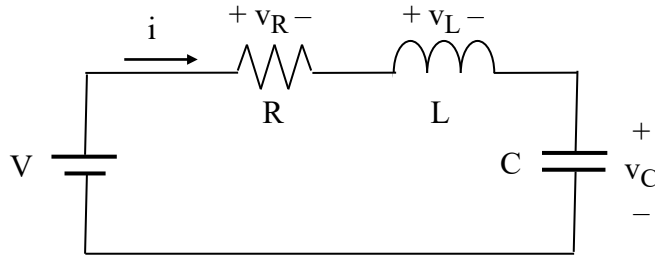


The simplest second-order system is one for which the L's can be combined, the C's can be combined, and the R's can be combined into one L, one C, and one R. Let us consider situations where there is no source, or a DC source.

### For an RLC Series Circuit

If the circuit is a series circuit, then there is one current.



Note that in the above circuit, the V and R could be converted to a Norton equivalent, permitting the incorporation of a current source into the problem instead of a voltage source.

Using KVL,

$$-V + v_R + v_L + v_C = -V + iR + L \frac{di}{dt} + \frac{1}{C} \int i dt = 0.$$

Taking the derivative,

$$R \frac{di}{dt} + L \frac{d^2 i}{dt^2} + \frac{i}{C} = 0.$$

Note that the constant source term V is gone, leaving us with the *characteristic equation* (i.e., source free), which yields the *natural response* of the circuit. Rewriting,

$$\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0. \quad (1)$$

Guess the natural response solution  $i = Ae^{st}$  and try it,

$$As^2 e^{st} + \frac{R}{L} A s e^{st} + \frac{1}{LC} A e^{st} = A e^{st} \left( s^2 + s \frac{R}{L} + \frac{1}{LC} \right) = 0,$$

And we reason that the only possibility for a general solution is when

$$s^2 + s \frac{R}{L} + \frac{1}{LC} = 0.$$

This leads us to

$$s = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}.$$

Defining

$$\alpha = \frac{R}{2L} \text{ as the damping coefficient (nepers/sec) for the series case} \quad (2)$$

$$\omega_o = \frac{1}{\sqrt{LC}} \text{ as the natural resonant frequency (radians/sec)} \quad (3)$$

then we have

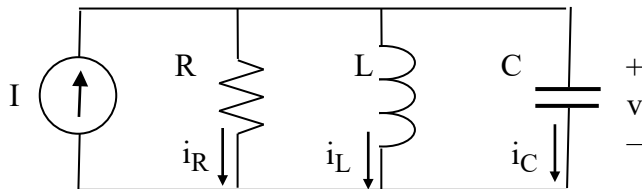
$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}. \quad (4)$$

Substituting (2) and (3) into (1) yields a useful form of (2), which is

$$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \omega_o^2 i = 0 \quad (5)$$

### For an RLC Parallel Circuit

If the circuit is a parallel circuit, then there is one voltage. KCL at the top node yields



$$-I + i_R + i_L + i_C = -I + \frac{v}{R} + \frac{1}{L} \int v dt + C \frac{dv}{dt} = 0.$$

Taking the derivative

$$\frac{1}{R} \frac{dv}{dt} + \frac{v}{L} + C \frac{d^2v}{dt^2} = 0.$$

Rewriting,

$$\frac{d^2v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{v}{LC} = 0. \quad (6)$$

Equation (6) has the same form as (5), except that

$$\alpha = \frac{1}{2RC} \text{ for the parallel case.} \quad (7)$$

Thus, it is clear that the solutions for series and parallel circuits have the same form, except that the damping coefficient  $\alpha$  is defined differently.

A nice property of exponential solutions such as  $Ae^{st}$  is that derivatives and integrals of  $Ae^{st}$  also have the  $e^{st}$  term. Thus, one can conclude that all voltages and currents throughout the series and parallel circuits have the same  $e^{st}$  terms.

### There are Three Cases

Equation (4)

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

has three distinct cases:

1. Case 1 is where  $\alpha > \omega_o$ , so that the term inside the radical is positive. **Overdamped.**
2. Case 2 is where  $\alpha < \omega_o$ , so that the term inside the radical is negative. **Underdamped.**
3. Case 3 is where  $\alpha = \omega_o$ , so that the term inside the radical is zero. **Critically Damped.**

Case 3 rarely happens in practice because the terms must be exactly equal. Case 3 should be thought of as a transition from Case 1 to Case 2, and the transition is very gradual.

### Case 1: Overdamped

When  $\alpha > \omega_o$ , then the solutions for (4) are both negative real and are

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2},$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}.$$

The natural response for any current (or voltage) in the circuit is then

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}.$$

Coefficients  $A_1$  and  $A_2$  are found from the boundary conditions as follows:

$$i(t=0) = A_1 e^{s_1 t} \Big|_{t=0} + A_2 e^{s_2 t} \Big|_{t=0} = A_1 + A_2,$$

$$\frac{di}{dt} \Big|_{t=0} = A_1 s_1 e^{s_1 t} \Big|_{t=0} + A_2 s_2 e^{s_2 t} \Big|_{t=0} = A_1 s_1 + A_2 s_2.$$

Thus, we have two equations and two unknowns. To solve, one must know  $i(t=0)$  and  $\frac{di}{dt} \Big|_{t=0}$ .

Solving, we get

$$A_1 = \frac{s_2 \bullet i(t=0) - \frac{di}{dt} \Big|_{t=0}}{s_2 - s_1}. \quad (\text{Note – see modification needed to include final response}) \quad (8)$$

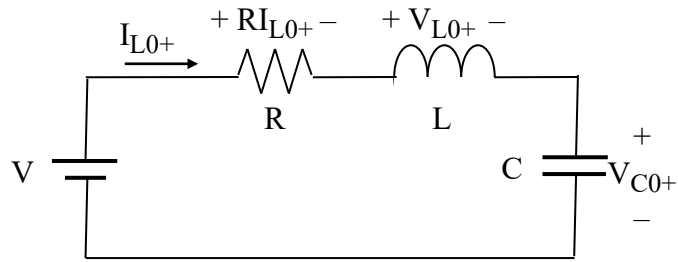
Then, find  $A_2$  from

$$A_2 = i(t=0) - A_1 \quad (\text{Note – see modification needed to include final response}) \quad (9)$$

**The key to finding  $A_1$  and  $A_2$  is always to know the inductor current and capacitor voltage at  $t = 0^+$ . Remember that**

- Unless there is an infinite impulse of current through a capacitor, the voltage across a capacitor (and the stored energy in the capacitor) remains constant during a switching transition from  $t = 0^-$  to  $t = 0^+$ .
- Unless there is an infinite impulse of voltage across an inductor, the current through an inductor (and the stored energy in the inductor) remains constant during a switching transition from  $t = 0^-$  to  $t = 0^+$ .

For example, consider the series circuit at  $t = 0^+$ . Current  $i(t = 0) = I_{0+}$ .



Once  $I_{L0+}$  and  $V_{C0+}$  are known, then we get  $\left. \frac{di_L}{dt} \right|_{t=0}$  as follows. From KVL

$$-V + RI_{0+} + V_{L0+} + V_{C0+} = 0$$

thus

$$V_{L0+} = V - RI_{0+} - V_{C0+}. \quad (10)$$

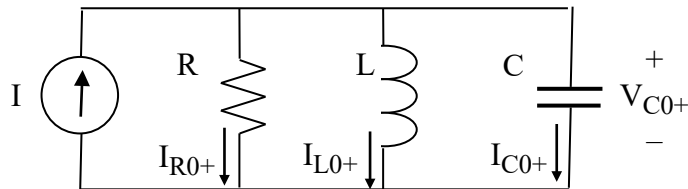
Since

$$V_{L0+} = L \left. \frac{di_L}{dt} \right|_{t=0}$$

then we have

$$\boxed{\left. \frac{di_L}{dt} \right|_{t=0} = \frac{V_{L0+}}{L}}. \quad (11)$$

Similarly, for the parallel circuit at  $t = 0^+$ , the voltage across all elements  $v(t = 0) = V_{C0+}$ .



We get  $\left. \frac{dv_C}{dt} \right|_{t=0}$  as follows. From KCL

$$-I + I_{R0+} + I_{L0+} + I_{C0+} = 0$$

thus

$$I_{C0+} = I - I_{R0+} - I_{L0+}. \quad (12)$$

Since

$$I_{C0+} = C \left. \frac{dv_C}{dt} \right|_{t=0}$$

then we have

$$\boxed{\left. \frac{dv_C}{dt} \right|_{t=0} = \frac{I_{C0+}}{C}}. \quad (13)$$

### Case 2: Underdamped

When  $\alpha < \omega_o$ , then the solutions for (4) are both complex and are

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -\alpha + \sqrt{(-1)(\omega_o^2 - \alpha^2)} = -\alpha + j\sqrt{\omega_o^2 - \alpha^2},$$

$$s_2 = -\alpha - j\sqrt{\omega_o^2 - \alpha^2}.$$

Define damped resonant frequency  $\omega_d$  as

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}, \quad (14)$$

so that

$$s_1 = -\alpha + j\omega_d,$$

$$s_2 = -\alpha - j\omega_d$$

The natural response for any current (or voltage) in the circuit is then

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t},$$

$$= A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}.$$

Expanding the  $e^{j\omega_d t}$  and  $e^{-j\omega_d t}$  terms using Euler's rule,  $e^{j\theta} = \cos(\theta) + j \sin(\theta)$ ,

$$\begin{aligned} i(t) &= A_1 e^{-\alpha t} [\cos(\omega_d t) + j \sin(\omega_d t)] + A_2 e^{-\alpha t} [\cos(-\omega_d t) + j \sin(-\omega_d t)], \\ &= A_1 e^{-\alpha t} [\cos(\omega_d t) + j \sin(\omega_d t)] + A_2 e^{-\alpha t} [\cos(\omega_d t) - j \sin(\omega_d t)], \\ &= e^{-\alpha t} [(A_1 + A_2) \cos(\omega_d t) + j(A_1 - A_2) \sin(\omega_d t)]. \end{aligned} \quad (15)$$

Since  $i(t)$  is a real value, then (15) cannot have an imaginary component. This means that  $(A_1 + A_2)$  is real, and  $j(A_1 - A_2)$  is real (which means that  $(A_1 - A_2)$  is imaginary). These conditions are met if  $A_2 = A_1^*$ . Thus, we can write (15) in the form of

$$i(t) = e^{-\alpha t} [B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)] \quad (16)$$

where  $B_1$  and  $B_2$  are real numbers.

To evaluate  $B_1$  and  $B_2$ , it follows from (16) that

$$i(t=0) = e^{-0} [B_1 \cos(0) + B_2 \sin(0)] = B_1,$$

so

$$B_1 = i(t=0) \quad (\text{Note – see modification needed to include final response}) \quad (17)$$

To find  $B_2$ , take the derivative of (16) and evaluate it at  $t = 0$ ,

$$\frac{di}{dt} = -\alpha e^{-\alpha t} [B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)] + e^{-\alpha t} [-B_1 \omega_d \sin(\omega_d t) + B_2 \omega_d \cos(\omega_d t)],$$

$$\left. \frac{di}{dt} \right|_{t=0} = -\alpha [B_1] + [B_2 \omega_d] = -\alpha B_1 + B_2 \omega_d,$$

so we can find  $B_2$  using

$$B_2 = \frac{\left. \frac{di}{dt} \right|_{t=0} + \alpha B_1}{\omega_d} \quad (18)$$

## Case 2 Solution in Polar Form

The form in (16),  $i(t) = e^{-\alpha t} [B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)]$  is most useful in evaluating coefficients  $B_1$  and  $B_2$ . But in practice, the answer is usually converted to polar form. Proceeding, write (16) as

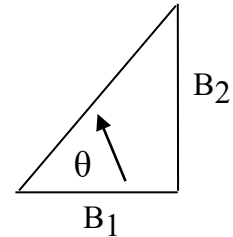
$$i(t) = \sqrt{B_1^2 + B_2^2} \bullet e^{-\alpha t} \left[ \frac{B_1}{\sqrt{B_1^2 + B_2^2}} \cos(\omega_d t) + \frac{B_2}{\sqrt{B_1^2 + B_2^2}} \sin(\omega_d t) \right]$$

$$i(t) = \sqrt{B_1^2 + B_2^2} \bullet e^{-\alpha t} [\cos(\theta) \cos(\omega_d t) + \sin(\theta) \sin(\omega_d t)]$$

The  $\frac{B_1}{\sqrt{B_1^2 + B_2^2}}$  and  $\frac{B_2}{\sqrt{B_1^2 + B_2^2}}$  terms are the cosine and sine, respectively, of the angle shown

in the right triangle. Unless  $B_1$  is zero, you can find  $\theta$  using

$$\theta = \tan^{-1} \left[ \frac{B_2}{B_1} \right].$$



But be careful because your calculator will give the wrong answer 50% of the time. The reason is that  $\tan(\theta) = \tan(\theta \pm 180^\circ)$ . So check the quadrant of your calculator answer with the quadrant consistent with the right triangle. If the calculator quadrant does not agree with the figure, then add or subtract  $180^\circ$  from your calculator angle, and re-check the quadrant.

The polar form for  $i(t)$  comes from a trigonometric identity. The expression

$$i(t) = \sqrt{B_1^2 + B_2^2} \bullet e^{-\alpha t} [\cos(\theta) \cos(\omega_d t) + \sin(\theta) \sin(\omega_d t)]$$

becomes the damped sinusoid

$$i(t) = \sqrt{B_1^2 + B_2^2} \bullet e^{-\alpha t} \cos(\omega_d t - \theta).$$

### Case 3: Critically Damped

When  $\alpha = \omega_o$ , then there is only one solution for (4), and it is  $s_1 = s_2 = -\alpha$ . Thus, we have repeated real roots. Proceeding as before, then there appears to be only one term in the natural response of  $i(t)$ ,

$$i(t) = A e^{-\alpha t}.$$



But this term does not permit the natural response to be zero at  $t = 0$ , which is a problem. Thus, let us propose that the solution for the natural response of  $i(t)$  has a second term of the form  $te^{-\alpha t}$ , so that

$$i(t) = A_1 te^{-\alpha t} + A_2 e^{-\alpha t} = (A_1 t + A_2) e^{-\alpha t}. \quad (19)$$

At this point, let us consider the general form of the differential equation given in (5), which shown again here is

$$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \omega_o^2 i = 0.$$

For the case with  $\alpha = \omega_o$ , the equation becomes

$$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \alpha^2 i = 0.$$

We already know that  $Ae^{-\alpha t}$  satisfies the equation, but let us now test the new term  $Ate^{-\alpha t}$ . Substituting in,

$$A(-\alpha e^{-\alpha t} - \alpha e^{-\alpha t} + \alpha^2 te^{-\alpha t}) + 2\alpha A(e^{-\alpha t} - \alpha te^{-\alpha t}) + \alpha^2 Ate^{-\alpha t} = 0?$$

Factoring out the  $Ae^{-\alpha t}$  term common to all yields

$$(-\alpha - \alpha + \alpha^2 t) + 2\alpha(1 - \alpha t) + \alpha^2 t = 0? \quad (\text{Yes!})$$

So, we confirm the solution of the natural response.

$$i(t) = A_1 te^{-\alpha t} + A_2 e^{-\alpha t} = (A_1 t + A_2) e^{-\alpha t} \quad (20)$$

To find coefficient  $A_2$ , use

$$i(t = 0) = (A_1 \cdot 0 + A_2) e^{-0} = A_2.$$

Thus,

$$A_2 = i(t = 0). \quad (\text{Note – see modification needed to include final response}) \quad (21)$$

To find  $A_1$ , take the derivative of (20),

$$\frac{di}{dt} = A_1 e^{-\alpha t} - \alpha A_1 t e^{-\alpha t} - \alpha A_2 e^{-\alpha t} = (A_1 - \alpha A_1 t - \alpha A_2) e^{-\alpha t}.$$

Evaluating at  $t = 0$  yields

$$\left. \frac{di}{dt} \right|_{t=0} = A_1 - \alpha A_2.$$

Thus,

$$A_1 = \left. \frac{di}{dt} \right|_{t=0} + \alpha A_2. \quad (22)$$

The question that now begs to be asked is “what happens to the  $te^{-\alpha t}$  term as  $t \rightarrow \infty$ ? Determine this using the series expansion for an exponential.

We know that

$$e^{\alpha t} = 1 + \frac{(\alpha t)}{1!} + \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} + \dots.$$

Then,

$$te^{-\alpha t} = \frac{t}{e^{\alpha t}} = \frac{t}{1 + \frac{(\alpha t)}{1!} + \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} + \dots}.$$

Dividing numerator and denominator by  $t$  yields,

$$te^{-\alpha t} = \frac{t}{e^{\alpha t}} = \frac{1}{\frac{1}{t} + \frac{(\alpha t)}{t \cdot 1!} + \frac{(\alpha t)^2}{t \cdot 2!} + \frac{(\alpha t)^3}{t \cdot 3!} + \dots} = \frac{1}{\frac{1}{t} + \frac{\alpha}{1!} + \frac{\alpha^2 t}{2!} + \frac{\alpha^3 t^2}{3!} + \dots}.$$

As  $t \rightarrow \infty$ , the  $\frac{1}{t}$  in the denominator goes to zero, and the other “ $t$ ” terms are huge. Since the numerator stays 1, and the denominator becomes huge, then  $te^{-\alpha t} \rightarrow 0$ .

### **The Total Response = The Natural Response Plus The Final Response**

If the circuit has DC sources, then the steady-state (i.e., “final”) values of voltage and current may not be zero. “Final” is the value after all the exponential terms have decayed to zero. And

yet, for all the cases examined here so far, the exponential terms all decayed to zero after a long time. So, how do we account for “final” values of voltages or currents?

Simply think of the circuit as having a total response that equals the sum of its natural response and final response. Add the final term, e.g.

$$i(t) = I_{final} + A_1 e^{s_1 t} + A_2 e^{s_2 t},$$

so

$$[i(t) - I_{final}] = A_1 e^{s_1 t} + A_2 e^{s_2 t} \text{ (for overdamped).} \quad (23)$$

Likewise,

$$i(t) = I_{final} + e^{-\alpha t} [B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)],$$

$$[i(t) - I_{final}] = e^{-\alpha t} [B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)] \text{ (for underdamped).} \quad (24)$$

Likewise,

$$i(t) = I_{final} + (A_1 t + A_2) e^{-\alpha t},$$

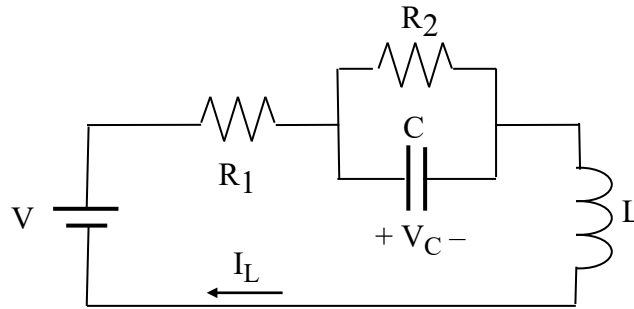
$$i(t) - I_{final} = (A_1 t + A_2) e^{-\alpha t} \text{ (for critically damped).} \quad (25)$$

You can see that the presence of the final term  $I_{final}$  will affect the A and B coefficients because the initial value of  $i(t)$  now contains the  $I_{final}$  term. **Take  $I_{final}$  into account when you evaluate the A's and B's by replacing  $i(t=0)$  in (8), (9), (17), and (21) with  $i(t=0) - I_{final}$ .**

How do you get the final values? If the problem has a DC source, then remember that after a long time when the time derivatives are zero, capacitors are “open circuits,” and inductors are “short circuits.” Compute the “final values” of voltages and currents according to the “open circuit” and short circuit” principles.

### The General Second Order Case

Second order circuits are not necessarily simple series or parallel RLC circuits. Any two non-combinable storage elements (e.g., an L and a C, two L's, or two C's) yields a second order circuit and can be solved as before, except that the  $\alpha$  and  $\omega_o$  are different from the simple series and parallel RLC cases. An example follows.



General Case Circuit #1 (Prob. 8.56)

Work problems like Circuit #1 by defining capacitor voltages and inductor currents as the  $N$  state variables. For Circuit #1,  $N = 2$ . You will write  $N$  circuit equations in terms of the state variables, and strive to get an equation that contains only one of them. **Use variables instead of numbers when writing the equations.**

For convenience, use the simple notation  $V_C$  and  $I_L$  to represent time varying capacitor voltage and inductor current, and  $\dot{V}_C$  and  $\dot{I}_L$  to represent their derivatives, and so on.

To find the *natural response*, turn off the sources. Write your  $N$  equations by using KVL and KCL, making sure to include each circuit element in your set of equations.

For Circuit #1, start with KVL around the outer loop,

$$R_1 I_L + V_C + L \dot{I}_L = 0. \quad (26)$$

Now, write KCL at the node just to the left of the capacitor,

$$-I_L + \frac{V_C}{R_2} + C \dot{V}_C = 0,$$

which yields

$$I_L = \frac{V_C}{R_2} + C \dot{V}_C. \quad (27)$$

Taking the derivative of (27) yields

$$\dot{I}_L = \frac{\dot{V}_C}{R_2} + C \ddot{V}_C. \quad (28)$$

We can now eliminate  $I_L$  and  $\dot{I}_L$  in (26) by substituting (27) and (28) into (26), yielding

$$R_1 \left[ \frac{V_C}{R_2} + C \dot{V}_C \right] + V_C + L \left[ \frac{\dot{V}_C}{R_2} + C \ddot{V}_C \right] = 0.$$

Gathering terms,

$$[LC] \ddot{V}_C + \left[ R_1 C + \frac{L}{R_2} \right] \dot{V}_C + \left[ \frac{R_1}{R_2} + 1 \right] V_C = 0,$$

and putting into standard form yields

$$\ddot{V}_C + \left[ \frac{R_1}{L} + \frac{1}{R_2 C} \right] \dot{V}_C + \frac{1}{LC} \left[ \frac{R_1}{R_2} + 1 \right] V_C = 0 \quad (29)$$

Comparing (29) to the standard form in (5), we see that the circuit is second-order and

$$\omega_o^2 = \frac{1}{LC} \left[ \frac{R_1}{R_2} + 1 \right], \quad 2\alpha = \left[ \frac{R_1}{L} + \frac{1}{R_2 C} \right], \quad \text{so } \alpha = \frac{1}{2} \left[ \frac{R_1}{L} + \frac{1}{R_2 C} \right].$$

The solution procedure for the natural response and total response of either  $v_C(t)$  or  $i_L(t)$  can then proceed in the same way as for series and parallel RLC circuits, using the  $\alpha$  and  $\omega_o$  values shown above.

Notice in the above two equations that when  $R_2 \rightarrow \infty$ ,

$$\omega_o^2 \rightarrow \frac{1}{LC}, \quad \text{and } \alpha \rightarrow \frac{R_1}{2L},$$

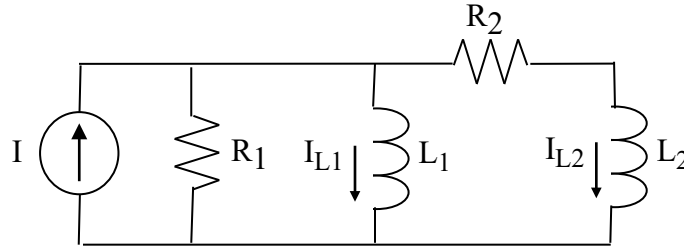
which correspond to a series RLC circuit. Does this make sense for this circuit?

Note that when  $R_1 \rightarrow 0$ , then

$$\omega_o^2 \rightarrow \frac{1}{LC}, \quad \text{and } \alpha \rightarrow \frac{1}{2R_2 C},$$

which correspond to a parallel RLC circuit. Does this make sense, too?

Now, consider Circuit #2.



General Case Circuit #2 (Prob. 8.60)

Turning off the independent source and writing KVL for the right-most mesh yields

$$-L_1 \dot{I}_{L1} + R_2 I_{L2} + L_2 \dot{I}_{L2} = 0. \quad (30)$$

KVL for the center mesh is

$$R_1 [I_{L1} + I_{L2}] + L_1 \dot{I}_{L1} = 0,$$

yielding

$$I_{L2} = -I_{L1} - \frac{L_1}{R_1} \dot{I}_{L1}. \quad (31)$$

Taking the derivative of (31) yields

$$\dot{I}_{L2} = -\dot{I}_{L1} - \frac{L_1}{R_1} \ddot{I}_{L1}. \quad (32)$$

Substituting into (31) and (32) into (30) yields

$$-L_1 \dot{I}_{L1} + R_2 \left[ -I_{L1} - \frac{L_1}{R_1} \dot{I}_{L1} \right] + L_2 \left[ -\dot{I}_{L1} - \frac{L_1}{R_1} \ddot{I}_{L1} \right] = 0.$$

Gathering terms,

$$\left[ -\frac{L_1 L_2}{R_1} \right] \ddot{I}_{L1} + \left[ -L_1 - \frac{R_2 L_1}{R_1} - L_2 \right] \dot{I}_{L1} - R_2 I_{L1} = 0,$$

and putting into standard form yields

$$\ddot{I}_{L1} + \left[ \frac{R_1}{L_2} + \frac{R_2}{L_2} + \frac{R_1}{L_1} \right] \dot{I}_{L1} + \left[ \frac{R_1 R_2}{L_1 L_2} \right] I_{L1} = 0. \quad (33)$$

Comparing (33) to (5),

$$\omega_o^2 = \frac{R_1 R_2}{L_1 L_2}, \quad 2\alpha = \left[ \frac{R_1}{L_2} + \frac{R_2}{L_2} + \frac{R_1}{L_1} \right], \quad \text{so } \alpha = \frac{1}{2} \left[ \frac{R_1}{L_2} + \frac{R_2}{L_2} + \frac{R_1}{L_1} \right]. \quad (34)$$

The solution procedure for the natural response and total response of either  $i_{L1}(t)$  or  $i_{L2}(t)$  can then proceed in the same way as for series and parallel RLC circuits, using the  $\alpha$  and  $\omega_o$  values shown above.

### Normalized Damping Ratio

Our previous expression for solving for  $s$  was

$$s^2 + 2\alpha s + \omega_o^2 = 0. \quad (35)$$

Equation (35) is sometimes written in terms of a “normalized damping ratio”  $\zeta$  as follows:

$$s^2 + 2\zeta\omega_o s + \omega_o^2 = 0. \quad (36)$$

Thus, the relationship between damping coefficient  $\alpha$  and normalized damping ratio  $\zeta$  is

$$\zeta = \frac{\alpha}{\omega_o}. \quad (37)$$

Normalized damping ratio  $\zeta$  has the convenient feature of being 1.0 at the point of critical damping. When  $\zeta < 1.0$ , the response is underdamped. When  $\zeta > 1.0$ , the response is overdamped. Examples for a unit step input are shown in the following figure.

