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Algebraic and Dynamic Lyapunov Equations on Time Scales

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***Abstract*—We revisit the canonical continuous-time and discrete-time matrix algebraic and matrix differential equations that play a central role in Lyapunov-based stability arguments. The goal is to generalize and extend these types of equations and subsequent analysis to dynamical systems on domains other than $\mathbb{R}$ or $\mathbb{Z}$, called "time scales", e.g. nonuniform discrete domains or domains consisting of a mixture of discrete and continuous components. In particular, we compare and contrast a generalization of the algebraic Lyapunov equation and the dynamic Lyapunov equation in this time scales setting.**

I. Introduction

One of the most widely used tools for investigating the stability of linear systems is the Second (Direct) Method of Lyapunov, presented in his dissertation of 1892. An excellent survey of Lyapunov's work can be found in [25]. The advantage of this particular approach is that it allows one to infer the stability of differential (and difference) equations without explicit knowledge of solutions.

We begin with a review of some tools used in Lyapunov's Second (Direct) Method in the context of linear differential equations on $\mathbb{R}$ and linear difference equations on $\mathbb{Z}$. Then, building on the work of DaCunha [11] we proceed to unify and extend this theory (from the algebraic equation to the dynamic equation) for application to dynamic linear systems defined on arbitrary time scale domains.

A time scale, $\mathbb{T}$, is any closed subset of the real line. Continuous time, $\mathbb{R}$, and discrete time, $\mathbb{Z}$, are special cases. Time scales have been useful in the analysis of switched linear systems [27], Fourier analysis of signals on time scales [26], and dynamic programming [35]. Control theory, including stability analysis, has recently been applied to signals on time scales [3], [4], [11], [12], [30], [31]. A brief introduction to time scales is in the Appendix. An online tutorial is available [39] and a detailed development is available in Bohner and Peterson [8].

This work is but a small subset of [33]; further application of generalized Lyapunov equations to the stability and control of switched linear systems on time scales can be found there.

II. Stability of Continuous-time Systems

We begin by considering the familiar linear state equation

$$\dot{x}(t) = A(t)x(t), \tag{II.1}$$

for $A \in \mathbb{R}^{n\times n}$, and $t \in \mathbb{R}$. Without loss of generality, assume that (II.1) has equilibrium $x = 0$. To establish asymptotic stability of this equilibrium, a standard approach is to seek a quadratic Lyapunov function associated with (II.1) as follows.

Let $V(x(t)) = x^T(t)Px(t)$. Then

$$\dot{V}(x(t)) = x^T(t)[A^T(t)P + PA(t)]x(t),$$

and thus it is sufficient to seek a $P \in \mathcal{S}_n^+$ which satisfies the *continuous-time algebraic Lyapunov equation*

$$A^T(t)P + PA(t) = -M(t), \tag{CALE}$$

where $M(t) \in \mathcal{S}_n^+$ is given. Here, $\mathcal{S}_n^+$ ($\mathcal{S}_n^-$) denotes the set of real, $n \times n$ positive (negative) definite symmetric matrices.

Theorem II.1. [1], [34] *The unique solution of* (CALE) *is given by*

$$P(t) = \int_{t_0}^{\infty} \Phi_A^T(s,t_0)M(t)\Phi_A(s,t_0)\,ds,$$

where $\Phi_A(t,t_0)$ is the transition matrix for system (II.1). *Moreover, $P \in \mathcal{S}_n^+$ whenever $M(t) \in \mathcal{S}_n^+$.*

On the other hand, suppose we seek a Lyapunov function of the form $V(x(t)) = x^T(t)P(t)x(t)$, the emphasis being that P is time varying. Then

$$\dot{V}(x(t)) = x^T(t)[A^T(t)P(t) + P(t)A(t) + \dot{P}(t)]x(t),$$

and so we seek a $P(t) \in \mathcal{S}_n^+$ which satisfies the *continuous-time differential Lyapunov equation*

$$A^T(t)P(t) + P(t)A(t) + \dot{P}(t) = -M(t), \tag{CDLE}$$

where $M(t) \in \mathcal{S}_n^+$ is specified.

Theorem II.2. [1] *The unique solution of* (CDLE) *subject to the initial condition $P(t_0) = P_0$ is given by*

$$P(t) = \Phi_A^{-T}(t,t_0)P(t_0)\Phi_A^{-1}(t,t_0) - \int_{t_0}^{t} \Phi_A^T(t,s)M(s)\Phi_A(t,s)\,ds. \tag{II.2}$$

Moreover, $P(t) \in \mathcal{S}_n^+$ whenever $M(t) \in \mathcal{S}_n^+$.

From the derivation of (CALE), we see that the constant solution of (CALE) is in fact a steady state solution of (CDLE) provided the initial condition $P(t_0)$ is chosen to be this constant solution of (CALE).

In light of the remark above, it is not surprising that (CALE) takes precedence over (CDLE) in the literature—matrix *algebraic* equations are much easier to solve than matrix *differential* equations. Even so, (CDLE) is interesting in its own right. There are problems in which the time varying nature of a system makes (CDLE) useful, especially in the context of periodic systems [1], [14]. However, when the time dependence is not of interest, or has minimal impact on the system (e.g., slowly time varying systems), it is more efficient to consider the algebraic equation (CALE) to obtain simpler, steady state solutions of the differential equation (CDLE).

III. Stability of Discrete-time Systems

Next, we quickly summarize the (uniform) discrete analogues of the concepts from the last section. Let $t \in \mathbb{Z}$ and consider the discrete linear system

$$\Delta x(t) = A(t)x(t), \tag{III.1}$$

for $A \in \mathbb{R}^{n\times n}$, and $t \in \mathbb{Z}$, where $\Delta x(t) := x(t+1) - x(t)$ is the usual forward difference operator. Note that we can rearrange (III.1), define $A_R(t) := A(t) + I$, and write (III.1) in its (possibly more familiar) equivalent recursive form

$$x(t+1) = A_R(t)x(t).$$

Let $V(x) = x^T(t)Px(t)$. Then

$$\Delta V(x(t)) = x^T(t)[A^T(t)P + PA(t) + A^T(t)PA(t)]x(t).$$

Therefore, to use Lyapunov's Theorem it is sufficient to seek a constant $P \in \mathcal{S}_n^+$ satisfying the *discrete-time algebraic Lyapunov equation*,

$$A^T(t)P + PA(t) + A^T(t)PA(t) = -M(t), \tag{DALE}$$

for a given $M(t) \in \mathcal{S}_n^+$.

Equivalently, (DALE) has the recursive form

$$A_R^T(t)PA_R(t) - P = -M(t), \tag{DALEr}$$

where $A_R(t) := A(t) + I$. This form seems to be more common in the literature on Lyapunov analysis of discrete linear systems [29], [34], [36].

Theorem III.1. [1], [34] *For $A(t) \equiv A$ and $M(t) \equiv M$, the unique solution of* (DALEr) *is the constant*

$$P = \sum_{j=0}^{\infty} (A_R^T)^j M A_R^j.$$

Moreover, $P \in \mathcal{S}_n^+$ whenever $M \in \mathcal{S}_n^+$. The sum converges provided $|\lambda| < 1$ for all $\lambda \in \operatorname{spec} A$.

On the other hand, if we start with a Lyapunov candidate of the form $V(x(t)) = x^T(t)P(t)x(t)$, we obtain

$$\begin{aligned}\Delta V(x(t)) = x^T(t)[&A^T(t)P(t+1) + P(t+1)A(t)\\ &+ A^T(t)P(t+1)A(t) + \Delta P(t)]x(t),\end{aligned}$$

so this time we seek a $P(t) \in \mathcal{S}_n^+$ satisfying the *discrete-time difference Lyapunov equation*,

$$\begin{aligned}&A^T(t)P(t+1) + P(t+1)A(t)\\ &\quad + A^T(t)P(t+1)A(t) + \Delta P(t) = -M(t),\end{aligned} \tag{DDLE}$$

for a given $M(t) \in \mathcal{S}_n^+$. The equivalent recursive form that appears more frequently in the literature is

$$A_R^T(t)P(t+1)A_R(t) - P(t) = -M(t). \tag{DDLEr}$$

Theorem III.2. [37] *The unique solution of* (DDLE) *satisfying $P(t_0) = P_0$ is given by*

$$P(t) = \Phi_A^{-T}(t,t_0)P(t_0)\Phi_A^{-1}(t,t_0) - \sum_{s=t_0}^{t} \Phi_A^T(s,t)M(s)\Phi_A(s,t), \tag{III.2}$$

where $\Phi_A(t,t_0)$ is the transition matrix for (III.1). *Moreover, $P(t) \in \mathcal{S}_n^+$ whenever $M(t) \in \mathcal{S}_n^+$.*

From the derivation of (DALE), we see that the constant solution of (DALE) is in fact a steady state solution of (DDLE) provided the initial condition $P(t_0)$ is chosen to be this constant solution of (DALE).

(DDLE) and its corresponding solution are most useful for stability analysis of linear systems when the time dependent nature of the equation is relevant. For example, (DDLE) is frequently seen in the context of the analysis of discrete periodic systems [5], [37], [38]. However, when the time-dependent aspect is not of interest (e.g., $A(t) \equiv A$ or A slowly time varying), it makes sense to simplify the problem (as we did in the continuous case) to an algebraic problem by seeking steady state solutions of (DDLE).

IV. A Unified Approach to Lyapunov Stability

Now we turn our attention to generalizing the previous concepts from $\mathbb{R}$ and $\mathbb{Z}$ to more general time domains (time scales), e.g. nonuniform discrete sets or sets with a combination of discrete and continuous components. In case the reader is unfamiliar with time scales, we have included an appendix with with a brief overview of time scales and basic time scales tools needed in this brief paper.

A. Stability of Dynamic Systems on Time Scales

Let $\mathbb{T}$ be a time scale, unbounded above with bounded graininess. We consider the dynamic linear system

$$x^{\Delta}(t) = A(t)x(t), \tag{IV.1}$$

for $A(t) \in \mathcal{R}(\mathbb{R}^{n\times n})$, and $t \in \mathbb{T}$, where x^{Δ} is the generalized Δ-derivative. Notice that (IV.1) reduces to the familiar systems in (II.1) and (III.1) when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively. Having seen how Lyapunov's Second Method allowed us to analyze stability of these systems on the familiar continuous and discrete domains, we would now like to apply this method to the analysis of (IV.1) defined on an arbitrary time scale.

In 2003, Pötzsche, Siegmund, and Wirth [32] developed spectral criteria for the exponential stability of (IV.1) in the scalar case and in the case that $A(t) \equiv A$. Then DaCunha

[11] extended these results by adapting the Second Method of Lyapunov to the analysis of a certain class of nonautonomous linear systems (slowly time varying systems) defined on time scales. Here we will explore and further extend those results, ultimately developing and solving a *time scale dynamic Lyapunov equation* which unifies the familiar Lyapunov equations on $\mathbb{R}$ and $\mathbb{Z}$ and is applicable to a much broader class of systems than those DaCunha studied.

We begin by giving generalized characterizations of stability for dynamic linear systems on time scales and then review a few necessary results from existing theory.

Definition IV.1. For $t \in \mathbb{T}$, an equilibrium $x = 0$ of (IV.1) is:

- *Lyapunov stable* if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $\|x(t_0)\| < \delta$, then $\|x(t)\| < \varepsilon$ for all $t \geq t_0$.
- *asymptotically stable* if it is Lyapunov stable and there exists a $\delta > 0$ such that if $\|x(t_0)\| < \delta$, then $\lim_{t\to\infty} \|x(t)\| = 0$.
- *exponentially stable* if it is asymptotically stable and there exist constants γ, λ, $\delta > 0$ with $-\lambda \in \mathcal{R}^+$ such that if $\|x(t_0)\| < \delta$, then $\|x(t)\| \leq \gamma e_{-\lambda}(t, t_0)\|x(t_0)\|$ for all $t \geq t_0$.

These characterizations of stability for system (IV.1) are generalizations of the corresponding characterizations of stability for systems defined on $\mathbb{R}$ and $\mathbb{Z}$. Specifically, the condition that $-\lambda \in \mathcal{R}^+$ reduces to $\lambda > 0$ and $0 < \lambda < 1$ for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively.

Define the (open) Hilger circle[1] via

$$\mathcal{H}_{\mu(t)} := \left\{ z \in \mathbb{C}_\mu : \left| z + \frac{1}{\mu(t)} \right| < \frac{1}{\mu(t)} \right\}.$$

Hoffacker and Gard [15] showed that, if $0 \leq \mu(t) \leq \mu_{\max}$ for all $t \in \mathbb{T}$, then there is a region $\mathcal{H}_{\min} \subset \mathbb{C}$, corresponding to $\mu_{\max}$ and given by

$$\mathcal{H}_{\min} := \left\{ z \in \mathbb{C}_{\mu_{\max}} : \left| z + \frac{1}{\mu_{\max}} \right| < \frac{1}{\mu_{\max}} \right\},$$

such that $\operatorname{spec} A \subset \mathcal{H}_{\min}$ is a sufficient (but not necessary) condition for the exponential stability of (IV.1) when $A(t) \equiv A$. Our goal here is to provide a similar sufficient condition for stability of (IV.1) for nonconstant system matrices by appealing to Lyapunov methods.

Definition IV.2. A function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *generalized* or *time scale Lyapunov function* for system (IV.1) if

(i) $V(x) \geq 0$ with equality if and only if $x = 0$, and
(ii) $V^\Delta(x(t)) \leq 0$.

Theorem IV.1 (Lyapunov's Second Theorem on $\mathbb{T}$, [23], [24])**.** *Given system* (IV.1) *with equilibrium $x = 0$, if there exists an associated Lyapunov function $V(x)$, then $x = 0$ is Lyapunov stable. Furthermore, if $V^\Delta(x(t)) < 0$, then $x = 0$ is asymptotically stable.*

[1]More appropriately, the *Hilger disk*, but this abuse of language is established in the literature now.

B. The Time Scale Algebraic Lyapunov Equation

We begin with the quadratic Lyapunov function candidate $V(x(t)) = x^T(t)Px(t)$. Differentiating with respect to $t \in \mathbb{T}$ yields

$$V^\Delta(x(t) = x^T[A^T(t)P + PA(t) + \mu(t)A^T(t)PA(t)]x.$$

Therefore, we seek a solution $P(t) \in \mathcal{S}_n^+$ to the *time scale algebraic Lyapunov equation*

$$A^T(t)P + PA(t) + \mu(t)A^T(t)PA(t) = -M(t), \quad \text{(TSALE)}$$

for a given $M(t) \in \mathcal{S}_n^+$.

This algebraic equation unifies the matrix algebraic Lyapunov equations on $\mathbb{R}$ and $\mathbb{Z}$ discussed earlier: (TSALE) reduces to (CALE) on $\mathbb{T} = \mathbb{R}$ and (DALE) on $\mathbb{T} = \mathbb{Z}$. However, the solutions to (TSALE) on an arbitrary time scale are fundamentally different than solutions to (CALE) and (DALE)—they are generally time varying—as the next theorem reveals.

Theorem IV.2 (Closed Form Solution of (TSALE), [11])**.** *For each fixed $t \in \mathbb{T}$, define*

$$\mathbb{S}_t := \begin{cases} \mu(t)\mathbb{N}_0, & \mu(t) \neq 0, \\ \mathbb{R}_0^+, & \mu(t) = 0. \end{cases}$$

The unique solution of (TSALE) *is given by*

$$P(t) = \int_{\mathbb{S}_t} \Phi_A^T(s, 0)M(t)\Phi_A(s, 0)\Delta s, \qquad \text{(IV.2)}$$

which converges provided $\lambda \in \mathcal{H}_{\min}$ for all $\lambda \in \operatorname{spec} A$ and all $t \geq T$. Moreover, $P(t) \in \mathcal{S}_n^+$ whenever $M(t) \in \mathcal{S}_n^+$.

The upshot here is that even though (IV.2) is a *bona fide* solution to (TSALE), it does not (in general) lead to constant solutions P—and P was assumed to be constant in the Lyapunov candidate at the start of this subsection. Thus, (TSALE) is not a "legitimate" Lyapunov equation in the sense that it is not an appropriate equation to use in a search for Lyapunov function candidates (even when $A(t) \equiv A$). We are forced to seek a Lyapunov function candidate with a time varying P, which we do next.

C. The Time Scale Dynamic Lyapunov Equation

This time we let $V(x(t)) = x^T(t)P(t)x(t)$. Differentiating with respect to $t \in \mathbb{T}$ yields

$$\begin{aligned} V^\Delta(x(t)) = x^T[&A^T(t)P(t) + P(t)A(t) \\ &+ \mu(t)A^T(t)P(t)A(t) \\ &+ (I + \mu(t)A^T(t))P^\Delta(t)(I + \mu(t)A(t))]x. \end{aligned}$$

Therefore, we seek a solution $P(t) \in \mathcal{S}_n^+$ of the *time scale dynamic Lyapunov equation*

$$\begin{aligned} A^T(t)P(t) + P(t)A(t) + \mu(t)A^T(t)P(t)A(t)& \\ + G^T(t)P^\Delta(t)G(t) = -M(t),& \end{aligned} \qquad \text{(TSDLE)}$$

where $G(t) := I + \mu(t)A(t)$ and $M(t) \in \mathcal{S}_n^+$ is specified. This equation unifies the matrix differential and difference

Lyapunov equations on $\mathbb{R}$ and $\mathbb{Z}$ discussed earlier: (TSDLE) reduces to (CDLE) on $\mathbb{T} = \mathbb{R}$ and to (DDLE) on $\mathbb{T} = \mathbb{Z}$. Just as importantly, (TSDLE) also generalizes those types of equations to arbitrary time scales.

Theorem IV.3 (Closed Form Solution of (TSDLE))**.** *The unique solution of* (TSDLE) *subject to the initial condition* $P(t_0) = P_0$ *is*

$$\begin{aligned} P(t) &= \Phi_A^{-T}(t,t_0)P(t_0)\Phi_A^{-1}(t,t_0) \\ &- \Phi_A^{-T}(t,t_0)\left[\int_{t_0}^{t} \Phi_A^T(s,t_0)M(s)\Phi_A(s,t_0)\,\Delta s\right]\Phi_A^{-1}(t,t_0), \end{aligned} \tag{IV.3}$$

where $\Phi_A(t,t_0)$ *is the transition matrix for* (IV.1)*.*

We have already seen that (TSDLE) is a generalized form unifying the Lyapunov *equations* (CDLE) and (DDLE) for systems on $\mathbb{R}$ and $\mathbb{Z}$, respectively, and extending them to arbitrary time domains. Equation (IV.3) is also a generalized form unifying the *solutions* of (CDLE) and (DDLE) since (IV.3) becomes (II.2) on $\mathbb{R}$ and (III.2) on $\mathbb{Z}$.

At this point, the analysis diverges from that of $\mathbb{R}$ and $\mathbb{Z}$: the solution of (TSALE) is time varying even when $A(t) \equiv A$ and $M(t) \equiv M$ are constant, since the domain of integration in the solution depends on $\mu(t)$. Only when operating on time scales of constant graininess, such as $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{T} = h\mathbb{Z}$, is the solution of (TSALE) constant. On $\mathbb{R}$ and $\mathbb{Z}$, (IV.2) agrees with the solutions of (CALE) and (DALE) and gives a steady state solution of (CDLE) and (DDLE) as desired. However, on an arbitrary $\mathbb{T}$, (IV.2) is not a stationary solution of (TSDLE) because $P(t)$ is not constant.

This underscores a crucial difference between algebraic Lyapunov equations on general time scales versus their $\mathbb{R}$ and $\mathbb{Z}$ counterparts: only when the time scale has constant graininess is a solution to an algebraic Lyapunov equation also a (stationary) solution to the dynamic Lyapunov equation.

Although the closed form solution of (TSDLE) is now available, to be useful in a Lyapunov argument, we must know the existence of a solution in the space $\mathcal{S}_n^+$; it is not clear from (IV.3) that $P(t) \in \mathcal{S}_n^+$. The next result clarifies the situation.

Theorem IV.4. *In Theorem IV.3, if the initial condition is*

$$P(t_0) = P_0 := \int_{t_0}^{\infty} \Phi_A^T(s,t_0)M(s)\Phi_A(s,t_0)\,\Delta s, \tag{IV.4}$$

then (IV.3) *becomes*

$$P(t) = \int_{t}^{\infty} \Phi_A^T(s,t)M(s)\Phi_A(s,t)\,\Delta s. \tag{IV.5}$$

The choice of initial condition given in Theorem IV.4 is necessary. Choosing any other initial condition results in the norm of (IV.3) being unbounded as $t \to \infty$.

With this choice of initial condition, $P(t)$ reduces to a useful form, especially for aligning our results with the literature, at least in the following sense. If $\mathbb{T}$ has constant graininess, $t_0 = 0$, and M is constant, then the initial matrix P_0 in (IV.4) is in fact a (constant) solution of (TSALE) which is in turn a stationary solution of (TSDLE). Therefore, precisely this choice of initial matrix (under the assumptions above) produces steady-state solutions of (TSDLE) from its algebraic counterpart (TSALE). This is what happens on $\mathbb{R}$ and $\mathbb{Z}$ but fails on general time scales.

Appendix

A. What Are Time Scales?

The theory of time scales springs from the 1988 doctoral dissertation of Stefan Hilger [20] that resulted in his seminal paper [19]. These works aimed to unify various overarching concepts from the (sometimes disparate) theories of discrete and continuous dynamical systems [28], but also to extend these theories to more general classes of dynamical systems. From there, time scales theory advanced fairly quickly, culminating in the excellent introductory text by Bohner and Peterson [7] and the more advanced monograph [8]. A succinct survey on time scales can be found in [2].

On the other hand, the form of (IV.5) allows us to deduce $P(t) \in \mathcal{S}_n^+$ whenever $M(t) \in \mathcal{S}_n^+$, which is essential if we want to apply Theorem IV.1.

A *time scale* $\mathbb{T}$ is any nonempty, (topologically) closed subset of the real numbers $\mathbb{R}$. Thus time scales can be (but are not limited to) any of the usual integer subsets (e.g. $\mathbb{Z}$ or $\mathbb{N}$), the entire real line $\mathbb{R}$, or any combination of discrete points unioned with closed intervals. For example, if $q > 1$ is fixed, the *quantum time scale* $\overline{q^{\mathbb{Z}}}$ is defined as

$$\overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}.$$

The quantum time scale appears throughout the mathematical physics literature, where the dynamical systems of interest are the q-difference equations [6], [9], [10]. Another interesting example is the *pulse time scale* $\mathbb{P}_{a,b}$ formed by a union of closed intervals each of length a and gap b:

$$\mathbb{P}_{a,b} := \bigcup_k \left[k(a+b), k(a+b)+a\right].$$

This time scale is used to study duty cycles of various waveforms. Other examples of interesting time scales include any collection of discrete points sampled from a probability distribution, any sequence of partial sums from a series with positive terms, or even the famous Cantor set.

The bulk of engineering systems theory to date rests on two time scales, $\mathbb{R}$ and $\mathbb{Z}$ (or more generally $h\mathbb{Z}$, meaning discrete points separated by distance h). However, there are occasions when necessity or convenience dictates the use of an alternate time scale. The question of how to approach the study of dynamical systems on time scales then becomes relevant, and in fact the majority of research on time scales so far has focused on expanding and generalizing the vast suite of tools available to the differential and difference equation theorist. We now briefly outline the portions of the time scales theory that are needed for this paper to be as self-contained as is practically possible.

TABLE I
CANONICAL TIME SCALES COMPARED TO THE GENERAL CASE.

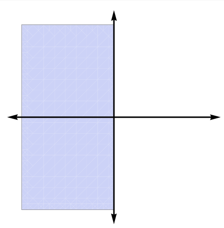
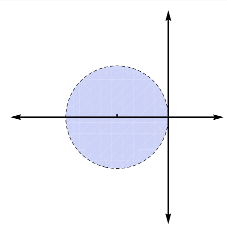
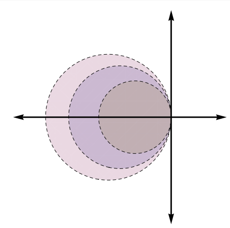
	continuous	(uniform) discrete	time scale
domain	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{T}$
forward jump	$\sigma(t) \equiv t$	$\sigma(t) \equiv t+1$	$\sigma(t)$ varies
step size	$\mu(t) \equiv 0$	$\mu(t) \equiv 1$	$\mu(t)$ varies
differential operator	$\dot{x}(t) := \lim\limits_{h\to 0} \dfrac{x(t+h)-x(t)}{h}$	$\Delta x(t) := x(t+1) - x(t)$	$x^{\Delta}(t) := \lim\limits_{\mu^*(t)\searrow\mu(t)} \dfrac{x(\sigma(t)) - x(t)}{\mu^*(t)}$
canonical equation	$\dot{x}(t) = Ax(t)$	$\Delta x(t) = Ax(t)$	$x^{\Delta}(t) = Ax(t)$
LTI stability region in $\mathbb{C}$			

TABLE II
DIFFERENTIAL AND INTEGRAL OPERATORS ON TIME SCALES.

time scale	differential operator	notes	integral operator	notes
$\mathbb{T}$	$x^{\Delta}(t) := \lim\limits_{\mu^*(t)\searrow\mu(t)} \dfrac{x(\sigma(t)) - x(t)}{\mu^*(t)}$	generalized derivative	$\int_a^b f(t)\,\Delta t$	generalized integral
$\mathbb{R}$	$x^{\Delta}(t) = \lim\limits_{h\to 0} \dfrac{x(t+h)-x(t)}{h}$	standard derivative	$\int_a^b f(t)\Delta t = \int_a^b f(t)\,dt$	standard Lebesgue integral
$\mathbb{Z}$	$x^{\Delta}(t) = \Delta x(t) := x(t+1) - x(t)$	forward difference	$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$	summation operator
$h\mathbb{Z}$	$x^{\Delta}(t) = \Delta_h x(t) := \frac{x(t+h)-x(t)}{h}$	h-forward difference	$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-h} f(t)h$	h-summation
$\overline{q^{\mathbb{Z}}}$	$x^{\Delta}(t) = \Delta_q x(t) := \frac{x(qt)-x(t)}{(q-1)t}$	q-difference	$\int_a^b f(t)\Delta t = \sum_{t=a}^{b/q} \frac{f(t)}{(q-1)t}$	q-summation
$\mathbb{P}_{a,b}$	$x^{\Delta}(t) = \begin{cases} \frac{dx}{dt}, & \sigma(t) = t, \\ \frac{x(t+b)-x(t)}{b}, & \sigma(t) > t \end{cases}$	pulse derivative	$\int_I f(t)\,\Delta t = \begin{cases} \int_I f(t)\,dt, & \sigma(t) = t, \\ f(t)\mu(t), & \sigma(t) > t \end{cases}$	pulse integral

B. The Time Scales Calculus

The *forward jump operator* is given by $\sigma(t) := \inf_{s\in\mathbb{T}}\{s > t\}$ and the *graininess function* $\mu(t)$ by $\mu(t) := \sigma(t) - t$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then the composition $f(\sigma(t))$ is often denoted by $f^{\sigma}(t)$.

A point $t \in \mathbb{T}$ is *right-scattered* if $\sigma(t) > t$ and *right dense* if $\sigma(t) = t$. A point $t \in \mathbb{T}$ is *left-scattered* if $\rho(t) < t$ and *left dense* if $\rho(t) = t$. If t is both left-scattered and right-scattered, we say t is *isolated* or *discrete*. If t is both left-dense and right-dense, we say t is *dense*. The set $\mathbb{T}^{\kappa}$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$, define $f^{\Delta}(t)$ as the number (when it exists), with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of t such that for all $s \in U$,

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s|. \quad \text{(A.1)}$$

The function $f^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is called the *delta derivative* or the *Hilger derivative* of f on $\mathbb{T}^{\kappa}$. Equivalently, (A.1) defines Δ-differential operator via

$$x^{\Delta}(t) := \lim_{\mu^*(t)\searrow\mu(t)} \frac{x(\sigma(t)) - x(t)}{\mu^*(t)}.$$

Since the graininess function induces a measure on $\mathbb{T}$, if we consider the Lebesgue integral over $\mathbb{T}$ with respect to the μ-induced measure, $\int_{\mathbb{T}} f(t)\,d\mu(t)$, then all of the standard results from measure theory are available [18].

A benefit of this general approach is that the realms of differential equations and difference equations can now be viewed as special cases of more general *dynamic equations on time scales*, i.e. equations involving the delta derivative(s) of some unknown function. The upshot here is that the concepts in Tables I and II apply just as readily to *any* closed subset of the real line as they do on $\mathbb{R}$ or $\mathbb{Z}$. Our goal is to leverage this general framework against wide classes of dynamical and control systems. Progress in this direction has been made in transforms theory [13], [26], control [12], [16], [17], dynamic programming [35], and biological models [21], [22].

The function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. We define the related sets $\mathcal{R} := \{p : \mathbb{T} \to \mathbb{R} : p \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T})$ and $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa\}$ and $\mathcal{R}^+ := \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}^\kappa\}$.

For $p(t) \in \mathcal{R}$, we define the *generalized time scale exponential function* $e_p(t, t_0)$ as the unique solution to the initial value problem $x^\Delta(t) = p(t)x(t)$, $x(t_0) = 1$, which exists when $p \in \mathcal{R}$. See [8].

Similarly, the unique solution to the matrix initial value problem $X^\Delta(t) = A(t)X(t)$, $X(t_0) = I$ is called the *transition matrix* associated with this system. This solution is denoted by $\Phi_A(t, t_0)$ and exists when $A \in \mathcal{R}$. A matrix is regressive if and only if all of its eigenvalues are in $\mathcal{R}$. Equivalently, the matrix $A(t)$ is regressive if and only if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^\kappa$.

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