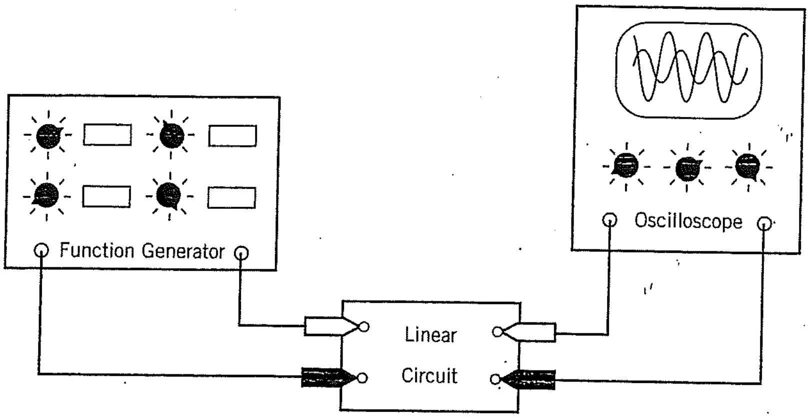


FIGURE 10.1-1
Measuring the input and output of a linear circuit.

10.2 *Sinusoidal Sources*

In this chapter, we will begin to consider electric circuits in which the source voltage or source current is sinusoidal. Such circuits play a prominent role in both communication systems and in power systems. There are so many important applications of these circuits that it is difficult to overstate their importance.

Consider a circuit having sinusoidal inputs. The inputs to a circuit are the independent voltage source voltages and the independent current source currents, so we are considering a circuit having sinusoidal source voltages and source currents. For now, assume that all of the sinusoidal inputs have the same frequency. Later we will consider the case where the inputs have different frequencies.

In Chapters 8 and 9, we've seen that the output or response of such a circuit consists of the sum of the natural response and the forced response, for example,

$$v(t) = v_n(t) + v_f(t)$$

When all of the inputs to the circuit are sinusoids having the same frequency, the forced response $v_f(t)$ is also a sinusoid having the same frequency as the inputs. As time goes on, the transient part of the response dies out. The part of the response that is left is called the steady-state response. Once the transient part of the response has died out, we say that the circuit is "at steady state." In the case of sinusoidal inputs having the same frequency, the steady-state response is equal to the forced response, a sinusoid at the input frequency.

We can choose the output of our circuit to be any voltage or current that is of interest to us. We conclude that when a circuit satisfies the two conditions that (1) all of the inputs are sinusoidal and have the same frequency and (2) the circuit is at steady state, then all of the currents and voltages are sinusoidal and have the same frequency as the inputs. Traditionally, sinusoidal currents have been called *alternating currents* (*ac*) and circuits that satisfy the above conditions are called *ac circuits*.

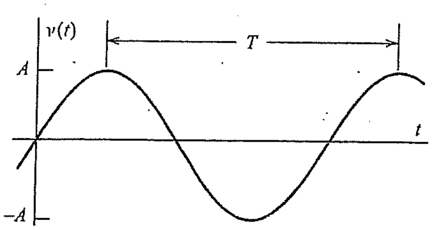


FIGURE 10.2-1 A sinusoidal function.

To summarize, an ac circuit is a steady-state circuit in which all of the inputs are sinusoidal and have the same frequency. All of the currents and voltages of an ac circuit are sinusoidal at the input frequency.

Consider the sinusoidal function

$$v(t) = A\sin(\omega t) \text{ V} \qquad (10.2\text{-}1)$$

shown in Figure 10.2-1. The parameter A in Eq. 10.2-1 and also in Figure 10.2-1 is called the amplitude of the sinusoid. The sinusoid is a periodic function defined by the property

$$v(t+T) = v(t) \qquad (10.2\text{-}2)$$

for all time. The constant T is called the "period of oscillation" or just the "period." The reciprocal of T defines the frequency or number of cycles per second, denoted by f, where

$$f = \frac{1}{T} \qquad (10.2\text{-}3)$$

The units of frequency are hertz (Hz) in honor of the scientist Heinrich Hertz, shown in Figure 10.2-2. The angular frequency of the sinusoidal function is

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad (10.2\text{-}4)$$

The units of angular frequency are radians per second.

Next, consider the effect of replacing t by $t + t_a$ where t_a is some arbitrary constant time. As shown in Figure 10.2-3, $v(t+t_a)$ is a sinusoid that is identical to $v(t)$ except that $v(t+t_a)$ is advanced from $v(t)$ by time t_a. We have

$$v(t+t_a) = A\sin(\omega(t+t_a)) = A\sin(\omega t + \omega t_a) = A\sin(\omega t + \theta) \text{ V}$$

Courtesy of the Institution of Electrical Engineers

FIGURE 10.2-2 Heinrich R. Hertz (1857–1894).

where θ is in radians and is called the phase angle of the sinusoid $A\sin(\omega t + \theta)$. The phase angle in radians is related to the time t_a by

$$\theta = \omega t_a = \frac{2\pi}{T} t_a = 2\pi \frac{t_a}{T} \qquad (10.2\text{-}5)$$

Similarly, replacing t by $t - t_d$ produces a sinusoid that is identical to $v(t)$ except that $v(t - t_d)$ is delayed from $v(t)$ by time t_d. We have

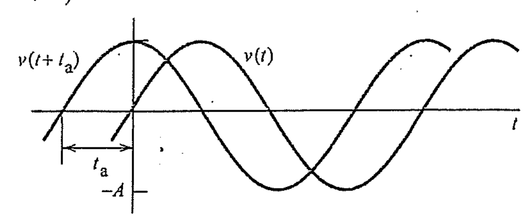


FIGURE 10.2-3 Advancing a sinusoid in time.

$$v(t - t_d) = A\sin(\omega(t - t_d)) = A\sin(\omega t - \omega t_d) = A\sin(\omega t + \theta) \text{ V}$$

where now the phase angle in radians is related to the time t_d by

$$\theta = -\omega t_d = -\frac{2\pi}{T} t_d = -2\pi \frac{t_d}{T} \qquad (10.2\text{-}6)$$

Notice that an advance or delay of a full period leaves a sinusoid unchanged, that is $v(t \pm T) = v(t)$. Consequently, an advance by time t_a is equivalent to a delay by time $T - t_a$. Similarly, a delay by time t_d is equivalent to an advance by time $T - t_d$.

Try it yourself in WileyPLUS

EXAMPLE 10.2-1 Phase Shift and Delay

Consider the sinusoids

$$v_1(t) = 10\cos(200t + 45°) \text{ V and } v_2(t) = 8\sin(200t + 15°) \text{ V}$$

Determine the time by which $v_2(t)$ is advanced or delayed with respect to $v_1(t)$.

Solution

The two sinusoids have the same frequency but different amplitudes. The time by which $v_2(t)$ is advanced or delayed with respect to $v_1(t)$ is the time between a peak of $v_2(t)$ and the nearest peak of $v_1(t)$. The period of the sinusoids is given by

$$200 = \frac{2\pi}{T} \quad \Rightarrow \quad T = \frac{\pi}{100} = 0.0314159 = 31.4159 \text{ ms}$$

Eq. 10.3-19 is required to be true for all values of time t. Let $t=0$. Then $e^{j\omega t}=e^{0}=1$ and Eq. 10.3-19 becomes

$$0=\mathrm{Re}\left\{\sum_i \mathbf{V}_i(\omega)\right\} \tag{10.3-20}$$

Next, let $t=\pi/(2\,\omega)$. Then $e^{j\omega t}=e^{-j\,\pi/2}=-j$ and Eq. 10.3-19 becomes

$$0=\mathrm{Re}\left\{-j\sum_i \mathbf{V}_i(\omega)\right\}=\mathrm{Im}\left\{\sum_i \mathbf{V}_i(\omega)\right\} \tag{10.3-21}$$

Together, Eqs. 10.3-19 and 10.3-21 indicate that the phasors 0 and $\sum_i \mathbf{V}_i(\omega)$ are equal. That is,

$$0=\sum_i \mathbf{V}_i(\omega)$$

In summary, if a set of sinusoidal voltages $v_i(t)$ satisfy KVL for an ac circuit, the corresponding phasor voltages $\mathbf{V}_i(\omega)$ satisfy the same KVL equation. Similarly, if a set of sinusoidal currents $i_i(t)$ satisfy KCL for an ac circuit, the corresponding phasor currents $\mathbf{I}_i(\omega)$ satisfy the same KCL equation.

SAME

Try it yourself in WileyPLUS

EXAMPLE 10.3-4 Kirchhoff's Laws for AC Circuits

The input to the circuit shown in Figure 10.3-3 is the voltage source voltage,

$$v_s(t)=25\cos(100t+15^\circ)\ \mathrm{V}$$

The output is the voltage across the capacitor,

$$v_C(t)=20\cos(100t-22^\circ)\ \mathrm{V}$$

Determine the resistor voltage $v_R(t)$.

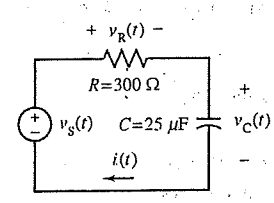


FIGURE 10.3-3 The circuit in Example 10.3-4

Solution

Apply KVL to get

$$v_R(t)=v_s(t)-v_C(t)=25\cos(100t+15^\circ)-20\cos(100t-22^\circ)$$

Writing the KVL equation using phasors, we have

$$\begin{aligned}\mathbf{V}_R(\omega)=\mathbf{V}_s(\omega)-\mathbf{V}_C(\omega)&=25\angle 15^\circ-20\angle -22^\circ\\ &=(24.15+j6.47)-(18.54-j7.49)\\ &=5.61+j13.96\\ &=15\angle 68.1^\circ\ \mathrm{V}\end{aligned}$$

Converting the phasor $\mathbf{V}_R(\omega)$ to the corresponding sinusoid, we have

$$\mathbf{V}_R(\omega)=15\angle 68.1^\circ\ \mathrm{V} \quad\Leftrightarrow\quad v_R(t)=15\cos(100t+68.1^\circ)\ \mathrm{V}$$

Alternate Solution

Alternately, we can solve the KVL equation using trigonometry instead of phasors. We'll need this trigonometric identity from Appendix C:

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Using the trigonometric identity, we determine

$$\begin{aligned} 25\cos(100t + 15°) &= 25[\cos(100t)\cos(15°) - \sin(100t)\sin(15°)] \\ &= 24.15\cos(100t) - 6.47\sin(100t) \end{aligned}$$

and

$$\begin{aligned} 20\cos(100t - 22°) &= 20[\cos(100t)\cos(22°) + \sin(100t)\sin(22°)] \\ &= 18.54\cos(100t) + 7.49\sin(100t) \end{aligned}$$

Substituting these results into the KVL equation gives

$$\begin{aligned} v_R(t) = v_s(t) - v_C(t) &= 25\cos(100t + 15°) - 20\cos(100t - 22°) \\ &= [24.15\cos(100t) - 6.47\sin(100t)] - [18.54\cos(100t) + 7.49\sin(100t)] \\ &= 5.61\cos(100t) - 13.96\sin(100t) \\ &= \sqrt{5.61^2 + 13.96^2}\cos\left(100t - \tan^{-1}\left(\frac{-16.96}{5.61}\right)\right) \\ &= 15\cos(100t + 68.1°)\text{ V} \end{aligned}$$

Using phasors instead of trigonometry to solve the KVL equation produced the same result but required less effort.

10.4 Impedances

NEW

We've seen that all of the currents and voltages of an ac circuit are sinusoids at the input frequency. Figure 10.4-1a shows an element of an ac circuit. The element voltage and element current are labeled as $v(t)$ and $i(t)$. We can write

$$v(t) = V_m\cos(\omega t + \theta)\text{ V and } i(t) = I_m\cos(\omega t + \phi)\text{ A} \tag{10.4-1}$$

where V_m and I_m are the amplitudes of the sinusoidal voltage and current, θ and ϕ are the phase angles of the voltage and current, and ω is the input frequency. The corresponding phasors are

$$\mathbf{V}(\omega) = V_m\angle\theta\text{ V and } \mathbf{I}(\omega) = I_m\angle\phi\text{ A}$$

Figure 10.4-1b shows the circuit element again, now labeled with the phasor voltage and current $\mathbf{V}(\omega)$ and $\mathbf{I}(\omega)$. Notice that the voltage and current adhere to the passive convention in both Figure 10.4-1a and Figure 10.4-1b.

The impedance of an element of an ac circuit is defined to be the ratio of the voltage phasor to the current phasor. The impedance is denoted as $\mathbf{Z}(\omega)$ so

$$\mathbf{Z}(\omega) = \frac{\mathbf{V}(\omega)}{\mathbf{I}(\omega)} = \frac{V_m\angle\theta}{I_m\angle\phi} = \frac{V_m}{I_m}\angle(\theta - \phi)\ \Omega \tag{10.4-2}$$

Consequently,

$$\mathbf{V}(\omega) = \mathbf{Z}(\omega)\mathbf{I}(\omega) \tag{10.4-3}$$

A resistor from an ac circuit is shown in Figure 10.4-4a. We know that the resistor voltage is a sinusoid at the input frequency so we can write

$$v_R(t) = A\cos(\omega t + \theta)$$

The resistor current is

$$i_R(t) = \frac{v_R(t)}{R} = \frac{A}{R}\cos(\omega t + \theta)$$

The impedance of the resistor is the ratio of the voltage phasor to the current phasor:

$$\mathbf{Z}_R(\omega) = \frac{\mathbf{V}_R(\omega)}{\mathbf{I}_R(\omega)} = \frac{A\angle\theta}{\frac{A}{R}\angle\theta} = R\ \Omega \quad (10.4\text{-}8)$$

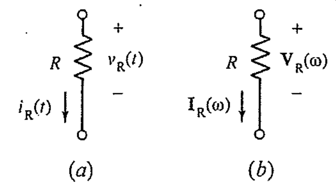


FIGURE 10.4-4 A resistor in an ac circuit represented (a) in the time domain and (b) in the frequency domain.

The impedance of a resistor is numerically equal to the resistance. Using Eq. 10.4-3, we write

$$\mathbf{V}_R(\omega) = R\,\mathbf{I}_R(\omega) \quad (10.4\text{-}9)$$

Try it yourself in WileyPLUS

EXAMPLE 10.4-1 Impedances

The input to the ac circuit shown in Figure 10.4-5 is the source voltage

$$v_S(t) = 12\cos(1000t + 15°)\ \text{V}$$

Determine (a) the impedances of the capacitor, inductor, and resistance and (b) the current $i(t)$.

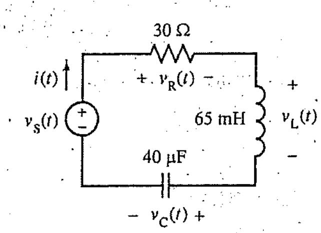


FIGURE 10.4-5 The AC circuit in Eample 10.4-1.

Solution

(a) The input frequency is $\omega = 1000$ rad/s. Using Eq. 10.4-4 shows that the impedance of the capacitor is

$$\mathbf{Z}_C(\omega) = \frac{1}{j\omega C} = \frac{1}{j1000\left(40 \times 10^{-6}\right)} = \frac{25}{j} = -j25\ \Omega$$

Using Eq. 10.4-6 shows that the impedance of the inductor is

$$\mathbf{Z}_L(\omega) = j\omega L = j1000(0.065) = j65\ \Omega$$

Using Eq. 10.4-8, the impedance of the resistor is

$$\mathbf{Z}_R(\omega) = R = 30\ \Omega$$

(b) Apply KVL to write

$$12\cos(1000t + 15°) = v_R(t) + v_L(t) + v_C(t)$$

Using phasors, we get

$$12\angle 15° = \mathbf{V}_R(\omega) + \mathbf{V}_L(\omega) + \mathbf{V}_C(\omega) \quad (10.4\text{-}10)$$

Using Eqs. 10.4-5, 10.4-7, and 10.4-9, we get

$$12\angle 15° = 30\,\mathbf{I}(\omega) + j65\,\mathbf{I}(\omega) - j25\,\mathbf{I}(\omega) = (30 + j40)\,\mathbf{I}(\omega) \quad (10.4\text{-}11)$$

Solving for $\mathbf{I}(\omega)$ gives

$$\mathbf{I}(\omega) = \frac{12\angle 15°}{30 + j40} = \frac{12\angle 15°}{50\angle 53.13°} = 0.24\angle 38.13°\ \text{A}$$

The corresponding sinusoid is

$$i(t) = 0.24\cos(1000t - 38.13°)\ \text{A}$$

Solution

The input frequency is $\omega = 1000$ rad/s. The impedance of the capacitor is

$$\frac{1}{j\omega C} = \frac{1}{j1000\left(20 \times 10^{-6}\right)} = \frac{50}{j} = -j50\ \Omega,$$

Figure 10.4-11 shows the circuit represented in the frequency domain using phasors and impedances. Notice that

(a) The voltage source voltage is described by the phasor corresponding to $v_s(t)$.
(b) The currents and voltages of the CCVS are described by phasors. The phasor corresponding to the controlled voltage is expressed as the product of the gain of the CCVS and the phasor corresponding to the controlling current.
(c) The resistors and the capacitor are described by their impedances.

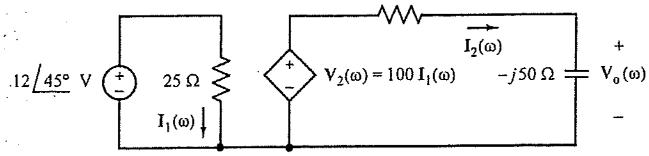


FIGURE 10.4-11 The circuit from Figure 10.4-10, represented in the frequency domain.

The controlling current of the CCVS in Figure 10.4-11 is

$$\mathbf{I}_1(\omega) = \frac{12\angle 45°}{25} = 0.48\angle 45°\ \text{A}$$

Apply KVL to the right-hand mesh in Figure 10.4-11 to get

$$100\mathbf{I}_1(\omega) = 25\mathbf{I}_2(\omega) - j50\mathbf{I}_2(\omega)$$

Solving for $\mathbf{I}_2(\omega)$ gives

$$\mathbf{I}_2(\omega) = \frac{100\left(0.48\angle 45°\right)}{25 - j50} = 0.85865\angle 108.44°\ \text{A}$$

Finally,

$$\mathbf{V}_o(\omega) = -j50 * \mathbf{I}_2(\omega) = 42.933\angle 18.44°\ \text{V}$$

The corresponding sinusoid is

$$v_o(t) = 42.933\cos(1000t + 18.44°)\ \text{V}$$

SAME

10.5 Series and Parallel Impedances

Figure 10.5-1*a* shows a circuit called "Circuit A" connected to two series impedances. Using KCL in Figure 10.5-1 shows that

$$\mathbf{I}_1 = \mathbf{I}_2 = \mathbf{I} \tag{10.5-1}$$

Using Ohm's law in Figure 10.5-1*a* shows that

$$\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}_1 = \mathbf{Z}_1\mathbf{I} \text{ and } \mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}_2 = \mathbf{Z}_2\mathbf{I}$$

Using KVL in Figure 10.5-1*a* shows that

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = (\mathbf{Z}_1 + \mathbf{Z}_2)\mathbf{I} \tag{10.5-2}$$

The impedance of the series combination of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ is given by

$$\frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2$$

We call this impedance the equivalent impedance of the parallel impedances and write

$$\mathbf{Z}_{eq} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}} \tag{10.5-8}$$

We say that the impedance $\mathbf{Z}_{eq}$ is equivalent to the parallel combination of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ because replacing $\mathbf{Z}_1$ and $\mathbf{Z}_2$ in Figure 10.5-2*a* by $\mathbf{Z}_{eq}$ in Figure 10.5-2*b* will not change the current or voltage of any element of Circuit A. Equation 10.5-8 generalizes to the case of n series impedances

$$\mathbf{Z}_{eq} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_n}} \tag{10.5-9}$$

Equivalently, we can write equation 10.5-9 in terms of admittances

$$\mathbf{Y}_{eq} = \frac{1}{\mathbf{Z}_{eq}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_n} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n \tag{10.5-10}$$

The currents the impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ in Figure 10.5-2*a* are given by

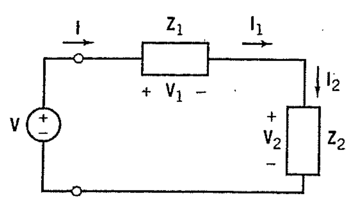
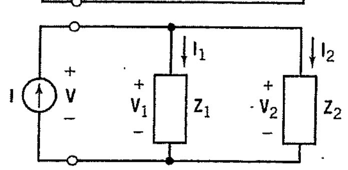
$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{1}{\mathbf{Z}_1}\frac{\mathbf{I}}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I} \text{ and } \mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I} \tag{10.5-11}$$

These equations show how **I**, the current in the parallel impedances, is divided between the individual impedances. They are called the current division equations.

The voltage division equations and current division equations are summarized in Table 10.5-1.

Table 10.5-1 Voltage and Current Division in the Frequency Domain

SAME

	CIRCUIT	EQUATIONS
Voltage division		$\mathbf{I}_1 = \mathbf{I}_2 = \mathbf{I}$ $\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$ $\mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$
Current division		$\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}$ $\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$ $\mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$

Try it yourself in WileyPLUS

EXAMPLE 10.5-1 Analysis of AC Circuits Using Impedances

Determine the steady-state current $i(t)$ in the *RLC* circuit shown in Figure 10.5-3*a*, using phasors and impedances.

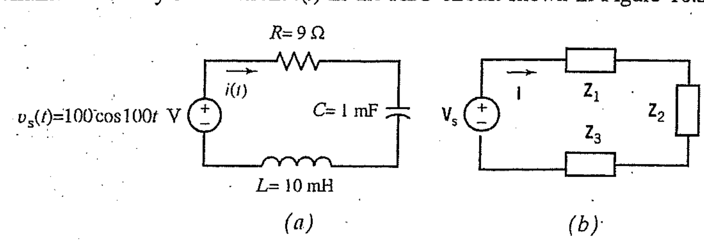


FIGURE 10.5-3 The circuit from Example 10.5-1 represented (*a*) in the time domain and (*b*) in the frequency domain.

Solution

First, we represent the circuit in using phasors and impedances as shown in Figure 10.5-3b. Noticing that the frequency of the sinusoidal input in Figure 10.5-3a is $\omega = 100$ rad/s, the impedances in Figure 10.5-3b are determined to be

$$\mathbf{Z}_1 = R = 9\,\Omega,\ \mathbf{Z}_2 = \frac{1}{j\omega C} = \frac{1}{j(100)(0.001)} = \frac{10}{j} = -j10\,\Omega$$

and

$$\mathbf{Z}_3 = j\omega L = j(100)(0.001) = j1\,\Omega$$

The input phasor in Figure 10.5-3b is

$$\mathbf{V}_s = 100\,\underline{/0^\circ}\ \mathrm{V}$$

Next, we use KVL in Figure 10.5-3b to obtain

$$\mathbf{Z}_1\mathbf{I} + \mathbf{Z}_2\mathbf{I} + \mathbf{Z}_3\mathbf{I} = \mathbf{V}_s$$

Substituting for the impedances and the input phasor gives

$$(9 - j10 + j1)\mathbf{I} = 100\,\underline{/0^\circ}$$

or

$$\mathbf{I} = \frac{100\,\underline{/0^\circ}}{9 - j9} = \frac{10\,\underline{/0^\circ}}{9\sqrt{2}\,\underline{/-45^\circ}} = 7.86\,\underline{/45^\circ}\ \mathrm{A}$$

Therefore, the steady-state current in the time domain is

$$i(t) = 7.86\cos(100t + 45^\circ)\ \mathrm{A}$$

SAME

Try it yourself in WileyPLUS

INTERACTIVE EXAMPLE

EXAMPLE 10.5-2 Voltage Division Using Impedances

Consider the circuit shown in Figure 10.5-4a. The input to the circuit is the voltage of the voltage source,

$$v_s(t) = 7.28\cos(4t + 77^\circ)\ \mathrm{V}$$

The output is the voltage across the inductor $v_o(t)$. Determine the steady-state output voltage $v_o(t)$.

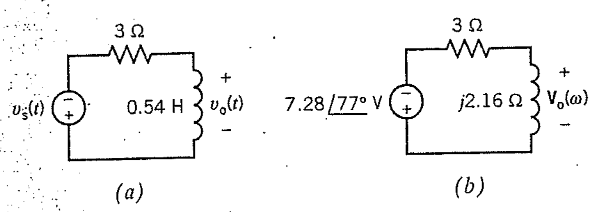


FIGURE 10.5-4 The circuit considered in Example 10.5-2 represented (a) in the time domain and (b) in the frequency domain.

Solution

The input voltage is sinusoid. The output voltage is also sinusoid and has the same frequency as the input voltage. The circuit has reached steady state. Consequently, the circuit in Figure 10.5-4a can be represented in the frequency domain, using phasors and impedances. Figure 10.5-4b shows the frequency-domain representation of the circuit from Figure 10.5-4a. The impedance of the inductor is $j\omega L = j(4)(0.54) = j2.16\,\Omega$, as shown in Figure 10.5-4b.

Apply the voltage divider principle to the circuit in Figure 10.5-4b to represent the output voltage in the frequency domain as

$$\mathbf{V}_o(\omega) = \frac{j2.16}{3+j2.16}\left(-7.28\,\angle 77^\circ\right) = \frac{2.16\,\angle 90^\circ}{3.70\,\angle 36^\circ}\left(-7.28\,\angle 77^\circ\right)$$

$$= \frac{(2.16)(-7.28)}{3.70}\,\angle (90^\circ + 77^\circ) - 36^\circ$$

$$= -4.25\,\angle 131^\circ = 4.25\,\angle 311^\circ \text{ V}$$

In the time domain, the output voltage is represented as

$$v_o(t) = 4.25\cos(4t + 311^\circ)\text{ V}$$

SAME

Try it yourself in WileyPLUS

EXAMPLE 10.5-3 AC Circuit Analysis

INTERACTIVE EXAMPLE

Consider the circuit shown in Figure 10.5-5a. The input to the circuit is the voltage of the voltage source,

$$v_s(t) = 7.68\cos(2t + 47^\circ)\text{ V}$$

The output is the voltage across the resistor,

$$v_o(t) = 1.59\cos(2t + 125^\circ)\text{ V}$$

Determine capacitance C of the capacitor.

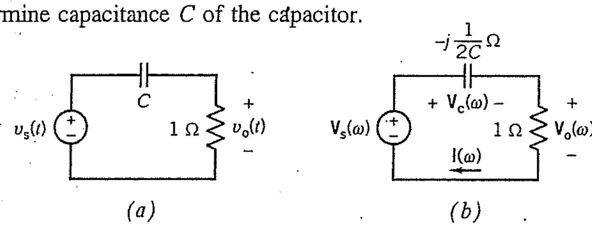


FIGURE 10.5-5 The circuit considered in Example 10.5-3 represented (a) in the time domain and (b) in the frequency domain.

Solution

The input voltage is sinusoid. The output voltage is also sinusoid and has the same frequency as the input voltage. Apparently, the circuit has reached steady state. Consequently, the circuit in Figure 10.5-5a can be represented in the frequency domain, using phasors and impedances. Figure 10.5-5b shows the frequency-domain representation of the circuit from Figure 10.5-5a. The impedance of the capacitor is

$$\frac{1}{j\omega C} = \frac{j}{j^2\omega C} = -\frac{j}{\omega C} = -\frac{j}{2\,C}$$

The phasors corresponding to the input and output sinusoids are

$$\mathbf{V}_s(\omega) = 7.68\,\angle 47^\circ\text{ V}$$

and

$$\mathbf{V}_o(\omega) = 1.59\,\angle 125^\circ\text{ V}$$

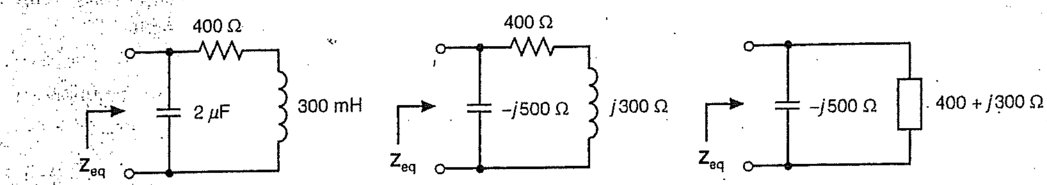
The current $\mathbf{I}(\omega)$ in Figure 10.5-5b is given by

$$\mathbf{I}(\omega) = \frac{\mathbf{V}_o(\omega)}{1} = \frac{1.59\,\angle 125^\circ}{1\,\angle 0^\circ} = 1.59\,\angle 125^\circ\text{ A}$$

Try it yourself in WileyPLUS

EXAMPLE 10.5-5 Equivalent Impedance

Determine the equivalent impedance of the circuit shown in Figure 10.5-7*a* at the frequency $\omega = 1000$ rad/s.



(*a*) (*b*) (*c*)

FIGURE 10.5-7 The circuit considered in Example 10.5-5 (*a*) in the time domain, (*b*) in the frequency domain, and (*c*) after replacing series impedances by an equivalent impedance.

Solution

Represent the circuit in the frequency domain as shown in Figure 10.5-7*b*. After replacing series impedances by an equivalent impedance, we have the circuit shown in Figure 10.5-7*c*. Z_{eq} is now seen to be the equivalent impedance of the parallel impedances in Figure 10.5-7*c*.

$$Z_{eq} = \frac{-j500(400+j300)}{-j500+400+j300} = \frac{150,000 - j200,000}{400 - j200} = \frac{250,000\angle -53.1^\circ}{447.2\angle -26.6^\circ} = 599.0\angle -26.5^\circ \Omega$$

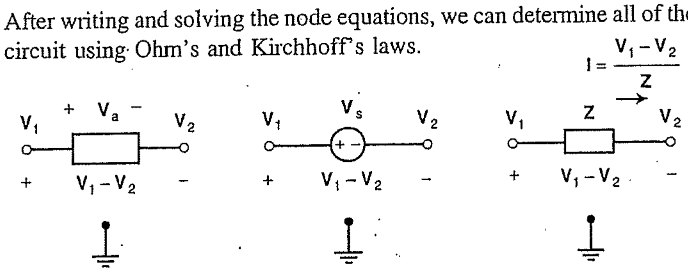
10.6 *Mesh and Node Equations* SAME

We can analyze an ac circuit by writing and solving a set of simultaneous equations. Two methods, the node equations and the mesh equations, are quite popular. Before writing either the node equations or mesh equations, we represent the ac circuit in the frequency domain using phasors and impedances.

The node equations are a set of simultaneous equations in which the unknowns are the node voltages. We write the node equations by

1. Expressing the element voltages and currents (for example, the current and voltage of an impedance) in terms of the node voltages.
2. Applying KCL at the nodes of the ac circuit.

After writing and solving the node equations, we can determine all of the voltages and currents of the ac circuit using Ohm's and Kirchhoff's laws.



(*a*) (*b*) (*c*)

FIGURE 10.6-1 Expressing element voltages and currents in terms of node voltages.

Figure 10.6-1 illustrates techniques for expressing the element voltages and currents in terms of the node voltages. Figure 10.6-1*a* shows a generic circuit element having node voltages $\mathbf{V}_1$ and $\mathbf{V}_2$ and element voltage $\mathbf{V}_a$. We see that

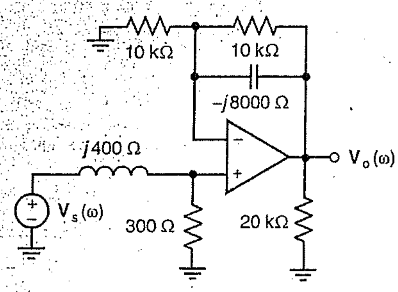


FIGURE 10.6-14 The frequency domain representation of the circuit from Figure 10.6-13.

Applying KCL at the noninverting node of the op amp, we get

$$\frac{\mathbf{V}_s - \mathbf{V}_a}{j400} = \frac{\mathbf{V}_a}{300} + 0 \quad \Rightarrow \quad \mathbf{V}_s = \mathbf{V}_a\left(1 + \frac{j400}{300}\right)$$

Solving for $\mathbf{V}_a$ gives

$$\mathbf{V}_a = \left(\frac{300}{300 + j\,400}\right)\mathbf{V}_s = \left(0.6\,\underline{/-53.1^\circ}\right)\left(0.125\,\underline{/15^\circ}\right) = 0.075\,\underline{/-38.1^\circ}\ \mathrm{V}$$

Next, apply KCL at the inverting node of the op amp to get

$$\frac{\mathbf{V}_a}{4000} + \frac{\mathbf{V}_a - \mathbf{V}_o}{10,000} + \frac{\mathbf{V}_a - \mathbf{V}_o}{-j8000} = 0$$

Multiplying by 80,000 gives

$$0 = 20\,\mathbf{V}_a + 8(\mathbf{V}_a - \mathbf{V}_o) + j10(\mathbf{V}_a - \mathbf{V}_o)$$

Solving for $\mathbf{V}_o$ gives

$$\mathbf{V}_o = \frac{28 + j10}{8 + j10}\mathbf{V}_a = \frac{29.73\,\underline{/19.65^\circ}}{12.81\,\underline{/51.34^\circ}}\left(0.075\,\underline{/-38.1^\circ}\right) = 0.174\,\underline{/-69.79^\circ}$$

In the time domain, the output voltage is

$$v_o(t) = 174\cos(500t - 69.79^\circ)\ \mathrm{mV}$$

10.7 Thévenin and Norton Equivalent Circuits

SAME

In this section, we will determine the Thévenin and Norton equivalent circuits of an ac circuit.

Figure 10.7-1 illustrates the use of Thévenin and Norton equivalent circuits. In Figure 10.7-1*a*, an ac circuit is partitioned into two parts—circuit A and circuit B—that are connected at a single pair of terminals. (This is the only connection between circuits A and B. In particular, if the overall circuit contains a dependent source, then either both parts of that dependent source must be in circuit A or both parts must be in circuit B.) In Figure 10.7-1*b*, circuit A is replaced by its Thévenin equivalent circuit, which consists of a voltage source in series with an impedance. In Figure 10.7-1*c*, circuit A is replaced by its Norton equivalent circuit, which consists of a current source in parallel with an impedance. Replacing circuit A by its Thévenin or Norton equivalent circuit does not change the voltage or current of any element in circuit B. This means that if you looked at a list of the values of the currents and voltages of all the circuit elements in circuit B, you could not tell whether circuit B was connected to circuit A or connected to its Thévenin equivalent or connected to its Norton equivalent circuit.

Finding the Thévenin or Norton equivalent circuit of circuit A involves three parameters: the open-circuit voltage $\mathbf{V}_{oc}$, the short-circuit current $\mathbf{I}_{sc}$, and the Thévenin impedance $\mathbf{Z}_t$. Figure 10.7-2

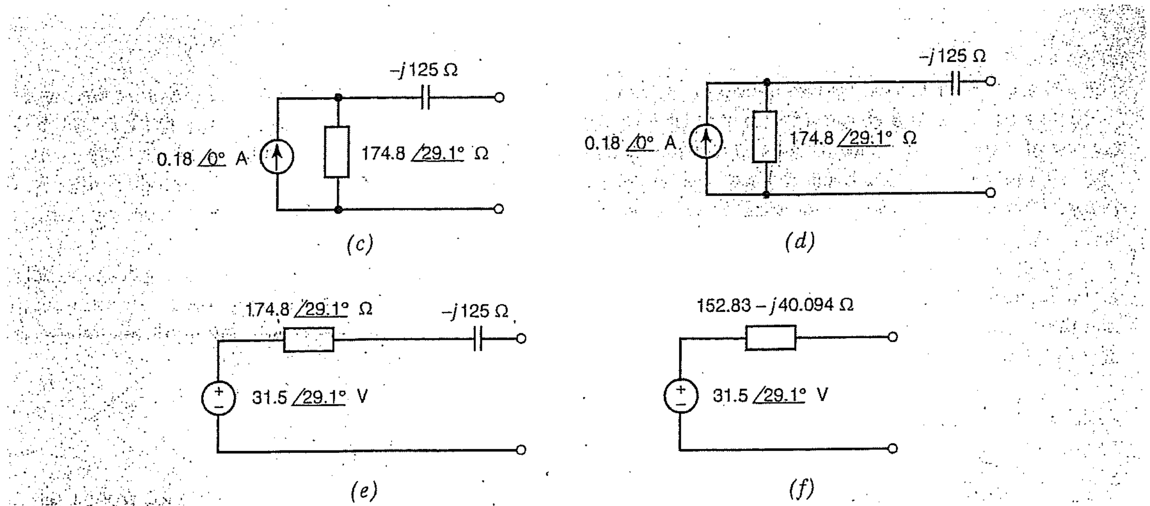


FIGURE 10.7-12 *(Continued)*

NEW.

10.8 *Superposition*

Suppose we encounter a circuit that is at steady state and all of its inputs are sinusoidal but not all of the input sinusoid have the same frequency. Such a circuit is not an ac circuit and the currents and voltages will not be sinusoidal. We can analyze this circuit using the principle of superposition.

The principle of superposition says that the output of a linear circuit due to several inputs working together is equal to the sum of the outputs working separately. The inputs to the circuit are the voltages of the independent voltage sources and the currents of the independent current sources.

When we set all but one input to zero, the other inputs become 0-V voltage sources and 0-A current sources. Because 0-V voltage sources are equivalent to short circuits and 0-A current sources are equivalent to open circuits, we replace the sources corresponding to the other inputs by open or short circuits. We are left with a steady-state circuit having a single sinusoidal input. Such a circuit is an ac circuit and we analyze it using phasors and impedances.

Thus, we use superposition to replace a circuit involving several sinusoidal inputs at different frequencies by several circuits each having a single sinusoidal input. We analyze each of the several ac circuits using phasors and impedances to obtain its sinusoidal output. The sum of those several sinusoidal outputs will be identical to the output of the original circuit. The following example illustrates this procedure.

EXAMPLE 10.8-1 Superposition

Determine the voltage $v_o(t)$ across the 8-Ω resistor in the circuit shown in Figure 10.8-1.

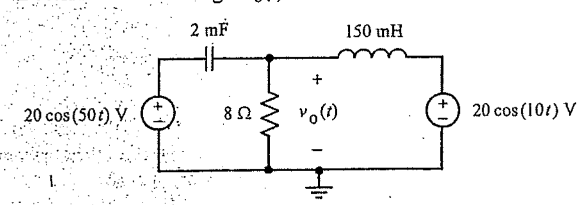


FIGURE 10.8-1 The circuit considered in Example 10.8-1.

Now let us determine the ratio of output-to-input voltage $\mathbf{V}_o/\mathbf{V}_s$ for the inverting amplifier shown in Figure 10.10-1a. This circuit can be analyzed by writing the node equation at node a as

$$\frac{\mathbf{V}_s - \mathbf{V}_1}{\mathbf{Z}_1} + \frac{\mathbf{V}_o - \mathbf{V}_1}{\mathbf{Z}_2} - \mathbf{I}_1 = 0 \tag{10.10-1}$$

When the operational amplifier is ideal, $\mathbf{V}_1$ and $\mathbf{I}_1$ are both 0. Then,

$$\frac{\mathbf{V}_s}{\mathbf{Z}_1} + \frac{\mathbf{V}_o}{\mathbf{Z}_2} = 0 \tag{10.10-2}$$

Finally,

$$\frac{\mathbf{V}_o}{\mathbf{V}_s} = -\frac{\mathbf{Z}_2}{\mathbf{Z}_1} \tag{10.10-3}$$

Next, we will determine the ratio of output-to-input voltage $\mathbf{V}_o/\mathbf{V}_s$ for the noninverting amplifier shown in Figure 10.10-1b. This circuit can be analyzed by writing the node equation at node a as

$$\frac{(\mathbf{V}_s + \mathbf{V}_1)}{\mathbf{Z}_1} - \frac{\mathbf{V}_o - (\mathbf{V}_s + \mathbf{V}_1)}{\mathbf{Z}_2} + \mathbf{I}_1 = 0 \tag{10.10-4}$$

When the operational amplifier is ideal, $\mathbf{V}_1$ and $\mathbf{I}_1$ are both 0. Then,

$$\frac{\mathbf{V}_s}{\mathbf{Z}_1} - \frac{\mathbf{V}_o - \mathbf{V}_s}{\mathbf{Z}_2} = 0$$

Finally,

$$\frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{\mathbf{Z}_1 + \mathbf{Z}_2}{\mathbf{Z}_1} \tag{10.10-5}$$

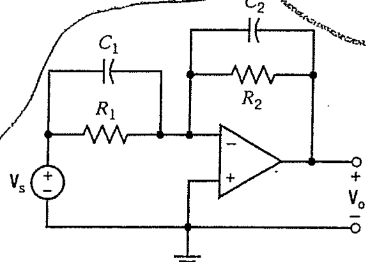


FIGURE 10.10-2 Operational amplifier with two RC circuits connected.

Typically, impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are obtained using only resistors and capacitors. Of course, in theory, we could use inductors, but their cost and size relative to capacitors result in little use of inductors with operational amplifiers.

An example of the inverting amplifier is shown in Figure 10.10-2, The impedance $\mathbf{Z}_n$, where n is equal to 1 or 2, is a parallel R_nC_n impedance so that

$$\mathbf{Z}_n = \frac{R_n \dfrac{1}{j\omega C_n}}{R_n + \dfrac{1}{j\omega C_n}} = \frac{R_n}{1 + j\omega C_n R_n} \tag{10.10-6}$$

Using Eqs. 10.10-3 and 10.10-6, one may obtain the ratio $\mathbf{V}_o/\mathbf{V}_s$.

EXAMPLE 10.10-1 AC Amplifier

Find the ratio $\mathbf{V}_o/\mathbf{V}_s$ for the circuit of Figure 10.10-2 when $R_1 = 1\,\text{k}\Omega$, $R_2 = 10\,\text{k}\Omega$, $C_1 = 0$, and $C_2 = 0.1\,\mu\text{F}$ for $\omega = 1000$ rad/s.

Solution

The circuit of Figure 10.10-2 is an example of the inverting amplifier shown in Figure 10.10-1a. Using Eqs. 10.10-3 and 10.10-6, we obtain

$$\frac{\mathbf{V}_o}{\mathbf{V}_s} = -\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = -\frac{\dfrac{R_2}{1 + j\omega C_2 R_2}}{\dfrac{R_1}{1 + j\omega C_1 R_1}} = -\frac{R_2(1 + j\omega C_1 R_1)}{R_1(1 + j\omega C_2 R_2)}$$

Substituting the given values of R_1, R_2, C_1, C_2, and ω gives

$$\frac{\mathbf{V}_o}{\mathbf{V}_s} = -\frac{10^4(1 + j10^0(0)10^3)}{10^3(1 + j10^3(0.1 \times 10^{-6})10^4)} = -\frac{10}{1+j} = 7.07\,\angle 135^\circ$$

EXERCISE 10.10-1 Find the ratio V_o/V_s for the circuit shown in Figure 10.10-2 when $R_1 = R_2 = 1\ k\Omega$, $C_2 = 0$, $C_1 = 1\ \mu F$, and $\omega = 1000$ rad/s.

Answer: $V_o/V_s = -1 - j$

10.11 *The Complete Response*

Next, we consider circuits with sinusoidal inputs that are subject to abrupt changes, as when a switch opens or closes. To find the complete response of such circuits, we:

- Represent the circuit by a differential equation.
- Find the general solution of the homogeneous differential equation. This solution is the natural response $v_n(t)$. The natural response will contain unknown constants that will be evaluated later.
- Find a particular solution of the differential equation. This solution is the forced response $v_f(t)$.
- Represent the response of the circuit as $v(t) = v_n(t) + v_f(t)$.
- Use the initial conditions, for example, the initial values of the currents in inductors and the voltages across capacitors to evaluate the unknown constants.

Consider the circuit shown in Figure 10.11-1. Before time $t = 0$, this circuit is at steady state, so all its voltages and currents are sinusoidal with a frequency of 5 rad/s. At time $t = 0$, the switch closes, disturbing the circuit. Immediately after $t = 0$, the currents and voltages are not sinusoidal. Eventually, the disturbance dies out and the circuit is again at steady state (most likely a different steady state). Once again, the currents and voltages are all sinusoidal with a frequency of 5 rad/s.

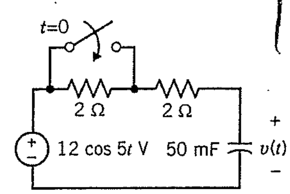


FIGURE 10.11-1 The circuit considered in Example 10.11-1.

Two different steady-state responses are used to find the complete response of this circuit. The steady-state response before the switch closes is used to determine the initial condition. The steady-state response after the switch closes is used as the particular solution of the differential equation representing the circuit.

EXAMPLE 10.11-1 Complete Response

Determine $v(t)$, the voltage across the capacitor in Figure 10.11-1, both before and after the switch closes.

Solution

Step 1: For $t < 0$, the switch is open and the circuit is at steady state.

The open switch acts like an open circuit, so the two 2-Ω resistors are connected in series. Replacing the series resistors with an equivalent resistor produces the circuit shown in Figure 10.11-2*a*. Next, we use impedances and phasors to represent the circuit in the frequency domain as shown in Figure 10.11-2*b*.

Using voltage division in the frequency domain gives

$$\mathbf{V}(\omega) = \left(\frac{-j4}{4-j4}\right)\left(12\angle 0^\circ\right) = \frac{48\angle -90^\circ}{5.66\angle -45^\circ} = 8.485\angle -45^\circ \text{ V}$$

In the time domain,

$$v(t) = 8.485 \cos(5t - 45^\circ) \text{ V}$$