Chapter 7

7.2 Capacitors

A capacitor is a circuit element that stores energy in an electric field. A capacitor can be constructed using two parallel conducting plates separated by distance d as shown in Figure 7.2-1. Electric charge is stored on the plates, and a uniform electric field exists between the conducting plates whenever there is a voltage across the capacitor. The space between the plates is filled with a dielectric material. Some capacitors use impregnated paper for a dielectric, whereas others use mica sheets, ceramics, metal films, or just air. A property of the dielectric material, called the dielectric constant, describes the relationship between the electric field strength and the capacitor voltage. Capacitors are represented by a parameter called the *capacitance*. The capacitance of a capacitor is proportional to the dielectric constant and surface area of the plates and is inversely proportional to the distance between the plates. In other words, the capacitance C of a capacitor is given by

$$\boxed{C = \frac{\in A}{d}}$$

where $\in$ is the dielectric constant, A the area of the plates, and d the distance between plates. ~~The unit of capacitance is~~ coulomb per volt and is called farad (F) in honor of Michael Faraday.

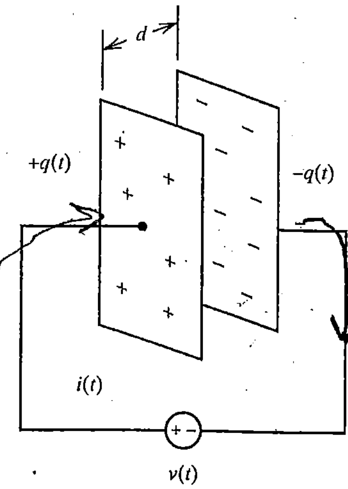


FIGURE 7.2-1 A capacitor connected to a voltage source.

A capacitor voltage $v(t)$ deposits a charge $+q(t)$ on one plate and a charge $-q(t)$ on the other plate. We say that the charge $q(t)$ is stored by the capacitor. The charge stored by a capacitor is proportional to the capacitor voltage $v(t)$. Thus, we write

$$q(t) = Cv(t) \tag{7.2-1}$$

where the constant of proportionality C is the capacitance of the capacitor.

Capacitance is a measure of the ability of a device to store energy in the form of a separated charge or an electric field.

In general, the capacitor voltage $v(t)$ varies as a function of time. Consequently, $q(t)$, the charge stored by the capacitor, also varies as a function of time. The variation of the capacitor charge with respect to time implies a capacitor current $i(t)$, given by

$$i(t) = \frac{d}{dt}q(t)$$

We differentiate Eq. 7.2-1 to obtain

$$\boxed{i(t) = C\frac{d}{dt}v(t)} \tag{7.2-2}$$

Equation 7.2-2 is the current–voltage relationship of a capacitor. The current and voltage in Eq. 7.7-2 adhere to the passive convention. Figure 7.2-2 shows two alternative symbols to represent capacitors in circuit diagrams. In both Figure 7.2-2a and b, the capacitor current and voltage adhere to the passive sign convention and are related by Eq. 7.2-2.

Now consider the waveform shown in Figure 7.2-3, in which the voltage changes from a constant voltage of zero to another constant voltage of 1 over an increment of time, Δt. Using Eq. 7.2-2, we obtain

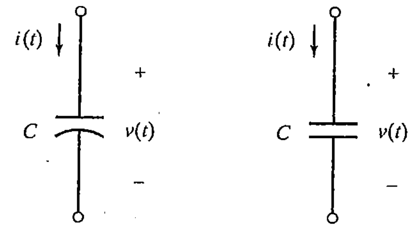


FIGURE 7.2-2 Circuit symbols of a capacitor.

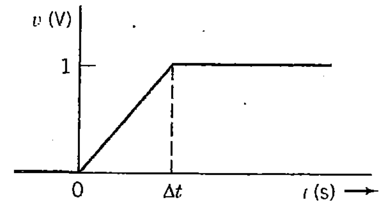


FIGURE 7.2-3 Voltage waveform in which the change in voltage occurs over an increment of time, Δt.

$$i(t) = \begin{cases} 0 & t < 0 \\ \dfrac{C}{Dt} & 0 < t < Dt \\ 0 & t > Dt \end{cases}$$

Thus, we obtain a pulse of height equal to $C/\Delta t$. As Δt decreases, the current will increase. Clearly, Δt cannot decline to zero or we would experience an infinite current. An infinite current is an impossibility because it would require infinite power. Thus, an instantaneous ($Dt = 0$) change of voltage across the capacitor is not possible. In other words, we cannot have a discontinuity in $v(t)$.

The voltage across a **capacitor** cannot change instantaneously.

Now, let us find the voltage $v(t)$ in terms of the current $i(t)$ by integrating both sides of Eq. 7.2-2. We obtain

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \tag{7.2-3}$$

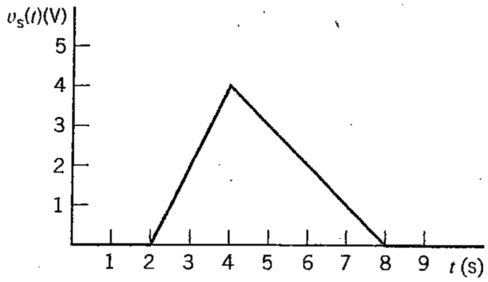
This equation says that the capacitor voltage $v(t)$ can be found by integrating the capacitor current from time $-\infty$ until time t. To do so requires that we know the value of the capacitor current from time $\tau = -\infty$ until time $\tau = t$. Often, we don't know the value of the current all the way back to $\tau = -\infty$. Instead, we break the integral up into two parts:

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + \frac{1}{C}\int_{-\infty}^{t_0} i(\tau)d\tau = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0) \tag{7.2-4}$$

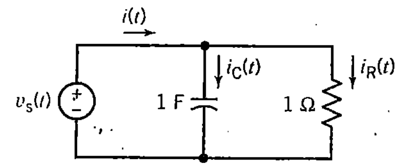
This equation says that the capacitor voltage $v(t)$ can be found by integrating the capacitor current from some convenient time $\tau = t_0$ until time $\tau = t$, provided that we also know the capacitor voltage at time t_0. Now we are required to know only the capacitor current from time $\tau = t_0$ until time $\tau = t$. The time t_0 is called the **initial time**, and the capacitor voltage $v(t_0)$ is called the **initial condition**. Frequently, it is convenient to select $t_0 = 0$ as the initial time.

Capacitors are commercially available in a variety of types and capacitance values. Capacitor types are described in terms of the dielectric material and the construction technique. Miniature metal film capacitors are shown in Figure 7.2-4. Miniature hermetically sealed polycarbonate capacitors are shown in Figure 7.2-5. Capacitance values typically range from picofarads (pF) to microfarads (μF).

EXERCISE 7.2-1 Determine the current $i(t)$ for $t > 0$ for the circuit of Figure E 7.2-1*b* when $v_s(t)$ is the voltage shown in Figure E 7.2-1*a*.



(*a*)



(*b*)

FIGURE E 7.2-1 (*a*) The voltage source voltage. (*b*) The circuit.

Hint: Determine $i_C(t)$ and $i_R(t)$ separately, then use KCL.

Answer: $v(t) = \begin{cases} 2t - 2 & 2 < t < 4 \\ 7 - t & 4 < t < 8 \\ 0 & \text{otherwise} \end{cases}$

7.3 Energy Storage in a Capacitor

Consider a capacitor that has been connected to a battery of voltage v. A current flows and a charge is stored on the plates of the capacitor, as shown in Figure 7.3-1. Eventually, the voltage across the capacitor is a constant, and the current through the capacitor is zero. The capacitor has stored energy by virtue of the separation of charges between the capacitor plates. These charges have an electrical force acting on them.

The forces acting on the charges stored in a capacitor are said to result from an electric field. An *electric field* is defined as the force acting on a unit positive charge in a specified region. Because the charges have a force acting on them along a direction x, we recognize that the energy required originally to separate the charges is now stored by the capacitor in the electric field.

The energy stored in a capacitor is

$$w_c(t) = \int_{-\infty}^{t} vi\, d\tau$$

$i = C\frac{dv}{dt}$

Remember that v and i are both functions of time and could be written as $v(t)$ and $i(t)$. Because

$$i = C\frac{dv}{dt}$$

we have

$$w_c = \int_{-\infty}^{t} vC\frac{dv}{dt}\, d\tau = C\int_{v(-\infty)}^{v(t)} v\, dv = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)}$$

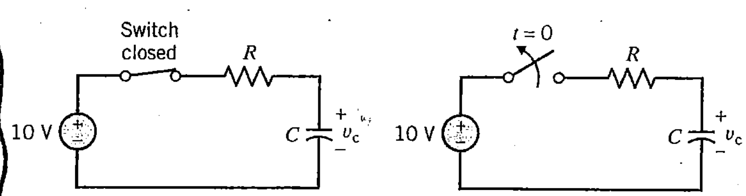


FIGURE 7.3-1 A circuit (*a*) where the capacitor is charged and $v_c = 10$ V and (*b*) the switch is opened at $t = 0$.

Because the capacitor was uncharged at $t = -\infty$, set $v(-\infty) = 0$. Therefore,

$$w_c(t) = \frac{1}{2}Cv^2(t) \text{ J} \tag{7.3-1}$$

Therefore, as a capacitor is being charged and $v(t)$ is changing, the energy stored, w_c, is changing. Note that $w_c(t) > 0$ for all $v(t)$, so the element is said to be passive.

Because $q = Cv$, we may rewrite Eq. 7.3-1 as

$$w_c = \frac{1}{2C}q^2(t) \text{ J} \tag{7.3-2}$$

The capacitor is a storage element that stores but does not dissipate energy. For example, consider a 100-mF capacitor that has a voltage of 100 V across it. The energy stored is

$$w_c = \frac{1}{2}Cv^2 = \frac{1}{2}(0.1)(100)^2 = 500 \text{ J}$$

As long as the capacitor is not connected to any other element, the energy of 500 J remains stored. Now if we connect the capacitor to the terminals of a resistor, we expect a current to flow until all the energy is dissipated as heat by the resistor. After all the energy dissipates, the current is zero and the voltage across the capacitor is zero.

As noted in the previous section, the requirement of conservation of charge implies that the voltage on a capacitor is continuous. Thus, the *voltage and charge on a capacitor cannot change instantaneously*. This statement is summarized by the equation

$$v(0^+) = v(0^-)$$

where the time just prior to $t = 0$ is called $t = 0^-$ and the time immediately after $t = 0$ is called $t = 0^+$. The time between $t = 0^-$ and $t = 0^+$ is infinitely small. Nevertheless, the voltage will not change abruptly.

To illustrate the continuity of voltage for a capacitor, consider the circuit shown in Figure 7.3-1. For the circuit shown in Figure 7.3-1*a*, the switch has been closed for a long time and the capacitor voltage has become $v_c = 10$ V. At time $t = 0$, we open the switch, as shown in Figure 7.3-1*b*. Because the voltage on the capacitor is continuous,

$$v_c(0^+) = v_c(0^-) = 10 \text{ V}$$

Try it yourself in WileyPLUS

EXAMPLE 7.3-1 Energy Stored by a Capacitor

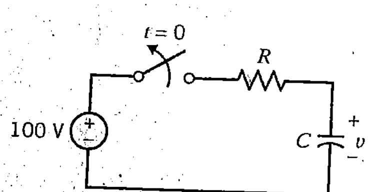


FIGURE 7.3-2 Circuit of Example 7.3-1 with $C = 10$ mF.

A 10-mF capacitor is charged to 100 V, as shown in the circuit of Figure 7.3-2. Find the energy stored by the capacitor and the voltage of the capacitor at $t = 0^+$ after the switch is opened.

Solution

The voltage of the capacitor is $v = 100$ V at $t = 0^-$. Because the voltage at $t = 0^+$ cannot change from the voltage at $t = 0^-$, we have

$$v(0^+) = v(0^-) = 100 \text{ V}$$

The energy stored by the capacitor at $t = 0^+$ is

$$w_c = \frac{1}{2}Cv^2 = \frac{1}{2}(10^{-2})(100)^2 = 50 \text{ J}$$

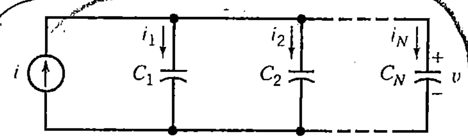


FIGURE 7.4-1 Parallel connection of N capacitors.

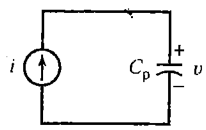


FIGURE 7.4-2 Equivalent circuit for N parallel capacitors.

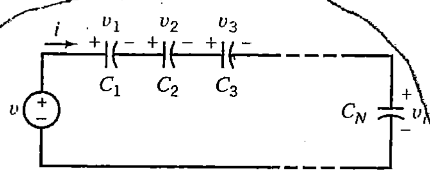


FIGURE 7.4-3 Series connection of N capacitors.

Series

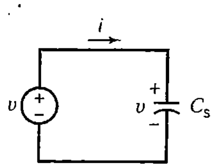


FIGURE 7.4-4 Equivalent circuit for N series capacitors.

First, let us consider the parallel connection of N capacitors as shown in Figure 7.4-1. We wish to determine the equivalent circuit for the N parallel capacitors as shown in Figure 7.4-2.

Using KCL, we have

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

Because

$$i_n = C_n \frac{dv}{dt}$$

and v appears across each capacitor, we obtain

$$\begin{aligned} i &= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + \cdots + C_N \frac{dv}{dt} \\ &= (C_1 + C_2 + C_3 + \cdots + C_N)\frac{dv}{dt} \\ &= \left(\sum_{n=1}^{N} C_n\right)\frac{dv}{dt} \end{aligned} \tag{7.4-1}$$

For the equivalent circuit shown in Figure 7.4-2,

$$i = C_p \frac{dv}{dt} \tag{7.4-2}$$

Comparing Eqs. 7.4-1 and 7.4-2, it is clear that

$$C_p = C_1 + C_2 + C_3 + \cdots + C_N = \sum_{n=1}^{N} C_n$$

Thus, the equivalent capacitance of a set of N parallel capacitors is simply the sum of the individual capacitances. It must be noted that all the parallel capacitors will have the same initial condition $v(0)$.

Now let us determine the equivalent capacitance C_s of a set of N series-connected capacitances, as shown in Figure 7.4-3. The equivalent circuit for the series of capacitors is shown in Figure 7.4-4.

Using KVL for the loop of Figure 7.4-3, we have

$$v = v_1 + v_2 + v_3 + \cdots + v_N \tag{7.4-3}$$

Comparing Eqs. 7.4-6 and 7.4-7, we find that

$$\frac{1}{C_s} = \sum_{n=1}^{N} \frac{1}{C_n} \tag{7.4-8}$$

Series Caps

For the case of two series capacitors, Eq. 7.4-8 becomes

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2}$$

or

$$C_s = \frac{C_1 C_2}{C_1 + C_2} \tag{7.4-9}$$

A **capacitor** in a **dc circuit** behaves as an **open circuit**.

7.5 Inductors

An inductor is a circuit element that stores energy in a magnetic field. An inductor can be constructed by winding a coil of wire around a magnetic core as shown in Figure 7.5-1. Inductors are represented by a parameter called the *inductance*. The inductance of an inductor depends on its size, materials, and method of construction. For example, the inductance of the inductor shown in Figure 7.5-1 is given by

$$L = \frac{\mu N^2 A}{l}$$

HENRY H

where N is the number of turns—that is, the number of times that the wire is wound around the core—A is the cross-sectional area of the core in square meters; l the length of the winding in meters; and μ is a property of the magnetic core known as the permeability. The unit of inductance is called

henry (H) in honor of the American physicist Joseph Henry. Practical inductors have inductances ranging from 1 μH to 10 H. Inductors are wound in various forms, as shown in Figure 7.5-2.

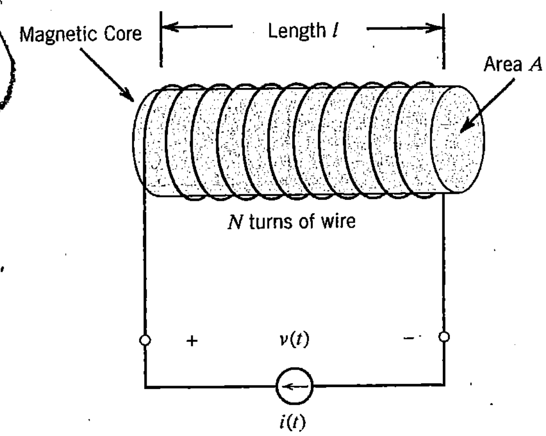


FIGURE 7.5-1 An inductor connected to a current source.

Inductance is a measure of the ability of a device to store energy in the form of a magnetic field.

In Figure 7.5-1, a current source is used to cause a coil current $i(t)$. We find that the voltage $v(t)$ across the coil is proportional to the rate of change of the coil current. That is,

$$v(t) = L\frac{d}{dt}i(t) \tag{7.5-1}$$

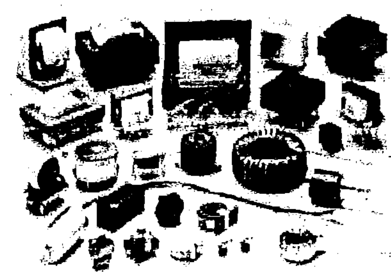
where the constant of proportionality is L, the inductance of the inductor.

Integrating both sides of Eq. 7.5-1, we obtain

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau \tag{7.5-2}$$

This equation says that the inductor current $i(t)$ can be found by integrating the inductor voltage from time $-\infty$ until time t. To do so requires that we know the value of the inductor voltage from time $\tau = -\infty$ until time $\tau = t$. Often, we don't know the value of the voltage all the way back to $\tau = -\infty$. Instead, we break the integral up into two parts:

$$i(t) = \frac{1}{L}\int_{-\infty}^{t_0} v(\tau)d\tau + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau \tag{7.5-3}$$



Courtesy of Vishay Intertechnology, Inc.

FIGURE 7.5-2 Elements with inductances arranged in various forms of coils.

The current in an inductance cannot change instantaneously.

$\frac{1}{2}LI^2$

7.7 Series and Parallel Inductors

A series and parallel connection of inductors can be reduced to an equivalent simple inductor. Consider a series connection of N inductors as shown in Figure 7.7-1. The voltage across the series connection is

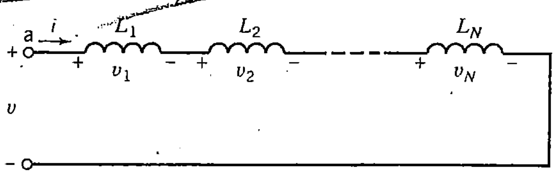


FIGURE 7.7-1 Series of N inductors.

$$\begin{aligned} v &= v_1 + v_2 + \cdots + v_N \\ &= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \cdots + L_N \frac{di}{dt} \\ &= \left(\sum_{n=1}^{N} L_N \right) \frac{di}{dt} \end{aligned}$$

Because the equivalent series inductor L_s, as shown in Figure 7.7-2, is represented by

$$v = L_s \frac{di}{dt}$$

we require that

$$L_s = \sum_{n=1}^{N} L_n \tag{7.7-1}$$

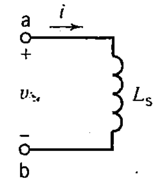


FIGURE 7.7-2 Equivalent inductor L_s for N series inductors.

Thus, an equivalent inductor for a series of inductors is the sum of the N inductors.

Now, consider the set of N inductors in parallel, as shown in Figure 7.7-3. The current i is equal to the sum of the currents in the N inductors:

$$i = \sum_{n=1}^{N} i_n$$

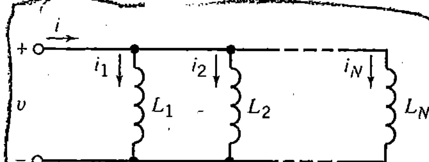


FIGURE 7.7-3 Connection of N parallel inductors.

However, because

$$i_n = \frac{1}{L_n} \int_{t_0}^{t} v \, d\tau + i_n(t_0)$$

we may obtain the expression

$$i = \left(\sum_{n=1}^{N} \frac{1}{L_n} \right) \int_{t_0}^{t} v \, d\tau + \sum_{n=1}^{N} i_n(t_0) \tag{7.7-2}$$

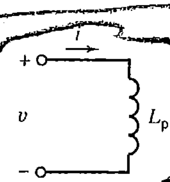


FIGURE 7.7-4 Equivalent inductor L_p for the connection of N parallel inductors.

The equivalent inductor L_p, as shown in Figure 7.7-4, is represented by the equation

$$i = \frac{1}{L_p} \int_{t_0}^{t} v \, d\tau + i(t_0) \tag{7.7-3}$$

When Eqs. 7.7-2 and 7.7-3 are set equal to each other, we have

$$\frac{1}{L_p} = \sum_{n=1}^{N} \frac{1}{L_n} \tag{7.7-4}$$

Table 7.13-2 Parallel and Series Capacitors and Inductors

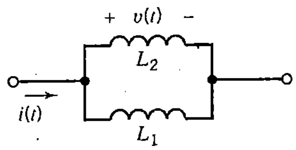
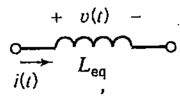
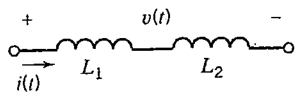
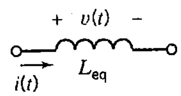
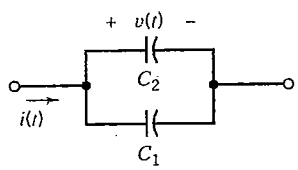
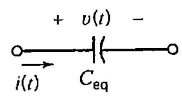
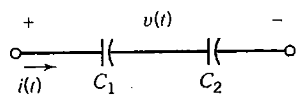
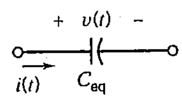
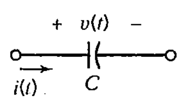
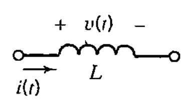
SERIES OR PARALLEL CIRCUIT	EQUIVALENT CIRCUIT	EQUATION
		$L_{eq} = \dfrac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2}}$
		$L_{eq} = L_1 + L_2$
		$C_{eq} = C_1 + C_2$
		$C_{eq} = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$

Table 7.13-1 Element Equations for Capacitors and Inductors

CAPACITOR	INDUCTOR
	
$i(t) = C\dfrac{d}{dt}v(t)$	$i(t) = \dfrac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i(t_0)$
$v(t) = \dfrac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$	$v(t) = L\dfrac{d}{dt}i(t)$

If a constant current of 25 mA flows through the capacitor, how long will it take for the capacitor to charge up to 150 μC?

Answer: $t = 3$ ms

P 7.2-2 The voltage $v(t)$ across a capacitor and current $i(t)$ in that capacitor adhere to the passive convention. Determine the current $i(t)$ when the capacitance is $C = 0.125$ F, and the voltage is $v(t) = 12\cos(2t + 30°)$ V.

$$= A\omega\cos\left(\omega t + \left(\theta + \frac{\pi}{2}\right)\right)$$

Answer: $i(t) = 3\cos(2t + 120°)$ A

P 7.2-3 ⊕ The voltage $v(t)$ across a capacitor and current $i(t)$ in that capacitor adhere to the passive convention. Determine the capacitance when the voltage is $v(t) = 12\cos(500t - 45°)$ V and the current is $i(t) = 3\cos(500t + 45°)$ mA.

Answer: $C = 0.5$ μF

2. The circuit includes one or more switches that open or close at time t_0. We denote the time immediately before the switch opens or closes as t_0^- and the time immediately after the switch opens or closes as t_0^+. Often, we will assume that $t_0 = 0$.
3. The circuit includes at least one capacitor or inductor.
4. We will assume that the switches in a circuit have been in position for a long time at $t = t_0$, the switching time. We will say that such a circuit is at *steady state* immediately before the time of switching. A circuit that contains only constant sources and is at steady state is called a *dc circuit.* All the element currents and voltages in a dc circuit are constant functions of time.

We are particularly interested in the current and voltage of energy storage elements after the switch opens or closes. (Recall from Section 2.9 that open switches act like open circuits and closed switches act like short circuits.) In Table 7.8-1, we summarize the important characteristics of the behavior of an inductor and a capacitor. In particular, notice that neither a capacitor voltage nor an inductor current can change instantaneously. (Recall from Sections 7.2 and 7.5 that such changes would require infinite power, something that is not physically possible.) However, instantaneous changes to an inductor voltage or a capacitor current are quite possible.

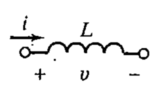
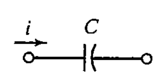
Suppose that a dc circuit contains an inductor. The inductor current, like every other voltage and current in the dc circuit, will be a constant function of time. The inductor voltage is proportional to the derivative of the inductor current, $v = L(di/dt)$, so the inductor voltage is zero. Consequently, the inductor acts like a short circuit.

An **inductor** in a **dc circuit** behaves as a **short circuit.**

Similarly, the voltage of a capacitor in a dc circuit will be a constant function of time. The capacitor current is proportional to the derivative of the capacitor voltage $i = C(dv/dt)$, so the capacitor current is zero. Consequently, the capacitor acts like a open circuit.

IMPORTANT

Table 7.8-1 Characteristics of Energy Storage Elements

VARIABLE	INDUCTORS	CAPACITORS
Passive sign convention		
Voltage	$v = L\frac{di}{dt}$	$v = \frac{1}{C}\int_0^t i\,d\tau + v(0)$
Current	$i = \frac{1}{L}\int_0^t v\,d\tau + i(0)$	$i = C\frac{dv}{dt}$
Power	$i = Li\frac{di}{dt}$	$p = Cv\frac{dv}{dt}$
Energy	$w = \frac{1}{2}Li^2$	$w = \frac{1}{2}Cv^2$
An instantaneous change is not permitted for the element's	Current	Voltage
Will permit an instantaneous change in the element's	Voltage	Current
This element acts as a (see note below)	Short circuit to a constant current into its terminals	Open circuit to a constant voltage across its terminals

Note: Assumes that the element is in a circuit with steady-state condition.