_N$, is shown to uniquely map into a frequency scale (a time scale in the frequency domain), $\mathbb{U}_N$, in the Fourier domain. Familiar Fourier transform theorems, including the shift, convolution and derivative theorems, are shown to generalize to the *time scale* DFT (TS-DFT).

II. Time Scales

Our introduction to time scales is limited to that needed to establish notation. A more detailed explanation are in our previous papers [4], [5], [6], [8], [9], [10], [13], [14], [16] and a complete rigorous treatment is in the text by Bohner and Peterson [1].

1) A *time scale*, $\mathbb{T}$, is any collection of closed intervals on the real line. Generally, the time scale can contain both discrete time points and continuous time intervals. Since our development of TS-DFT is only on time scales containing discrete points, we henceforth restrict attention to time scales containing discrete points,[2] denoted $\mathbb{D}$. Discrete time, $\mathbb{Z}$, is a special case.
2) The *graininess*, $\mu(t)$, is the distance between adjacent points in a time scale at time $t \in \mathbb{T}$ and is defined generally by
$$\mu(t) = \left(\inf_{\tau > t, \tau \in \mathbb{T}} \tau \right) - t.$$
For $\mathbb{D}$,
$$\mu(t_n) = t_{n+1} - t_n.$$
3) The *Hilger derivative* of a function $x(t)$ at $t \in \mathbb{T}$ is
$$x^{\Delta}(t) := \frac{x(t+\mu(t)) - x(t)}{\mu(t)}.$$
When $\mu(t) = dt$ $(= 0)$, the Hilger derivative is interpreted in the limiting sense and
$$x^{\Delta}(t) = \frac{d}{dt} x(t).$$
For $\mathbb{D}$, we have
$$x^{\Delta}(t_n) = \frac{x(t_{n+1}) - x(t_n)}{\mu(t_n)}.$$
4) If $y(t) = x^{\Delta}(t)$, then the *definite time scale integral* is
$$\int_a^b y(t) \Delta t = x(b) - x(a).$$
For $\mathbb{D}$, we have [1]
$$\int_{t_p}^{t_q} y(t) \Delta t = \sum_{n=p}^{q-1} y(t_n) \mu(t_n).$$
5) When $x(0) = 1$, the solution to the *Hilger differential equation*,
$$x^{\Delta}(t) = z x(t),$$

[1] Our development is distinct from the time scale Fourier transform proposed by Hilger [12], [14], [16]. Our treatment more closely resembles Laplace transform generalizations where two signals on a time scale $\mathbb{T}$, when convolved, result in a signal on the same time scale, $\mathbb{T}$ [1], [6], [8], [9].

[2] We use $\mathbb{D}$ to denote a time scale with an arbitrary, possibly infinite, set of discrete isolated points. The notation $\mathbb{D}_N$ indicates the time scale has N points.

is $x(t) = e_z(t)$ where the *generalized exponential* is

$$e_z(t) := \exp\left(\int_{\tau=0}^{t} \frac{\ln(1 + z\mu(\tau))}{\mu(\tau)} \Delta\tau\right).$$

For $\mathbb{D}$ and $n > 0$,

$$e_z(t_n) = \prod_{m=0}^{n-1} (1 + z\mu(t_m)). \quad (1)$$

Since $\mu(t_m)$ is real,

$$e_z^*(t_n) = e_{z^*}(t_n) \quad (2)$$

The properties of the generalized exponential parallel those of z^n for the z-transform and $e^{j\omega t}$ for the Fourier transform are responsible for the utility of the TS-DFT.

III. Time Scale Exponential Basis Sets

Consider a time scale, $\mathbb{D}_N$, of $N+1$ real temporal points, $\{t_n | 0 \leq n \leq N\}$ with $t_0 = 0 \leq t_n < t_{n+1} \leq t_N$. (The point t_N is required to determine the graininess of the point t_{N-1}.) Let $x(t_n)$ and $h(t_n)$ be images on $\mathbb{D}_N$. Define the inner product

$$\langle x(t_n) | h(t_n) \rangle = \sum_{n=0}^{N-1} x(t_n) h^*(t_n) w(t_n) \quad (3)$$

where $w(t_n) > 0$ is a weighting function and the asterisk denotes complex conjugation. Generally, $w(t_n)$ is arbitrary but, to make the integration constant with time scale integration, we will henceforth use the graininess as the weight, *i.e.* $w(t_n) = \mu(t_n)$ The corresponding norm is

$$||x(t)|| = \sqrt{\langle x(t_n) | x(t_n) \rangle}.$$

Two time scale exponentials are orthogonal if

$$\langle e_z(t_n) | e_\varsigma(t_n) \rangle = 0 \text{ for } z \neq \varsigma. \quad (4)$$

If we set $\varsigma = 0$, then applying (1) and (3) to (4) gives

$$\begin{aligned} &\langle e_z(t_n) | e_\varsigma(t_n) \rangle|_{\varsigma=0} = \langle e_z(t_n) | 1 \rangle \\ &= \textstyle\sum_{n=0}^{N-1} e_z(t_n) \mu(t_n) \\ &= \mu(t_0) + \textstyle\sum_{n=1}^{N-1} \mu(t_n) \prod_{m=0}^{n-1} (1 + z\mu(t_m)) \\ &= 0 \text{ for } z \neq \varsigma = 0. \end{aligned} \quad (5)$$

The solution of the $N-1$st order polynomial can be used to generate orthogonal time scale exponentials. Motivated by (5), we dub the roots of the polynomial

$$\mu(t_0) + \sum_{n=1}^{N-1} \mu(t_n) \prod_{m=0}^{n-1} (1 + z\mu(t_m)) = 0 \quad (6)$$

the *frequency roots* of the time scales, $\mathbb{D}_N$.

A. Example Frequency Roots

Here are some examples of frequency roots from the polynomial in (6).

1) On a Time Scale of Uniformly Spaced Points: The time scale of uniformly spaced points is the time scale conventionally associated with the DFT. Then $\mu(t_n) = 1$ and (5) becomes

$$\begin{aligned} \langle e_z(t_n) | 1 \rangle &= 1 + \sum_{n=1}^{N-1} \prod_{m=0}^{n-1} (1 + z) \\ &= \sum_{n=0}^{N-1} (1 + z)^n = 0. \end{aligned}$$

This is a geometric series with solution

$$\langle e_z(t_n^\sigma) | 1 \rangle = \frac{(1 + z)^N - 1}{z} = 0. \quad (7)$$

Note that the zeroth order term in the numerator is zero, so since $z \neq \varsigma = 0$, (7) is an $(N-1)$st polynomial with $N-1$ frequency roots. For $z \neq \varsigma = 0$ (required by (4)) and $z \neq -1$ (the regressivity condition for $\mathbb{Z}$ [1]), this equation is satisfied when $(1+z)^N = \exp(-j2\pi k)$ where k is an arbitrary integer. Thus the $N-1$ polynomial frequency roots are

$$z_k = -1 + e^{-j2\pi k/N}; \; 0 < k < N. \quad (8)$$

As shown in Figure 1, these are points equally spaced on a unit circle centered at $z = -1$.

Other time scales do not lend themselves to the ease of analysis afforded by the time scale of uniformly spaced points.

2)On a log time scale we have $\mathbb{D}_N = \{t_n = \log_2(n)\}$. Example frequency roots of this time scale are shown in Figure 1.

3)The Harmonic Time Scale is defined as

$$t_n = \begin{cases} 0 & ; \quad n = 0 \\ \sum_{k=1}^{n} \frac{1}{k} & ; \quad n > 0. \end{cases}$$

The frequency roots are shown in Figure 1 for $N = 16$.

4)The Geometric Time Scale for a parameter $q > 0$, is defined as[3]

$$t_n = \begin{cases} 0 & ; \quad n = 0 \\ q^n & ; \quad n > 0. \end{cases}$$

The frequency roots are shown in Figure 2 for $N = 16$.

For $q < 1$, the values of the time scale are the same as in (9) except they are arranged in ascending order.

5)The Poisson Time Scale chooses points in a Poisson process [16] with parameter λ. The origin, $t_0 = 0$, is then included. Example frequency roots are shown in Figure 3.

B. Basis Examples

Here are some examples of time scale exponential basis sets for some example time scales. For the exponential basis plots in Figure 4, $N = 8$. Points are linearly connected. The location of points on the time scale are marked along the time axis with dots.

1) On a Time Scale with Uniformly Spaced Points the TS-DFT becomes the conventional DFT [16]. For the uniformly

[3] This differs from the *quantum time scale* [2] which includes the origin and points q^n for $n \in \mathbb{Z}$.

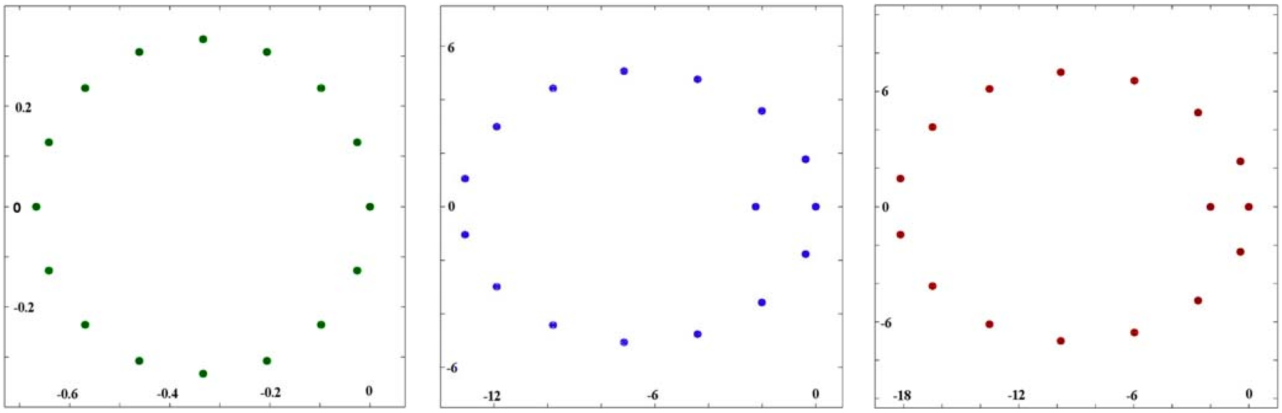

Fig. 1. Frequency roots of some time scales. In each plot, the horizontal and vertical scales are the same. LEFT: The frequency roots of the time scales with equally spaced points lie on a shifted circle in the z plane, dubbed the *Hilger circle* [1]. MIDDLE: The frequency roots of the log time scale, plotted on the complex z plane, discussed in Section III-A1 for $N = 16$. RIGHT: The frequency roots of the harmonic scale, plotted on the complex z plane, as discussed in Section III-A1 for $N = 16$.

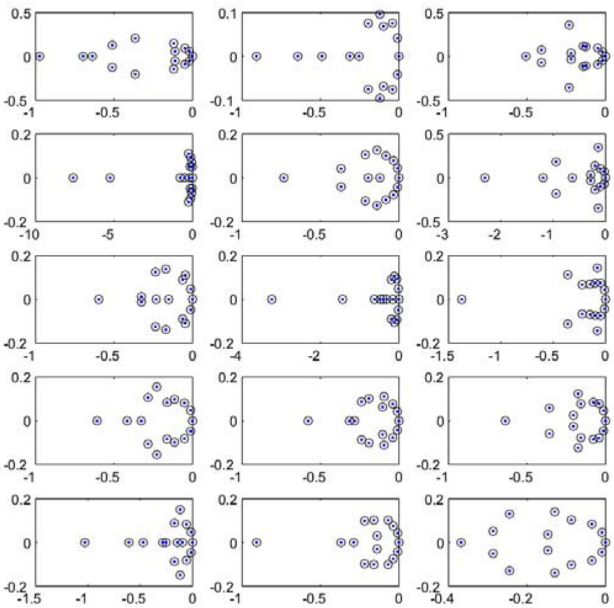

Fig. 3. The frequency roots of 15 realizations of a Poisson time scale with parameter $\lambda = 1$ point per interval. The shape of the root locations varies considerably. Note that scales can differ from plot to plot.

spaced points described in (III-A1), the orthonormalized exponential basis functions are

$$e_{z_k}(t_n) = e^{j2\pi nk/N}; \text{ for } 0 \le n, k < N. \tag{9}$$

Proof: Since $\mu(t_n) = 1$, we have from (1)

$$e_{z_k}(t_n) = \prod_{m=0}^{n-1}(1+z) = (1+z)^n.$$

Substituting (8) gives (9). These are the familiar normalized basis functions for the *discrete Fourier transform* (DFT) and are shown in Figure 4.

1) Other Basis Sets: The orthonormalized basis functions for the log time scale, the harmonic time scale, and the geometric time scale for $q = 1.2$ are shown in Figure 4.

C. Orthogonal Expansions and Inversion

When the orthogonal basis $\{e_{z_k}(t_n)|0 \le n, k < N\}$ is complete, we can expand any function, $x(t_n)$, on the time scale as

$$x(t_n) = \sum_{\ell=0}^{N-1} c_\ell e_{z_\ell}(t_n)$$

where c_n are the series expansion coefficients. Let

$$\begin{aligned} X(z_k) &:= \sum_{n=0}^{N-1} x(t_n) e^*_{z_k}(t_n)\mu(t_n) \\ &= \sum_{n=0}^{N-1}\left[\sum_{\ell=0}^{N-1} c_\ell e_{z_\ell}(t_n)\right] e^*_{z_k}(t_n)\mu(t_n) \\ &= \sum_{\ell=0}^{N-1} c_\ell \left[\sum_{n=0}^{N-1} e_{z_\ell}(t_n) e^*_{z_k}(t_n)\mu(t_n)\right] \\ &= \sum_{\ell=0}^{N-1} c_\ell \left[\|e_{z_k}\|^2 \delta[\ell - k]\right] \\ &= c_k \|e_{z_k}\|^2. \end{aligned}$$

Thus

$$c_k = \frac{X(z_k)}{\|e_{z_k}\|^2}$$

where

$$\|e_{z_k}\|^2 := \sum_{n=0}^{N-1} |e_{z_k}(t_n)|^2 \mu(t_n).$$

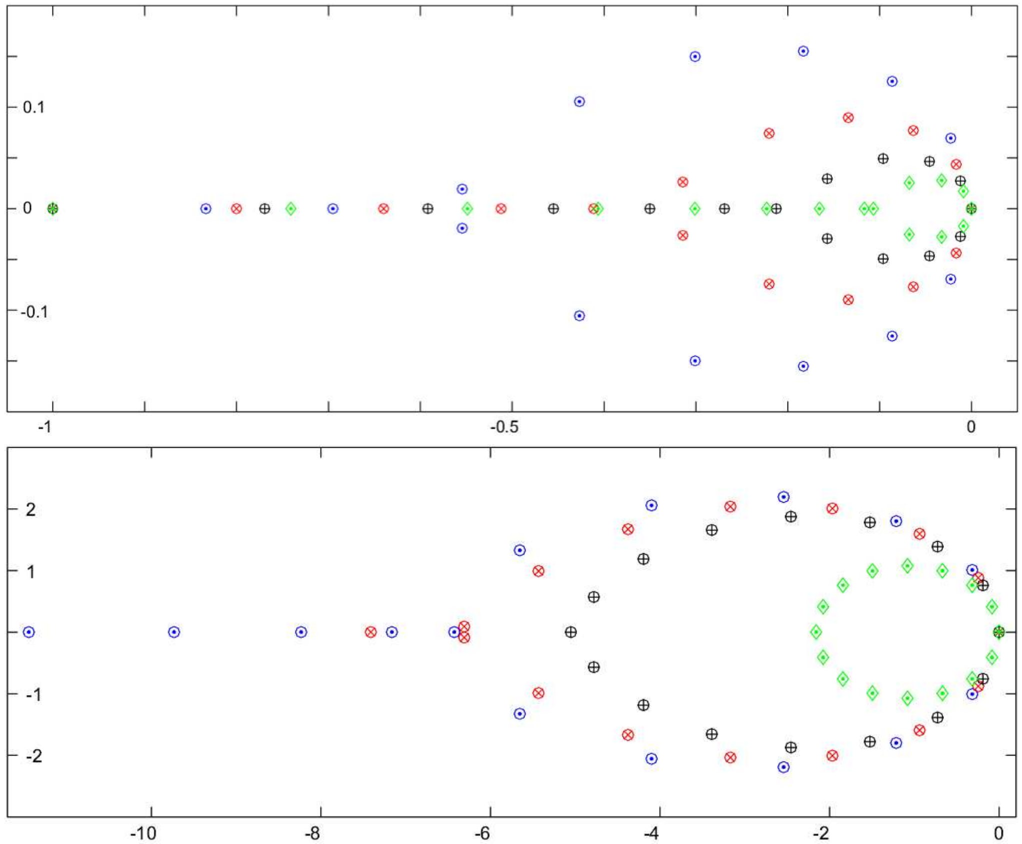

Fig. 2. TOP: The frequency roots of the geometric time scale in (9), plotted on the complex z plane, as discussed in Section III-A1 for $N = 16$. The values of q are 1.20 $\odot$, 1.25 $\otimes$, 1.30 $\oplus$, and 1.35 $\diamond$. BOTTOM: The frequency roots of the geometric time scale, plotted on the complex z plane, as discussed in Section III-A1 for $N = 16$. The values of q are 0.85 $\odot$, 0.875 $\otimes$, 0.90 $\oplus$, and 0.99 $\diamond$.

We find the following notation useful.[4]

$$\partial t_n := \mu(t_n)$$

and

$$\partial z_k := \frac{1}{\|e_{z_k}\|^2}. \tag{10}$$

Note that if t_n has units of time, then ∂t_n has units of time and ∂z_k has units of reciprocal time.

From this analysis, we define the *time scale DFT* (TS-DFT) and its inverse.

► *TS-DFT* ◄

$$x(t_n) \leftrightarrow X(z_k) = \sum_{n=0}^{N-1} x(t_n) e^*_{z_k}(t_n) \partial t_n. \tag{11}$$

► *Inverse TS-DFT* ◄

$$x(t_n) = \sum_{k=0}^{N-1} X(z_k)\; e_{z_k}(t_n) \partial z_k \leftrightarrow X(z_k). \tag{12}$$

[4]An alternate possibly more representative notation might be $\mu_{\mathbb{D}}(t_n)$ in lieu of ∂t_n and $\mu_{\mathbb{U}}(z_k)$ instead of ∂z_k. We have opted for the shorter more compact notation.

Thus $x(t_n)$ is a finite duration signal on a time scale $\mathbb{D}_N$ with graininess $\partial t_n = \mu(t_n)$. This imposes a frequency scale, $\mathbb{U}_N$, with values $X(z_k)$ and graininess ∂z_k given by (10). Thus

$$u_k = \begin{cases} 0 & ; \quad k = 0 \\ \sum_{\ell=1}^{k} \partial z_\ell = z_{k-1} + \partial z_k & ; \quad 1 \leq k \leq N. \end{cases} \tag{13}$$

define the point locations on the time scale $\mathbb{U}_N$. The image $X(z_k)$ is assigned to the point[5] u_k.

► *Conjugate Symmetry.* When $x(t_n)$ is real, $X^*(z_k) = X(z_k^*)$. *Proof*: The proof results immediately upon applying (2) to the TS-DFT definition in (11).

D. TS-DFT Transform Theorems

Here are some theorems that parallel the conventional DFT [16].

► *Area Theorem.* Since $t_0 = 0$,

$$X(z_0) = \sum_{n=0}^{N-1} x(t_n) \partial t_n$$

[5] An alternate notation might be $X(u_k)$ in lieu of $X(z_k)$. We choose to continue with the notation $X(z_k)$.

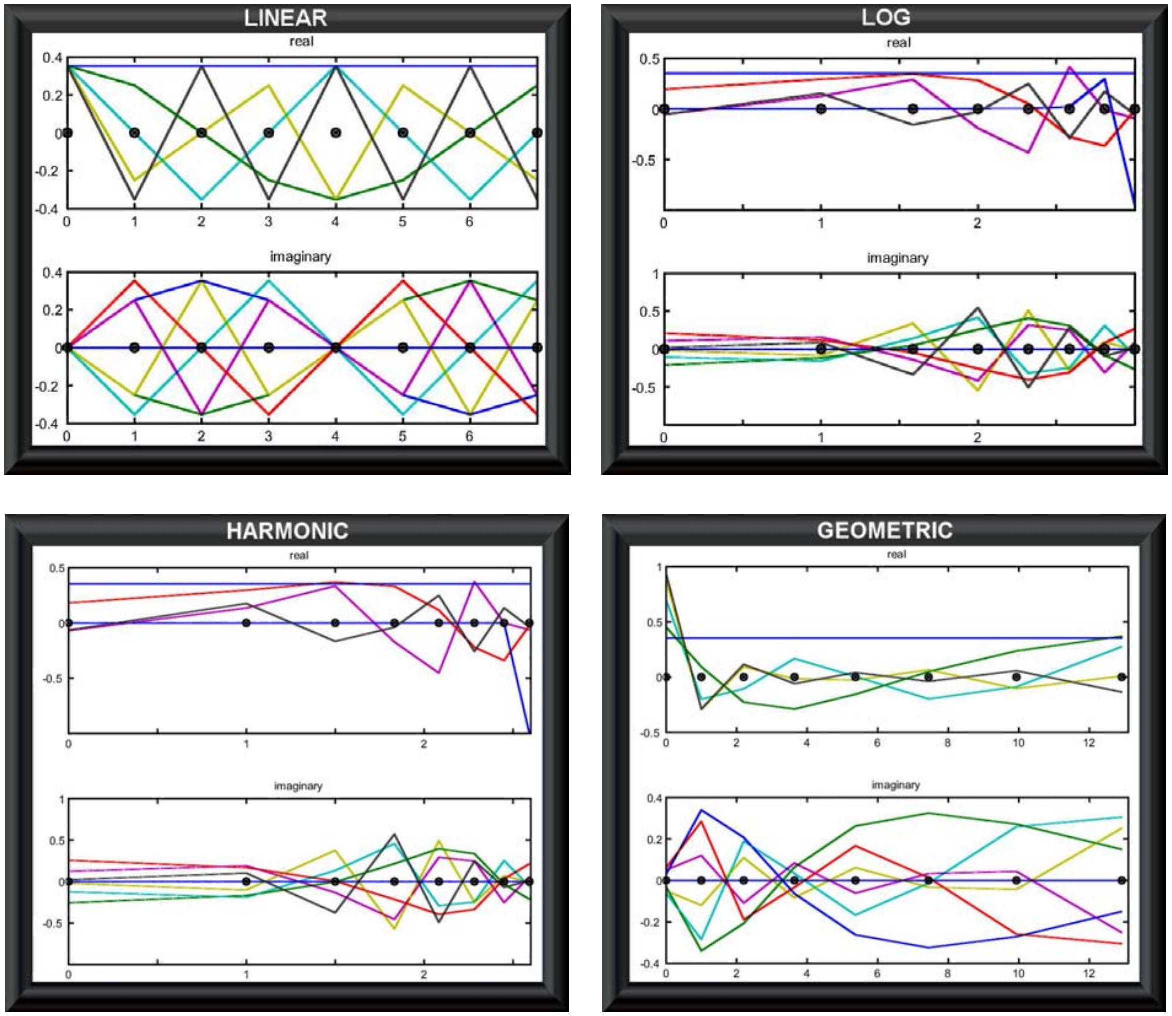

Fig. 4. The real (top) and imaginary (bottom) components of the orthonormalized basis functions for the linear time scale using $N = 8$. The LINEAR plots are the familiar sins and cosines of the DFT kernel. The GEOMETRIC time scale is for $q = 1.2$

Proof: Since $z_0 = 0$ and $e_z(0) = 1$, this follows immediately from the TS-DFT definition in (11). Likewise,

$$x(t_0) = \sum_{k=0}^{N-1} X(z_k)\partial z_k.$$

► *Conjugate Symmetry.* When $x(t_n)$ is real, $X^*(z_k) = X(z_k^*)$.
Proof: The proof results immediately upon applying (2) to the TS-DFT definition in (11).

► *Power Theorem.*

$$\sum_{n=0}^{N-1} x(t_n)h^*(t_n)\partial t_n = \sum_{k=0}^{N-1} X(z_k)H^*(z_k)\partial z_k. \qquad (14)$$

Proof: Follows from the definition of the TS-DFT in (11) and its inverse in (12).

$$\sum_{k=0}^{N-1} X(z_k)H^*(z_k)\partial z_k$$

$$= \sum_{k=0}^{N-1} X(z_k)\left[\sum_{n=0}^{N-1} h(t_n)e_{z_k}^*(t_n)\partial t_n\right]^* \partial z_k$$

$$= \sum_{n=0}^{N-1}\left[\sum_{k=0}^{N-1} X(z_k)\, e_{z_k}(t_n)\partial z_k\right] h^*(t_n)\partial t_n$$

$$= \sum_{n=0}^{N-1} x(t_n)h^*(t_n)\partial t_n$$

► *Parseval's Theorem* is a special case of the power theorem when $x = h$.

$$\sum_{n=0}^{N-1} \|x(t_n)\|^2\partial t_n = \sum_{k=0}^{N-1} \|X(z_k)\|^2\partial z_k.$$

► *Derivative theorem.*

$$x^{\Delta}(t_n) \leftrightarrow z_k X(z_k), \qquad (15)$$

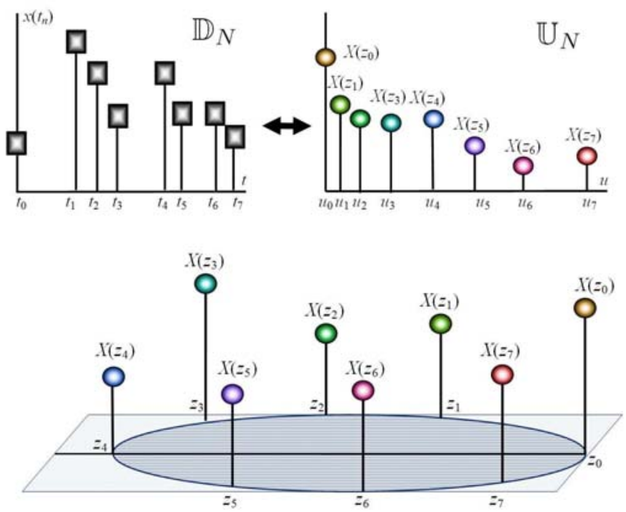

Fig. 5. A graphical illustration of the TS-DFT. On the upper left is a signal, $x(t_n)$, on a time scale $\mathbb{D}_N$ where, here, $N = 8$. The time scale $\mathbb{D}_N$ dictates the frequency roots, z_k, as illustrated in Figures 1, 2 and 3 and the exponential basis sets illustrated in Figure 4. The basis set applied to the signal $x(t_n)$ gives the values of the TS-DFT, namely $X(z_k)$, as illustrated in the bottom figure. The norms of the basis set components determines the ∂z_k's in (10) which, in turn, determines the frequency scale, $\mathbb{U}_N$, in (13). This is shown in the upper right.

Proof: From (12),

$$\begin{aligned} x^{\Delta}(t_n) &= \sum_{k=0}^{N-1} X(z_k)\, e^{\Delta}_{z_k}(t_n)\partial z_k \\ &= \sum_{k=0}^{N-1} [z_k X(z_k)]\; e_{z_k}(t_n)\partial z_k, \end{aligned}$$

from which (15) follows.

IV. Shifts on a Time Scale

► The *boxminus shift operator*, $\boxminus$, on an arbitrary function, $h(t_n)$ on $\mathbb{D}_N$, is defined by its TS-DFT.

$$h(t_n \boxminus t_m) \leftrightarrow H(z_k)e^{*}_{z_k}(t_m). \tag{16}$$

► *The Hilger delta* [6] is defined as

$$\delta(t_n) := \frac{\delta[n]}{\partial t_0}$$

where $\delta[n]$ is the *Kronecker delta*[6] and we have used $t_0 = 0$.
► The *TS-DFT of the Hilger delta* is

$$\delta(t_n) \leftrightarrow 1. \tag{17}$$

Proof: The proof follows directly from the TS-DFT in (11).
► *The shifted Hilger delta* and its TS-DFT is

$$\delta(t_n \boxminus t_m) = \frac{\delta[n-m]}{\partial t_m} \leftrightarrow e^{*}_{z_k}(t_m). \tag{18}$$

Proof: Follows from application of (16) to (17).

[6] $\delta[n] = 1$ for $n = 0$ and is otherwise zero.

TS-DFT	$x(t_n)$ on $\mathbb{D}_N$	$\leftrightarrow$	$X(z_k)$ on $\mathbb{U}_N$
transform	$X(z_k)$	$=$	$\sum x(t_n)e^{*}_{z_k}(t_n)\partial t_n$
inverse	$x(t_n)$	$=$	$\sum X(z_k)e_{z_k}(t_n)\partial z_k$
area theorem	$\sum x(t_n)\partial t_n$	$=$	$X(z_0)$
symmetry (x real)	$X^{*}(z_k)$	$=$	$X(z_k^{*})$
power theorem	$\sum x(t_n)h^{*}(t_n)\partial t_n$	$=$	$\sum X(z_k)H^{*}(z_k)\partial z_k$
Parseval's theorem	$\sum \lVert x(t_n)\rVert^2 \partial t_n$	$=$	$\sum \lVert X(z_k)\rVert^2 \partial z_k$
box minus shift theorem	$h(t_n \boxminus t_m)$	$\leftrightarrow$	$H(z_k)e^{*}_{z_k}(t_m)$
inverted $\boxminus$ shift theorem	$x^{*}(t_m \boxminus t_n)$	$\leftrightarrow$	$X^{*}(z_k)e^{*}_{z_k}(t_m)$
box minus theorem	$x^{*}(\boxminus t_n)$	$\leftrightarrow$	$X^{*}(z_k)$
box plus shift theorem	$h(t_n \boxplus t_m)$	$\leftrightarrow$	$H(z_k)e_{z_k}(t_m)$
convolution[a]	$x(t_n) * h(t_n)$	$\leftrightarrow$	$X(z_k)H(z_k)$
correlation[b]	$x(t_n) \star h(t_n)$	$\leftrightarrow$	$X^{*}(z_k)H(z_k)$
derivative	$x^{\Delta}(t_n)$	$\leftrightarrow$	$z_k X(z_k)$
frequency response	$h(t_n) * e_{z_k}(t_n)$	$=$	$H(z_k)e_{z_k}(t_n)$

TABLE I
SOME TS-DFT THEOREMS. ALL SUMS ARE FROM 0 TO $N-1$, *i.e* OVER n WE HAVE $\sum = \sum_{n=0}^{N-1}$ AND, OVER k, $\sum = \sum_{k=0}^{N-1}$. (A) CONVOLUTION IS DEFINED IN (28) AND (B) CORRELATION IN (30).

► *The sifting property of Hilger delta* follows immediately as

$$\sum_{m=0}^{N-1} x(t_m)\;\delta(t_n \boxminus t_m)\;\partial t_m = x(t_n). \tag{19}$$

► *Basis Exponential TS-DFT.* The TS-DFT of a basis exponential is

$$e_{z_\ell}(t_n) \leftrightarrow \delta(z_k \boxminus z_\ell). \tag{20}$$

where

$$\delta(z_k \boxminus z_\ell) = \frac{\delta[k-\ell]}{\partial z_\ell}$$

is the Hilger delta on the time scale $\mathbb{D}$.
A special case is for $t_n = t_0 = 0$.

$$1 \leftrightarrow \delta(z_k).$$

Proof: Substitute (20) into (11) and use the orthogonal property in (4).

DC Values	$e_{z_0}(t_n)$	=	1
	$e_{z_k}(t_0)$	=	1
Hilger delta	$\delta(t_n) := \delta[n]/\partial t_0$	$\leftrightarrow$	1
shifted Hilger delta	$\delta(t_n \boxminus t_m)$	$\leftrightarrow$	$e^*_{z_k}(t_m)$
sifting property	$\sum_{m=0}^{N-1} x(t_m)$		
	$\times\delta(t_n \boxminus t_m)\mu(t_m)$	=	$x(t_n)$
convolution identity	$x(t_n) * \delta(t_n)$	=	$x(t_n)$
one	1	$\leftrightarrow$	$\delta(z_k) = \delta[k]/\partial z_\ell$
basis exponential	$e_{z_\ell}(t_n)$	$\leftrightarrow$	$\delta(z_k \boxminus z_\ell)$
conjugate symmetry	$e^*_{z_k}(t_n)$	=	$e_{z_k^*}(t_n)$
box minus shift	$e_{z_k}(t_n \boxminus t_m)$	=	$e_{z_k}(t_n)e^*_{z_k}(t_m)$
	$e_{z_k}(\boxminus t_n)$	=	$e^*_{z_k}(t_n)$
	$e_{z_k}(t_n \boxminus t_n)$	=	$\lvert e_{z_k}(t_n)\rvert^2$
box plus shift	$e_{z_k}(t_n \boxplus t_m)$	=	$e_{z_k}(t_n)e_{z_k}(t_m)$
	$e_{z_k}(\boxplus t_n)$	=	$e_{z_k}(t_n)$
	$e_{z_k}(t_n \boxplus t_n)$	=	$(e_{z_k}(t_n))^2$

TABLE II
PROPERTIES OF EXPONENTIALS AND HILGER DELTAS.

► *Basis Exponential Shift.* The box minus basis exponential shift can be written as

$$e_{z_\ell}(t_n \boxminus t_m) = e_{z_\ell}(t_n)e^*_{z_\ell}(t_m). \tag{21}$$

Proof: Applying (16) to (20) gives

$$\begin{aligned} e_{z_\ell}(t_n \boxminus t_m) &\leftrightarrow e^*_{z_k}(t_m)\delta(z_k \boxminus z_\ell) \\ &= e^*_{z_\ell}(t_m)\delta(z_k \boxminus z_\ell) \end{aligned}$$

But, from (20),

$$e_{z_\ell}(t_n)e^*_{z_\ell}(t_m) \leftrightarrow e^*_{z_k}(t_m)\delta(z_k \boxminus z_\ell).$$

Since the transforms in both cases are the same, (21) follows.

Interpreting

$$e_{z_k}(\boxminus t_m) = e_{z_k}(0 \boxminus t_m),$$

it follows from the basis exponential shift identity in (21) that

$$e_{z_k}(\boxminus t_m) = e^*_{z_k}(t_m). \tag{22}$$

► The TS-DFT of an inverted box minus shift is

$$x^*(t_m \boxminus t_n) \leftrightarrow X^*(z_k)e^*_{z_k}(t_m) \tag{23}$$

Proof:

$$\begin{aligned} x(t_m \boxminus t_n) &= \sum_{k=0}^{N-1} X(z_k)e_{z_k}(t_m \boxminus t_n)\partial z_k \\ &= \sum_{k=0}^{N-1} \left[X(z_k)e_{z_k}(t_m)\right] e^*_{z_k}(t_n)\partial z_k \end{aligned}$$

Conjugating both sides gives (23).

A special case of (23) is

$$x^*(\boxminus t_n) \leftrightarrow X^*(z_k).$$

► *Boxplus Operation.* The *box plus* operation is defined as

$$x(t_m \boxplus t_n) := x\left(t_m \boxminus (\boxminus t_n)\right).$$

► *Boxplus Semigroup Property.*

$$e_{z_k}(t_n \boxplus t_m) = e_{z_k}(t_n)e_{z_k}(t_m). \tag{24}$$

Proof:

$$\begin{aligned} e_{z_k}(t_n \boxplus t_m) &= e_{z_k}\left(t_n \boxminus (\boxminus t_m)\right) \\ &= e_{z_k}(t_n)e^*_{z_k}(\boxminus t_m) \\ &= e_{z_k}(t_n)e_{z_k}(t_m). \end{aligned}$$

Interpreting $e_{z_k}(\boxplus t_m) = e_{z_k}(0 \boxplus t_m)$, it follows that

$$e_{z_k}(\boxplus t_m) = e_{z_k}(t_m). \tag{25}$$

► *Box Plus Commutivity.* The commutative property of the box plus shift is

$$x(t_n \boxplus t_m) = x(t_m \boxplus t_n). \tag{26}$$

Proof:

$$x(t_n \boxplus t_m) = \sum_{k=0}^{N-1} X(z_k)e_{z_k}(t_n \boxplus t_m)\partial z_k.$$

From (24),

$$e_{z_k}(t_n \boxplus t_m) = e_{z_k}(t_m \boxplus t_n).$$

From which (26) immediately follows.

► *Box Plus TS-DFT.* The TS-DFT of a box plus shift is

$$x(t_n \boxplus t_m) \leftrightarrow X(z_k)e_{z_k}(t_m). \tag{27}$$

Proof:

$$\begin{aligned} x(t_n \boxplus t_m) &= \sum_{k=0}^{N-1} X(z_k)e_{z_k}(t_n \boxplus t_m)\partial z_k \\ &= \sum_{k=0}^{N-1} \left[X(z_k)e_{z_k}(t_m)\right] e_{z_k}(t_n)\partial z_k \end{aligned}$$

from which (27) follows.

► *Box Plus Identity.*

$$x(\boxplus t_n) = x(t_n).$$

Proof: Using (25),

$$\begin{aligned} x(\boxplus t_n) &= \sum_{k=0}^{N-1} X(z_k) e_{z_k}(\boxplus t_n) \partial z_k \\ &= \sum_{k=0}^{N-1} X(z_k) e_{z_k}(t_n) \partial z_k = x(t_n). \end{aligned}$$

V. Time Scale Convolution and Correlation

► *Discrete time scale convolution* between two functions is defined as

$$x(t_n) * h(t_n) := \sum_{m=0}^{N-1} x(t_m) h(t_n \boxminus t_m) \partial t_m. \quad (28)$$

► *The TS-DFT of a convolution* is the product of the transforms.

$$x(t_n) * h(t_n) \leftrightarrow X(z_k) H(z_k). \quad (29)$$

Proof: Let $y = x * h$. Then

$$\begin{aligned} Y(z_k) &= \sum_{n=0}^{N-1} y(t_n) e^*_{z_k}(t_n) \partial t_n \\ &= \sum_{n=0}^{N-1} \left[\sum_{m=0}^{N-1} x(t_m) h(t_n \boxminus t_m) \partial t_m \right] e^*_{z_k}(t_n) \partial t_n \\ &= \sum_{m=0}^{N-1} x(t_m) \left[\sum_{n=0}^{N-1} h(t_n \boxminus t_m) e^*_{z_k}(t_n) \partial t_n \right] \partial t_m \\ &= \sum_{m=0}^{N-1} x(t_m) \left[H(z_k) e^*_{z_k}(t_m) \right] \partial t_m \\ &= \left[\sum_{m=0}^{N-1} x(t_m) e^*_{z_k}(t_m) \partial t_m \right] H(z_k) = X(z_k) H(z_k). \end{aligned}$$

► *Discrete time scale convolution* is

- commutative

$$x * h = h * x,$$

- associative

$$g * (h * x) = (g * h) * x,$$

- and distributive over addition

$$x * (g + h) = x * g + x * h.$$

Proof: The proof follows immediately from the TS-DFT of a convolution in (29).

► *Discrete time scale correlation* between two functions is defined as

$$x(t_n) \star h(t_n) := \sum_{m=0}^{N-1} x^*(t_m) h(t_n \boxplus t_m) \partial t_m. \quad (30)$$

► *Transformation of Correlation.*

$$x(t_n) \star h(t_n) \leftrightarrow X^*(z_k) H(z_k). \quad (31)$$

Proof:

$$\begin{aligned} x(t_n) \star h(t_n) &= \sum_{m=0}^{N-1} x^*(t_m) h(t_n \boxplus t_m) \partial t_m \\ &\leftrightarrow \sum_{n=0}^{N-1} \left[\sum_{m=0}^{N-1} x^*(t_m) h(t_n \boxplus t_m) \partial t_m \right] \\ &\quad \times \; e^*_{z_k}(t_n) \partial t_n \\ &= \sum_{m=0}^{N-1} x^*(t_m) \\ &\quad \times \left[\sum_{n=0}^{N-1} h(t_n \boxplus t_m) e^*_{z_k}(t_n) \partial t_n \right] \partial t_m \\ &= \sum_{m=0}^{N-1} x^*(t_m) \left[H(z_k) e_{z_k}(t_m) \right] \partial t_m \\ &= H(z_k) \left[\sum_{m=0}^{N-1} x(t_m) e^*_{z_k}(t_m) \partial t_m \right]^* \quad (32) \end{aligned}$$

from which (31) immediately follows.

► An alternate expression for correlation in (30) is

$$x(t_n) \star h(t_n) := \sum_{m=0}^{N-1} x^*(t_m \boxminus t_n) h(t_m) \partial t_m. \quad (33)$$

Proof: Using the power theorem

$$\begin{aligned} &\sum_{m=0}^{N-1} x^*(t_n \boxminus t_m) h(t_n) \partial t_n \\ &= \sum_{k=0}^{N-1} \left[X(z_k) e^*_{z_k}(t_n) \right]^* H(z_k) \partial z_k \\ &= \sum_{k=0}^{N-1} \left[X^*(z_k) H(z_k) \right] e_{z_k}(t_n) \partial z_k \end{aligned}$$

which is the same result as (31).

► *Correlation obeys the following laws.*

1) *Reversal.* If $y = x \star h$, and $\lambda = h \star x$, then

$$\lambda(t_n) = y^*(\boxminus t_n).$$

2) Order

$$\begin{aligned} g \star (h \star x) &= h \star (g \star x) \\ &= (h * g) \star x. \end{aligned}$$

3) Distributive over addition

$$x \star (g + h) = x \star g + x \star h.$$

Proof:

1) Let $y(t_n) = x(t_n) \star h(t_n)$. Then

$$y(t_n) = \sum_{k=0}^{N-1} X(z_k) H^*(z_k) e_{z_k}(t_n) \partial z_k,$$

I.	convolution	$x * h$	$\leftrightarrow$	XH
	correlation	$x \star h$	$\leftrightarrow$	X^*H
II.	commutative	$h * x$	$=$	$x * h$
	associative	$g * (h * x)$	$=$	$(g * h) * x$
	distributive	$x * (g + h)$	$=$	$x * g + x * h$
III.	reversal	$y(t_n) = x \star h$	$\rightarrow$	$h \star x = y^*(\boxminus t_n)$
	order	$g \star (h \star x)$	$=$	$h \star (g \star x)$
			$=$	$(h * g) \star x$
	distributive	$x \star (g + h)$	$=$	$x \star g + x \star h$
IV.	correlation	$x(t_n) \star h(t_n)$	$=$	$x^*(\boxminus t_n) * h(t_n)$
	convolution	$x(t_n) * h(t_n)$	$=$	$x^*(\boxminus t_n) \star h(t_n)$

TABLE III
PROPERTIES OF CONVOLUTION AND CORRELATION. (I) TS-DFT. (II) CONVOLUTION. (III) CORRELATION. (IV) RELATIONSHIPS BETWEEN CONVOLUTION AND CORRELATION.

and

$$y^*(t_n) = \sum_{k=0}^{N-1} X^*(z_k)H(z_k)e^*_{z_k}(t_n)\partial z_k.$$

Thus

$$\begin{aligned} y^*(\boxminus t_n) &= \sum_{k=0}^{N-1} X(z_k)H^*(z_k)e^*_{z_k}(\boxminus t_n)\partial z_k \\ &= \sum_{k=0}^{N-1} H^*(z_k)X(z_k)e_{z_k}(t_n)\partial z_k. \\ &= h(t_n) \star x(t_n) = \lambda(t_n). \end{aligned}$$

2) Follows immediately from

$$g \star (h \star x) \leftrightarrow G^*H^*X.$$

3) Follows immediately from the definition of correlation in (30).

► *The relationship between convolution and correlation.*

$$x(t_n) \star h(t_n) = x^*(\boxminus t_n) * h(t_n)$$

and

$$x(t_n) * h(t_n) = x^*(\boxminus t_n) \star h(t_n).$$

$$\begin{aligned} g \star (h \star x) &= h \star (g \star x) \\ &= (h * g) \star x. \end{aligned}$$

VI. FINAL REMARKS

The TS-DFT establishes a generalization of the DFT to cases where points are not spaced uniformly. The generalization preserves properties of the conventional DFT, including derivative, shift, convolution and correlation relationships. The TS-DFT and its inverse are defined in (11) and (12). TS-DFT theorems are listed in Tables I, II and III.

Much work remains in the development of the foundations of the TS-DFT. The study of the mapping of time scales, $\mathbb{D}_N$, to frequency scales, $\mathbb{U}_N$, remains, for example, an open problem, as does filtering, sampling, and the mechanics of convolution and correlation [15].

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