NSF Grant #0726996
NSF PROGRAM NAME: Dynamical Systems

On Switched Linear Systems with Nonuniform Time Domains

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Abstract: The authors' research focuses generally on the study of dynamical systems that evolve over time domains (or "time scales") that are not necessarily topologically equivalent to $\mathbb{R}$ and $\mathbb{Z}$. This paper examines the stability of a certain class of dynamical systems, the "switched linear system," on an arbitrary discrete time scale. Notably, a region of temporal stability is derived, which constrains the graininess of the time scale.

1. Background: Recently, researchers have increasingly turned their attention to the problems posed by dynamical systems that are modeled as mixtures of discrete-event switching logic with standard differential or difference equations. These systems are often termed "switched systems." At their most basic, switched systems consist of differential/difference equations whose parameters (e.g. system coefficients) change discontinuously. Classic examples are the vehicle transmission system, in which the engine/transmission dynamics change essentially instantaneously through gear shifts; biological systems in which cell regulatory dynamics change drastically at various protein concentration thresholds; and real-time distributed control networks, in which closed-loop controllers share a congested communications link connecting sensors, actuators, other controllers and other communications clients, leading to highly variable timing characteristics. Two excellent overviews are given in the references [4, 7].

The problems posed by distributed control networks have been examined by several groups, and present a particularly interesting challenge because the underlying time domain—the times at which communication packets are sent/received, sensor samples are reported, etc.—is neither continuous nor discrete in the usual sense. In other words, neither the continuous real line $\mathbb{R}$ nor the integers $\mathbb{Z}$ appropriately capture the "temporal dynamics" of the system.

Fortunately, recent developments in the mathematics community seem well suited for problems of non-uniform time domains. The study of *dynamic equations on time scales* (DETS) has led to new understanding of how to model and analyze dynamical systems on virtually any time domain that is a subset of $\mathbb{R}$, through the use of generalized differential equations [1]. Such domains are termed "time scales", and given the symbol $\mathbb{T}$. As expected, when $\mathbb{T} = \mathbb{R}$, DETS reduces to standard continuous differential equations; when $\mathbb{T} = \mathbb{Z}$ (or $\mathbb{T} = h\mathbb{Z}$ with h a real number) DETS reduces to standard difference equations. Between those two extremes are many interesting time scales including mixtures of continuous closed intervals interspersed with discrete points. A complete exegesis of the subject of DETS is beyond the scope of this paper, but several introductory resources exist, including a brief web-based tutorial. Table 1 highlights some differences and similarities between $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{T}$.

The present work is concerned primarily with discrete time scales (no continuous intervals) with non-uniform step sizes. These are the time scales that naturally fit the problems of real-time networked systems. But perhaps even more compelling is the fact that time scales besides $\mathbb{R}$ and $\mathbb{Z}$ are themselves a design parameter—to one degree or another, the timing of digital communications systems is under the influence of the designer. A natural question, and one we seek to address in this work, is: what constraints must be placed on the design of the system's time domain itself? We show that there is a "temporal region of stability" that constrains the construction of $\mathbb{T}$, in addition to the well-known regions of stability that constrain the eigenvalue placement for switched linear systems. We follow with numerical case studies that illustrate the nature of the temporal region of stability, and conclude with some open questions.

2. Problem Statement: Let $\mathcal{A} := \{A_1, A_2\}$ be a set of matrices in $\mathbb{R}^{n\times n}$ with non-repeated eigenvalues, and $s : \mathbb{T} \to \{1,2\}$ be a switching signal, where $\mathbb{T}$ is a time scale. The switched linear system

$$x^{\Delta}(t) = A_{s(t)}x(t),\ t \geq 0,\ x(0) = x_0,\ t \in \mathbb{T}, \quad (1)$$

has unique solution $x : \mathbb{T} \to \mathbb{R}^n$. Throughout the ensuing discussion, we make the following assumptions:

- **(A1)** Switching signal s is arbitrary over $\mathbb{T}$, ergo the "arbitrary" or "unconstrained" switching problem.
- **(A2)** All eigenvalues of $A_i \in \mathcal{A}$ lie strictly within the Hilger circle for all $t \in \mathbb{T}$. (Each A_i is "stable", meaning that $x^{\Delta}(t) = A_i x(t)$ has $\|x(t)\| < \infty$ for all $t \geq 0$.)
- **(A3)** All elements of $\mathcal{A}$ commute, i.e. $A_iA_j - A_jA_i = 0$ for all $A_i, A_j \in \mathcal{A}$.
- **(A4)** $\mathbb{T}$ has the following properties: (i) $0 \in \mathbb{T}$, (ii) $\mathbb{T}$ is unbounded above, and (iii) $\mathbb{T}$ has graininess $0 < \mu_{\min} \leq \mu(t) \leq \mu_{\max}$ for all $t \in \mathbb{T}$.

In this work, we constrain the number of switching systems to two for clarity and brevity; this constraint is not generally necessary, however.
To prove the stability of (1), we construct the Lyapunov candidate

$$V(t) = x^T(t)P(t)x(t),$$

with $P(t) = P^T(t) > 0$. (We henceforth assume that all quantities are time-varying except for A_i, unless otherwise indicated.) Thus, stability of the arbitrarily switched system (1) requires

$$A_i^TP + PA_i + \mu A_i^TPA_i + (I+\mu A_i^T)P^{\Delta}(I+\mu A_i) < 0, \quad (2)$$

for $i = 1, 2$. The approach taken next involves two steps:

(a) Set
$$A_i^TP + PA_i + \mu A_i^TPA_i = -M_i, \quad (3)$$
where $M_i = M_i^T > 0$ and solve for P.

(b) Restrict (or design) the time scale $\mathbb{T}$ such that
$$(I + \mu A_i)^TP^{\Delta}(I + \mu A_i) - M_i < 0. \quad (4)$$

We next comment on the feasibility of these two steps.

3. Switched System Stability: First, in step (a), it is required to solve (3) for P. From the initial assumption that A_i is stable, the work of DaCunha [2] gives

$$P(t) = \int_{\mathbb{S}} \Phi_{A_i}(s,0)^T M_i(t)\Phi_{A_i}(s,0)\Delta s, \quad (5)$$

where $\Phi_{A_i}(t,0)$ is the transition matrix that solves $z^{\Delta}(s) = A_iz(s)$ with $s \in \mathbb{S}$ and $z(0) = z_0$, and $M_i = M_i^T > 0$. The region of integration is $\mathbb{S} = \mu(t)\mathbb{N}_0$, i.e. $\mathbb{S} = \{0, \mu(t), 2\mu(t), 3\mu(t), \dots\}$. The transition matrix is always full rank, and, because $\mathbb{S}$ is itself a uniform-graininess time scale, we have $\Phi_{A_i}(s,0) = e_A(s,0)$.
It is not obvious at the outset that solving (3) for $i = 1$ will necessarily give the same solution as for $i = 2$. To arrive at a so-called "common solution" for P, we propose the following theorem.

Theorem 1 (Common Solution) *P is a common solution of (3) for $i = 1, 2$ if*

$$M_1(t) = \int_{\mathbb{S}} \Phi_{A_2}(s,0)^TQ(t)\Phi_{A_2}(s,0)\Delta s, \quad (6)$$
$$M_2(t) = \int_{\mathbb{S}} \Phi_{A_1}(s,0)^TQ(t)\Phi_{A_1}(s,0)\Delta s, \quad (7)$$

where $Q(t) = Q^T(t) > 0$ is arbitrary.

Proof: By direct substitution of either (6) or (7) into (5), P is independent of i because

$$\begin{aligned} P(t) &= \int_{\mathbb{S}} \Phi_{A_1}(s,0)^TM_1(t)\Phi_{A_1}(s,0)\Delta s \\ &= \int_{\mathbb{S}}\int_{\mathbb{S}} \Phi_{A_1}(s,0)^T\Phi_{A_2}(\omega,0)^TQ(t)\cdots \\ &\quad \cdots\Phi_{A_2}(\omega,0)\Phi_{A_1}(s,0)\Delta\omega\Delta s \\ &= \int_{\mathbb{S}}\int_{\mathbb{S}} \Phi_{A_2}(\omega,0)^T\Phi_{A_1}(s,0)^TQ(t)\cdots \\ &\quad \cdots\Phi_{A_1}(s,0)\Phi_{A_2}(\omega,0)\Delta s\Delta\omega \\ &= \int_{\mathbb{S}} \Phi_{A_2}(\omega,0)^TM_2(t)\Phi_{A_2}(\omega,0)\Delta\omega. \end{aligned}$$

Table 1: Mathematical expressions on $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{T}$

$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{T}$	**Comments**
$\dot{x}(t) = f(x(t))$	$x(t+1) = f(x(t))$	$x^{\Delta}(t) = f(x(t))$	The generalized (Hilger) dynamic derivative
$x(t) = e^{a(t-t_0)}$	$x(t) = a^{t-t_0}$	$x(t) = e_a(t, t_0)$	Solutions of 1st-order linear differential equations
$a \in \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$	$a \in \{\lambda \in \mathbb{C} : \|\lambda\| < 1\|\}$	$a \in \{\lambda \in \mathbb{C} : \|1+\mu\lambda\| < 1\}$	Left-half $\mathbb{C}$ plane; unit circle; Hilger circle
$e^{(a+b)t}$	$(ab)^t$	$e_{a \oplus b}(t, 0)$	Multiplication of two exponential functions

The commutativity of the transition matrices in lines 2 and 3 above is made possible by the assumption A3. (The stability of switched linear systems with a commuting family of A_i is an established result from continuous-time analysis too.) In line 1, it is seen that P solves (3) for $i = 1$. On the other hand, in line 4 we see that P solves (3) for $i = 2$. Thus P is a common solution. □

To justify that step (b) is possible, we observe that

$$P^{\Delta} = \frac{P^{\sigma} - P}{\mu}.$$

In view of the Common Solution Theorem, P is a function of time though the domain of integration and the matrix $Q(t)$. If Q is set constant, then P^{σ} depends only on μ^{σ}, through the domain of integration $\mathbb{S}^{\sigma} = \mu^{\sigma}(t)\mathbb{N}_0$. Thus it is always possible to reduce the magnitude of P^{Δ} (and therefore of $(I+\mu A_i)^T P^{\Delta}(I+\mu A_i)$) by setting μ^{σ} "near" μ. In effect, we may design the time scale "on the fly," determining the next graininess as a function of the current graininess in such a manner that (4) holds at all times. The following analysis will make extensive use of a lemma.

Lemma 2 (Quadratic Integral) *Let $A \in \mathbb{C}^{n \times n}$ be diagonal and stable on time scale $\mathbb{S} = h\mathbb{N}_0$. Then*

$$\int_{\mathbb{S}} \Phi_A(s,0)^T \Phi_A(s,0) \Delta s = \left[-2 \operatorname{Re} A - hA^T A\right]^{-1},$$

(where transpose of a complex matrix implies complex conjugation).

Proof: Since A is diagonal, it is sufficient to prove the scalar case with $a = A(1,1)$. On constant-graininess time scale $\mathbb{S}$, the transition matrix equates to the time scale exponential, $\Phi_a(s,0) = e_a(s,0)$. Thus,

$$\begin{aligned} \int_{\mathbb{S}} \Phi_A(s,0)^T \Phi_A(s,0) \Delta s &= \int_{\mathbb{S}} e_{\bar{a}}(s,0) e_a(s,0) \Delta s \\ &= \int_{\mathbb{S}} e_{\bar{a} \oplus a}(s,0) \Delta s. \end{aligned}$$

The exponent calculates to

$$\bar{a} \oplus a = 2 \operatorname{Re}(a) + h\,|a|^2 = \frac{|1+ha|^2 - 1}{h}.$$

Since (by assumption) $|1+ha| < 1$, it follows that $\bar{a} \oplus a$ is strictly within the Hilger circle on $\mathbb{S}$, and therefore stable. Consequently, the integral is

$$\int_{\mathbb{S}} e_{\bar{a} \oplus a}(s,0) \Delta s = \frac{1}{-2 \operatorname{Re}(a) - h\,|a|^2}.$$

The lemma claim follows immediately. □

To realize the benefit of the Quadratic Integral Lemma, we first note an important theorem [3] which states that real matrices commute if and only if they are simultaneously diagonalizable. Therefore, under assumption A3, there exists a (possibly complex-valued) constant similarity matrix S such that $A_i = S^{-1} J_i S$ for $i = 1, 2$ with J_i a diagonal matrix of eigenvalues. (S is guaranteed to be invertible because all A_i were defined with non-repeating eigenvalues.)

The central question addressed next is, at any time t with graininess $\mu(t)$, what is the **maximum allowable** "next graininess" $\mu^{\sigma}(t)$ such that (4) holds for all i? This question suggests that there are stable pairs $\{\mu, \mu^{\sigma}\}$ at every point in time. The union of all such pairs forms a two-dimensional "temporal region of stability" $\mathcal{R}$ in the $\mu^{\sigma}(t)$ vs. $\mu(t)$ plane.

Theorem 3 (Region of Stability) *Define*

$$K_i := \left[2 \operatorname{Re} J_i + \mu J_i^* J_i\right], \qquad i = 1, 2,$$

with $*$ denoting complex-conjugate transposition. Define region $\mathcal{R} \subset \mathbb{R}^2$ consisting of pairs $\{\mu, \mu^{\sigma}\}$ such that

$$K_1^{-1} K_2^{-1} K_1^{\sigma} K_2^{\sigma} > (I + \mu J_1)^* (I + \mu J_1), \tag{8}$$

with $\mu_{\min} \le \mu(t) \le \mu_{\max}$ for all $t \in \mathbb{T}$, and $A_i = S^{-1} J_i S$. Then (1) remains stable under arbitrary switching if $\{\mu^{\sigma}(t), \mu(t)\} \in \mathcal{R}$ for all $t \in \mathbb{T}$.

Proof: First we note that all K matrices are diagonal. Next, stability of (1) requires that (4) must hold, and therefore any $\{\mu, \mu^\sigma\}$ pair that preserves (4) is a member of $\mathcal{R}$. We proceed to compute the left-hand side of (4). To begin, we choose $Q = S^*S$. Thus,

$$\begin{aligned} P(t) &= \int_{\mathbb{S}}\int_{\mathbb{S}} \left(S^*\Phi_{J_1}(s,0)^*\Phi_{J_2}(\omega,0)^*S^{-*}\right) S^* \cdots \\ &\quad \cdots S\left(S^{-1}\Phi_{J_2}(\omega,0)\Phi_{J_1}(s,0)S\right)\Delta\omega\Delta s \\ &= S^*\left[\int_{\mathbb{S}} \Phi_{J_1}(s,0)^*\Phi_{J_1}(s,0)\Delta s \cdots \right. \\ &\quad \left. \cdots \int_{\mathbb{S}} \Phi_{J_2}(\omega,0)^*\Phi_{J_2}(\omega,0)\Delta\omega\right] S \\ &= S^*\left[-2\operatorname{Re} J_1 - \mu J_1^* J_1\right]^{-1} \cdots \\ &\quad \cdots \left[-2\operatorname{Re} J_2 - \mu J_2^* J_2\right]^{-1} S. \end{aligned}$$

Similarly,

$$\begin{aligned} P^\sigma &= S^*\left[-2\operatorname{Re} J_1 - \mu^\sigma J_1^* J_1\right]^{-1} \cdots \\ &\quad \cdots \left[-2\operatorname{Re} J_2 - \mu^\sigma J_2^* J_2\right]^{-1} S. \end{aligned}$$

After eliminating the S terms, (4) for $i = 1$ now reads

$$\begin{aligned} 0 &> \left[2\operatorname{Re} J_1 + \mu^\sigma J_1^* J_1\right]^{-1}\left[2\operatorname{Re} J_2 + \mu^\sigma J_2^* J_2\right]^{-1} \\ &\quad - \left[2\operatorname{Re} J_1 + \mu J_1^* J_1\right]^{-1}\left[2\operatorname{Re} J_2 + \mu J_2^* J_2\right]^{-1} \\ &\quad + \left[\frac{1}{\mu} + 2\operatorname{Re} J_1 + \mu J_1^* J_1\right]^{-1}\left[2\operatorname{Re} J_2 + \mu J_2^* J_2\right]^{-1}, \end{aligned}$$

and similarly for $i = 2$. After some simplification, expression (8) arises, forming a set of $2n$ inequalities that implicitly bound μ^σ in terms of μ. □

For any given choice of μ, it is the minimum of the $2n$ inequalities that determines the bound on μ^σ. Note that increasing the magnitude of the left-hand sides requires *decreasing* μ^σ, implying that μ^σ is *upper-bounded* by these inequalities. Furthermore, it is significant that the switching sequence (i.e. switching from A_2 to A_1 or vice versa) does not matter: J_1 and J_2 may be swapped with no change in the inequalities above. We now come to an interesting result.

Corollary 4 (Unit Slope) *Under arbitrary switching, the system of (1) will remain stable if $\mu^\sigma(t) \leq \mu(t)$ for all $t \in \mathbb{T}$.*

Proof: The result follows from (8) by noting that the norm of the right-hand sides must always be less than 1. By assumption, all diagonals of J_i are strictly within all Hilger circles defined by $\mathbb{T}$. Thus, $\|I + \mu J_i\| < 1$. On the other hand, the norm of the left-hand side is greater than or equal to 1 when $\mu^\sigma(t) \leq \mu(t)$. Thus, in a plot of μ^σ vs. μ, the region including and below the unit-slope line $\mu^\sigma = \mu$ contains stable $\{\mu, \mu^\sigma\}$ pairs. □

4. Case Studies: Armed with equation (8), we may now view examples of the region of stability in Figures 1 and 2. Each region is contained within a bounding box defined by $0 < \mu(t) \leq \mu_{\max}$ and $0 < \mu^\sigma(t) \leq \mu_{\max}$.

In the figures, systems of dimension $n = 2$ are chosen, and two features are computed: (a) the bounding curves defined by (8) as if is was an equality, and (b) the shaded region $\mathcal{R}$ of $\{\mu, \mu^\sigma\}$ pairs that satisfy (4). It is immediately evident that $\mathcal{R}$ is, as expected, the area underneath the minimum of the bounding curves.

As predicted by the Unit Slope Lemma, no bounding curves dip below the line $\mu^\sigma = \mu$, but some enclose significantly more area than others. In practical terms, then, it is sometimes possible to design a time scale in which μ^σ (the next time step or sample period) is significantly larger than the current sample period. Sometimes, however, μ^σ can only be fractionally larger than μ. In effect, then, the dynamics of each switched system dictate how much variability, or volatility, is allowable in the graininess of $\mathbb{T}$. Equation (8) sheds some light here: cases where a bounding curve lies very near to $\mu^\sigma = \mu$ are those for which $\|I + \mu J_i\| \cong 1$. These cases arise when an eigenvalue of A_i is very near the Hilger circle. Since there is a different Hilger Circle for each different μ, the only eigenvalues that are always "near" the Hilger circle are those near zero.

5. Conclusions and Further Work: There are several intriguing avenues for further investigation. One unresolved question involves the necessity of the region $\mathcal{R}$. The preceding discussion demonstrates sufficiency—that the system of (1) will be stable if $\{\mu, \mu^\sigma\} \in \mathcal{R}$—but in point of fact, upon simulating the dynamics of (1) with random (stable) A_i, we have not found evidence of unstable behavior even when a majority of the $\{\mu, \mu^\sigma\}$ pairs in $\mathbb{T}$ reside outside of $\mathcal{R}$.

At first glance, it might seem that solving (2) for P, rather than (3), would obviate the need to satisfy (4). Equation (2) (or rather, the left-hand side of (2) when equated to some symmetric, negative definite matrix) is termed the *time scale dynamic Lyapunov equation* (TSDLE) whereas (3) is the *time scale algebraic Lyapunov equation* (TSALE). A solution to the TSDLE does in fact

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Figure 1: Region $\mathcal{R}$ is shaded below for two systems with eigenvalues $\{-0.78, -0.6\}$ and $\{-0.5 \pm 0.5j\}$. Equation (8) is solved as a set of 4 quadratic equalities, yielding a set of 8 curves that are functions of μ and μ^σ; $\mathcal{R}$ must be the area below all of them. Four curves are outside the plot axes and four can be seen below.

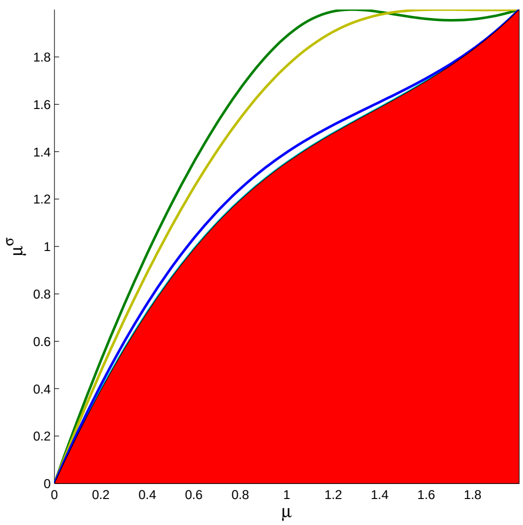
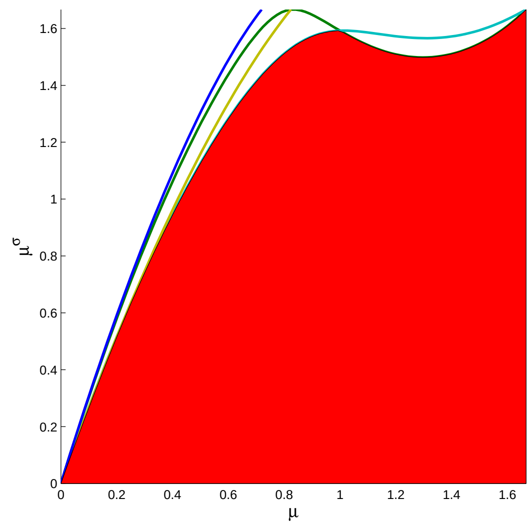


Figure 2: Region $\mathcal{R}$ is shaded below for two systems with eigenvalues $\{-1.2, -0.8\}$ and $\{-1 \pm 0.2j\}$.



exist [6]; however, it has two important drawbacks. First, to date there is no known method to construct (or even prove the existence of) a TSDLE solution that is common to two or more systems, i.e. a P that is independent of i, unless the graininess is constant. Second, the (unique) solution of the TSDLE demands complete knowledge of the entire time scale, raising questions of causality for systems that dictate, or "design," the time scale step-by-step.

Also of interest is the question of constrained switching, in which the choice of each successive system matrix A_i is not arbitrary but based upon knowledge of t, $\mu(t)$, the state variable x, or some other factor. Switching constraints may alter the shape and size of the region of stability.

Lastly, two comments on extensions of this work. First, it is possible to extend the results to $m > 2$ multiple switched systems. The analysis is similar, and the results appear in a nearly-completed Ph.D. dissertation [6]. Second, it is possible to extend the results to systems which are simultaneously triangularizable, but not diagonalizable (i.e. systems with repeated eigenvalues and generalized eigenvectors). The analysis of this case appears in [5].

In summary, this paper presents the sufficient conditions for stability of switched linear systems on non-uniform discrete time domains. In addition to the previously known requirement that all A_i are stable and commutative, we show that the time scale itself must be restricted or designed to preserve stability. A sufficient condition is $\{\mu, \mu^\sigma\} \in \mathcal{R}$, the region of stability defined by (8). Switched systems that operate on non-uniform discrete time domains arise in distributed control systems, where controllers, sensors, actuators and other systems compete for bandwidth on a common communications link.

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