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9.1 *Introduction*

In this chapter, we consider second-order circuits. A second-order circuit is a circuit that is represented by a second-order differential equation. As a rule of thumb, the order of the differential equation that represents a circuit is equal to the number of capacitors in the circuit plus the number of inductors. For example, a second-order circuit might contain one capacitor and one inductor, or it might contain two capacitors and no inductors.

For example, a second-order circuit could be represented by the equation

$$\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2\, x(t) = f(t)$$

where $x(t)$ is the output of the circuit, and $f(t)$ is the input to the circuit. The output of the circuit, also called the response of the circuit, can be the current or voltage of any device in the circuit. The output is frequently chosen to be the current of an inductor or the voltage of a capacitor. The voltages of independent voltage sources and/or currents of independent current sources provide the input to the circuit. The coefficients of this differential equation have names: α is called the damping coefficient, and ω_0 is called the resonant frequency.

To find the response of the second-order circuit, we:

- Represent the circuit by a second-order differential equation.
- Find the general solution of the homogeneous differential equation. This solution is the natural response $x_n(t)$. The natural response will contain two unknown constants that will be evaluated later.

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9.2 Differential Equation for Circuits with Two Energy Storage Elements

$i_c = C\frac{dv}{dt}$

$v_L = L\frac{di}{dt}$

In Chapter 8, we considered circuits that contained only one energy storage element, and these could be described by a first-order differential equation. In this section, we consider the description of circuits with two irreducible energy storage elements that are described by a second-order differential equation. Later, we will consider circuits with three or more irreducible energy storage elements that are described by a third-order (or higher) differential equation. We use the term *irreducible* to indicate that all parallel or series connections or other reducible combinations of like storage elements have been reduced to their irreducible form. Thus, for example, any parallel capacitors have been reduced to an equivalent capacitor C_p.

In the following paragraphs, we use two methods to obtain the second-order differential equation for circuits with two energy storage elements. Then, in the next section, we obtain the solution to these second-order differential equations.

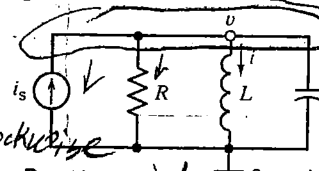

FIGURE 9.2-1 A parallel *RLC* circuit.

First, let us consider the circuit shown in Figure 9.2-1, which consists of a parallel combination of a resistor, an inductor, and a capacitor. Writing the nodal equation at the top node, we have

$$\frac{v}{R} + i + C\frac{dv}{dt} = i_s \qquad (9.2\text{-}1)$$

$-i_s + C\frac{dv}{dt} + \frac{1}{L}\int v\,dt + \frac{v}{R} = 0$

Then we write the equation for the inductor as

$-\frac{di_s}{dt} + C\frac{d^2v}{dt^2} + \frac{v}{L} + \frac{1}{R}\frac{dv}{dt} = 0$

$$v = L\frac{di}{dt} \qquad (9.2\text{-}2)$$

$\boxed{\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = \frac{1}{C}\frac{di_s}{dt}}$

Substitute Eq. 9.2-2 into Eq. 9.2-1, obtaining

$$\frac{L}{R}\frac{di}{dt} + i + CL\frac{d^2i}{dt^2} = i_s \qquad (9.2\text{-}3)$$

which is the second-order differential equation we seek. Solve this equation for $i(t)$. If $v(t)$ is required, use Eq. 9.2-2 to obtain it.

This method of obtaining the second-order differential equation may be called the *direct method* and is summarized in Table 9.2-1.

In Table 9.2-1, the circuit variables are called x_1 and x_2. In any example, x_1 and x_2 will be specific element currents or voltages. When we analyzed the circuit of Figure 9.2-1, we used $x_1 = v$ and $x_2 = i$. In contrast, to analyze the circuit of Figure 9.2-2, we will use $x_1 = i$ and $x_2 = v$, where i is the inductor current and v is the capacitor voltage.

Now let us consider the *RLC* series circuit shown in Figure 9.2-2 and use the direct method to obtain the second-order differential equation. We chose $x_1 = i$ and $x_2 = v$. First, we seek an equation for $dx_1/dt = di/dt$. Writing KVL around the loop, we have

$$L\frac{di}{dt} + v + Ri = v_s \qquad (9.2\text{-}4)$$

where v is the capacitor voltage. This equation may be written as

$$\frac{di}{dt} + \frac{v}{L} + \frac{R}{L}i = \frac{v_s}{L} \qquad (9.2\text{-}5)$$

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$i_C = C\frac{dv}{dt}$, $v_C = \frac{1}{C}\int i_C\,dt$ $W_C = \frac{1}{2}Cv_C^2$, v_C cannot change instantly

$v_L = L\frac{di}{dt}$, $i_L = \frac{1}{L}\int v_L\,dt$ $W_L = \frac{1}{2}Li_L^2$, i_L cannot change instantly

Table 9.2-1 The Direct Method for Obtaining the Second-Order Differential Equation of a Circuit

Step 1	Identify the first and second variables, x_1 and x_2. These variables are capacitor voltages and/or inductor currents.
Step 2	Write one first-order differential equation, obtaining $\frac{dx_1}{dt} = f(x_1, x_2)$.
Step 3	Obtain an additional first-order differential equation in terms of the second variable so that $\frac{dx_2}{dt} = Kx_1$ or $x_1 = \frac{1}{K}\frac{dx_2}{dt}$.
Step 4	Substitute the equation of step 3 into the equation of step 2, thus obtaining a second-order differential equation in terms of x_2.

$-v_s + L\frac{di}{dt} + \frac{1}{C}\int i\,dt + iR = 0$

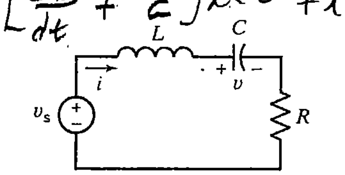

FIGURE 9.2-2 A series *RLC* circuit.

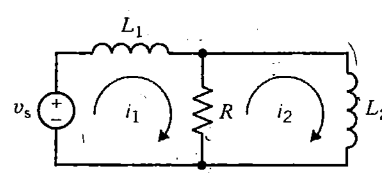

FIGURE 9.2-3 Circuit with two inductors.

$-\frac{dv_s}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} + R\frac{di}{dt} = 0$

$-v_s + L_1\frac{di_1}{dt} + R(i_1 - i_2) = 0$

Recall $v = x_2$ and obtain an equation in terms of $\frac{dx_2}{dt}$. Because

$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{1}{L}\frac{dv_s}{dt}$

$$C\frac{dv}{dt} = i \qquad (9.2\text{-}6)$$

$R(i_2 - i_1) + L_2\frac{di_2}{dt} = 0$

or

$$C\frac{dx_2}{dt} = x_1 \qquad (9.2\text{-}7)$$

substitute Eq. 9.2-6 into Eq. 9.2-5 to obtain the desired second-order differential equation:

$$C\frac{d^2v}{dt^2} + \frac{v}{L} + \frac{RC}{L}\frac{dv}{dt} = \frac{v_s}{L} \qquad (9.2\text{-}8)$$

$-v_s + L_1 s i_1 + R(i_1 - i_2) = 0$

Equation 9.2-8 may be rewritten as

$R(i_2 - i_1) + L_2 s i_2 = 0$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{v_s}{LC} \qquad (9.2\text{-}9)$$

It remains to obtain one second-order differential equation. This is done in the second step of the operator method. The differential operator s, where $s = d/dt$, is used to transform differential equations into algebraic equations. Upon replacing d/dt by s, Eqs. 9.2-13 and 9.2-14 become

$$si_1 + i_1 - i_2 = v_s$$

and

$$-i_1 + i_2 + 2si_2 = 0$$

$-V_s + L_1 s i_1 + R(i_1 - i_2) = 0$

$R(i_2 - i_1) + L_2 s i_2 = 0$

These two equations may be rewritten as

$$(s+1)i_1 - i_2 = v_s$$

and

$$-i_1 + (2s+1)i_2 = 0$$

$$\begin{bmatrix} L_1 s + R & -R \\ -R & L_2 s + R \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix}$$

We may solve for i_2, obtaining

$$\boxed{i_2 = \frac{1v_s}{(s+1)(2s+1) - 1}} = \frac{v_s}{2s^2 + 3s}$$

Gaussian Elimination

$\text{Row } 2_{new} = \text{Row } 2 + \left[\frac{R}{L_1 s + R}\right] \text{Row } 1$

Therefore,

$$(2s^2 + 3s)i_2 = v_s$$

Now, replacing s^2 by $\dfrac{d^2}{dt^2}$ and s by $\dfrac{d}{dt}$, we obtain the differential equation

New value $-R + \left[\frac{R}{L_1 s + R}\right]\left[L_1 s + R\right] = 0$

$$2\frac{d^2 i_2}{dt^2} + 3\frac{di_2}{dt} = v_s \qquad (9.2\text{-}15)$$

$(L_2 s + R) + \left[\frac{R}{L_1 s + R}\right]\left[-R\right]$

The operator method for obtaining the second-order differential equation is summarized in Table 9.2-2.

$= L_2 s + R - \frac{R^2}{L_1 s + R}$

Matches

Table 9.2-2 Operator Method for Obtaining the Second-Order Differential Equation of a Circuit

OK Right

V_s

$0 + \left(\frac{R}{L_1 s + R}\right) V_s$

Step 1	Identify the variable x_1 for which the solution is desired.
Step 2	Write one differential equation in terms of the desired variable x_1 and a second variable, x_2.
Step 3	Obtain an additional equation in terms of the second variable and the first variable.
Step 4	Use the operator $s = d/dt$ and $1/s = \int dt$ to obtain two algebraic equations in terms of s and the two variables x_1 and x_2.
Step 5	Using Cramer's rule, solve for the desired variable so that $x_1 = f(s, \text{sources}) = P(s)/Q(s)$, where $P(s)$ and $Q(s)$ are polynomials in s.
Step 6	Rearrange the equation of step 5 so that $Q(s)x_1 = P(s)$.
Step 7	Convert the operators back to derivatives for the equation of step 6 to obtain the second-order differential equation.

$$\begin{bmatrix} L_1 s + R & -R \\ 0 & L_2 s + R - \frac{R^2}{L_1 s + R} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_s \\ \frac{R V_s}{L_1 s + R} \end{bmatrix}$$

Mult by $(L_1 s + R)$, $\left[(L_1 s + R)(L_2 s + R) - R^2\right] i_2 = R V_s$

$(L_1 L_2 s^2 + L_1 s R + R L_2 s - R^2) i_2 = R V_s$, $i_2 = \dfrac{R V_s}{L_1 L_2 s^2 + s(L_1 R + L_2 R) - R^2}$

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$-V_s + R1\, i_1 + L_2\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) = 0$

$L_2\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) + L_1\frac{di_2}{dt} + R2\, i_2 = 0$

Try it yourself in WileyPLUS

EXAMPLE 9.2-1 Representing a Circuit by a Differential Equation

Find the differential equation for the current i_2 for the circuit of Figure 9.2-4.

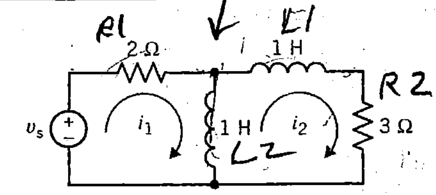

FIGURE 9.2-4 Circuit for Example 9.2-1.

Solution

Write the two mesh equations, using KVL to obtain

$$2i_1 + \frac{di_1}{dt} - \frac{di_2}{dt} = v_s$$

$2i_1 + si_1 - si_2 = V_s$

$$-\frac{di_1}{dt} + 3\, i_2 + 2\frac{di_2}{dt} = 0$$

$-si_1 + 3i_2 + 2si_2 = 0$

Using the operator $s = d/dt$, we have

$$(2+s)i_1 - si_2 = v_s$$

and

$$-si_1 + (3+2s)i_2 = 0$$

$\begin{bmatrix}2+s & -s\\ -s & 3+2s\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix} = \begin{bmatrix}V_s\\ 0\end{bmatrix}$

Using Cramer's rule to solve for i_2, we obtain

GAUSS ELIM.

Row 2: $-\left[\frac{-s}{2+s}\right]$ Row 1

$$i_2 = \frac{sv_s}{(2+s)(3+2s) - s^2} = \frac{sv_s}{s^2 + 7s + 6} \qquad (9.2\text{-}16)$$

$(-s) - \left[\frac{-s}{2+s}\right][2+s] = 0$

$0 - \frac{-s}{2+s}V_s = \frac{sV_s}{2+s}$

Rearranging Eq. 9.2-16, we obtain

$$(s^2 + 7s + 6)i_2 = sv_s \qquad (9.2\text{-}17)$$

Therefore, the differential equation for i_2 is

$(3+2s) - \left[\frac{-s}{2+s}\right](-s) = 3+2s - \frac{s^2}{2+s}$

$$\frac{d^2i_2}{dt^2} + 7\frac{di_2}{dt} + 6i_2 = \frac{dv_s}{dt} \qquad (9.2\text{-}18)$$

Row 2, $\left[(3+2s) - \frac{s^2}{2+s}\right] i_2 = \frac{sV_s}{2+s}$, $\left[(3+2s)(2+s) - s^2\right] i_2 = sV_s$

Try it

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Hint: Use the direct method. IN CLASS, 0324

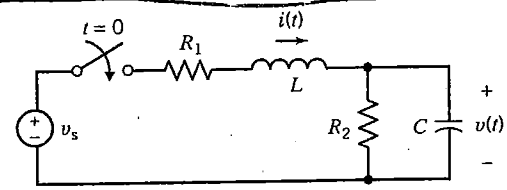

Figure P 9.2-5

he

the input for $t > 0$.

Hint: Use the operator method. IN CLASS 0324

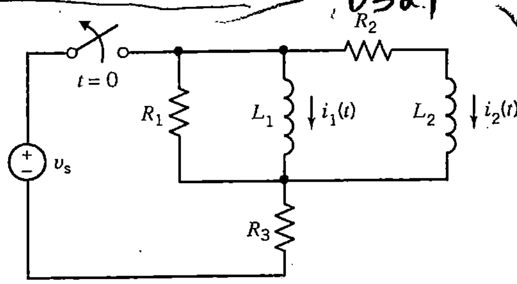

Figure P 9.2-7

Figure P 9.4-3

IN CLASS 0324

P 9.4-4 The circuit shown in Figure P 9.4-4 contains a switch that is sometimes open and sometimes closed. Determine the damping factor α, the resonant frequency ω_0, and the damped resonant frequency ω_d of the circuit when (a) the switch is open and (b) the switch is closed.

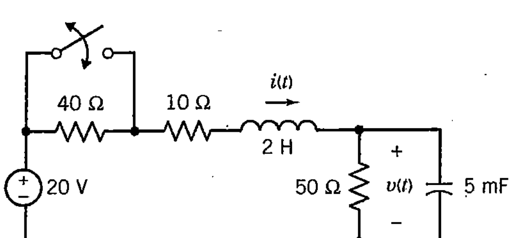

Figure P 9.4-4

P 9.4-5 The circuit shown in Figure P 9.4-5 is used in airplanes to detect smokers who surreptitiously light up before they can take a single puff. The sensor activates the switch, and

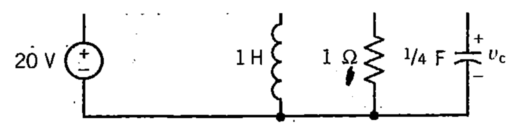

Figure P 9.5-2

IN CLASS 0324

P 9.5-3 Police often use stun guns to incapacitate potentially dangerous felons. The handheld device provides a series of high-voltage, low-current pulses. The power of the pulses is far below lethal levels, but it is enough to cause muscles to contract and put the person out of action. The device provides a pulse of up to 50,000 V, and a current of 1 mA flows through an arc. A model of the circuit for one period is shown in Figure P 9.5-3. Find $v(t)$ for $0 < t < 1$ ms. The resistor R represents the spark gap. Select C so that the response is critically damped.

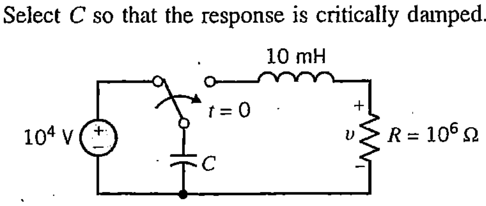

Figure P 9.5-3