Compound Interest

P: Principal Amount
F: Future Value
i: Annual Interest Rate (w/o compounding)
N: Compounding periods per year (effect)
M: Number of years

APR: Effective Annual Rate, including effect of compounding

\[ F = P \left(1 + \frac{i}{N}\right)^{MN} \]

\[ \frac{F}{P} = \left(1 + \frac{i}{N}\right)^{MN} \]

M compounding intervals per year

If \( i = 0.05 \) (i.e., 5\%), \( N = 1 \) (compounded annually)

\[ \frac{F}{P} = \left(1 + \frac{0.05}{1}\right)^{1\cdot M} = (1 + 0.05)^M = 1.05^M \]

\( N = 2 \) (semi-annually)

\[ \frac{F}{P} = \left(1 + \frac{0.05}{2}\right)^{2\cdot M} = (1.025)^{2M} = 1.0506^M \]

\( N = 4 \) (quarterly)

\[ \frac{F}{P} = \left(1 + \frac{0.05}{4}\right)^{4M} = (1.0125)^{4M} = 1.0509^M \]

\( N = 12 \) (monthly)

\[ \frac{F}{P} = \left(1 + \frac{0.05}{12}\right)^{12M} = (1.004167)^{12M} = 1.0512^M \]

\( N = 365 \) (daily)

\[ \frac{F}{P} = \left(1 + \frac{0.05}{365}\right)^{365M} = (1.000137)^{365M} = 1.0513^M \]
\[ N = \text{huge (every second, msec, usec ---)} \]

\[
\frac{F}{P} = (1 + \frac{0.05}{N})^N = ?
\]

Use natural log series expansion

\[
\ln\left(\frac{F}{P}\right) = NM \ln\left(1 + \frac{0.05}{N}\right)
\]

\[
\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \quad |x| < 1
\]

Our \( x \) is \( \frac{0.05}{N} \) (very tiny)

So \( \ln(1 + \text{tiny}) = \text{tiny} - \frac{\text{tiny}^2}{2} + \frac{\text{tiny}^3}{3} \)

Thus \( \ln\left(\frac{F}{P}\right) = NM \cdot \frac{0.05}{N} = M(0.05) \)

So, now raise both sides as a power of \( e \)

\[
e^{\ln\left(\frac{F}{P}\right)} = \frac{F}{P} = e^{M(0.05)} = e^M
\]

Years

\[ M = 1, \quad \frac{F}{P} = e^{0.05 \cdot 1} = 1.0513 \] (the asymptote)

\[ M = 2, \quad \frac{F}{P} = e^{0.05 \cdot 2} = 1.1025 \]

\[ \vdots \]

Exponential Growth (the magic of compound interest)

Engineering approach - easier to use!
How long to double the principal?

\[ \frac{F}{P} = 2 = e^{\bar{i}M} \] 
\[ \ln 2 = \bar{i}M, \quad M = \frac{\ln 2}{\bar{i}} \]
\[ \bar{i} = 10\%, \quad M = \frac{\ln 2}{0.1} = \frac{0.693}{0.1} = 6.93 \text{ years} \]

"Rule of 70", \[ \frac{F}{P} = 2 \] when

For 10\%, \[ M = \frac{69.3}{\bar{i}(\%)} = \frac{70}{10} = 7 \text{ years} \]

For 5\%, \[ M = \frac{70}{5} = 14 \text{ years} \]

What interest rate will double your money in 5 years (approx)

\[ M = \frac{70}{\bar{i}(\%)} \] 
\[ \bar{i}(\%) = \frac{70}{5} = 14\% \]

How many years to get \( \frac{F}{P} = 1.5 \) with 9\% interest

\[ \frac{F}{P} = 1.5 = e^{\bar{i}M} \]
\[ \ln(1.5) = \bar{i}M, \quad M = \frac{\ln(1.5)}{0.09} = 10.14 \text{ years} \]

Many engineering problems follow exponential decay.
Glass of Cold Water, Set Outside (temp = 100°F)

\[ T_{\text{temp}} = 32 + (100 + (100 - 32)) e^{-\frac{t}{\tau}} \]

- \( t = 0, \) put glass outside
- \( t = \tau, \) \( T_{\text{final}} \approx T_{\text{initial}} \)
- \( t \to \infty, \) \( T_{\text{final}} \approx T_{\text{ambient}} \)

How quick—described by time constant \( \tau \)

\[
\begin{align*}
0 & \quad 0.368 & \quad 0.632 \\
1\tau & \quad 0.135 & \quad 0.865 \\
2\tau & \quad 0.050 & \quad 0.950 \\
\tau & \quad 0 & \quad 1
\end{align*}
\]

1\( \tau \) has moved 36.8\% of the \( AT = 68^\circ \)
3\( \tau \) has moved 95\% of the \( AT = 68^\circ \)