Definitions

Node: A point or set of points at the same potential that have at least two branches connected to them.

Branch: A circuit element that connects nodes.

Major Node: A node with three or more branches connected to it.

Super Node: Two major nodes with an ideal voltage source between them.

Reference Node: The node to which all other node potentials are referenced. The relative voltage of the reference node is zero.

Solution Procedure

1. Draw a neat circuit diagram and try to eliminate as many branch crossings as possible.

2. Choose a reference node. All other node voltages will be referenced to it. Ideally, it should be the node with the most branches connected to it, so that the number of terms in the admittance matrix is minimal.

3. If the circuit contains voltage sources, do either of the following:
   - Convert them to current sources (if they have series impedances)
   - Create super nodes by encircling the corresponding end nodes of each voltage source.

4. Assign a number to every major node (except the reference node) that is not part of a super node (N1 of these).

5. Assign a number to either end (but not both ends) of every super node that does not touch the reference node (N2 of these)

6. Apply KCL to every numbered node from Step 4 (N1 equations)

7. Apply KCL to every numbered super node from Step 5 (N2 equations)

8. The dimension of the problem is now N1 + N2. Solve the set of linear equations for the node voltages. At this point, the circuit has been “solved.”

9. Using your results, check KCL for at least one node to make sure that your currents sum to zero.

9. Use Ohm’s Law, KCL, and the voltage divider principle to find other node voltages, branch currents, and powers as needed.
Definitions

Branch: A circuit element that connects nodes

Planar Network: A network whose circuit diagram can be drawn on a plane in such a manner that no branches pass over or under other branches

Loop: A closed path

Mesh: A loop that is the only loop passing through at least one branch

Solution Procedure

1. Draw a neat circuit diagram and make sure that the circuit is planar (if not planar, then the circuit is not a candidate for mesh analysis)

2. If the circuit contains current sources, do either of the following:
   A. Convert them to voltage sources (if they have internal impedances), or
   B. Create super meshes by making sure in Step 3 that two (and not more than two) meshes pass through each current source. SM super meshes.

3. Draw clockwise mesh currents, where each one passes through at least one new branch. M meshes.

4. Apply KVL for every mesh that is not part of a super mesh (M – 2SM equations)

5. For meshes that form super meshes, apply KVL to the portion of the loop formed by the two meshes that does not pass through the current source (SM equations)

6. For each super mesh, write an equation that relates the corresponding mesh currents to the current source (SM equations)

7. The dimension of the problem is now M. Solve the set of M linear equations for the mesh currents. At this point, the network has been “solved.”

8. Using your results, check KVL around at least one mesh to make sure that the net voltage drop is zero.

9. Use Ohm’s law, loop currents, KVL, KCL and the voltage divider principle to find node voltages, branch currents, and powers as needed.
Building the Admittance Matrix

Most power system networks are analyzed by first forming the admittance matrix. The admittance matrix is based upon Kirchhoff’s current law (KCL), and it is easily formed and very sparse for most large networks.

Consider the three-bus network shown in Figure that has five branch impedances and one current source.

![Three-Bus Network Diagram](image)

Figure 1. Three-Bus Network

Applying KCL at the three independent nodes yields the following equations for the bus voltages (with respect to ground):

At bus 1, \( \frac{V_1}{Z_E} + \frac{V_1 - V_2}{Z_A} = 0 \),

At bus 2, \( \frac{V_2}{Z_B} + \frac{V_2 - V_1}{Z_A} + \frac{V_2 - V_3}{Z_C} = 0 \),

At bus 3, \( \frac{V_3}{Z_D} + \frac{V_3 - V_2}{Z_C} = I_3 \).

Collecting terms and writing the equations in matrix form yields

\[
\begin{bmatrix}
\frac{1}{Z_E} + \frac{1}{Z_A} & -\frac{1}{Z_A} & 0 \\
-\frac{1}{Z_A} & \frac{1}{Z_A} + \frac{1}{Z_B} + \frac{1}{Z_C} & -\frac{1}{Z_C} \\
0 & -\frac{1}{Z_C} & \frac{1}{Z_C} + \frac{1}{Z_D}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
I_3
\end{bmatrix},
\]

or in matrix form,

\[ YV = I \]
where $Y$ is the admittance matrix, $V$ is a vector of bus voltages (with respect to ground), and $I$ is a vector of current injections.

Voltage sources, if present, can be converted to current sources using the usual network rules. If a bus has a zero-impedance voltage source attached to it, then the bus voltage is already known, and the dimension of the problem is reduced by one.

A simple observation of the structure of the above admittance matrix leads to the following rule for building $Y$:

1. The diagonal terms of $Y$ contain the sum of all branch admittances connected directly to the corresponding bus.

2. The off-diagonal elements of $Y$ contain the negative sum of all branch admittances connected directly between the corresponding buses.
These rules make \( Y \) very simple to build using a computer program. For example, assume that the impedance data for the above network has the following form, one data input line per branch:

<table>
<thead>
<tr>
<th>From Bus</th>
<th>To Bus</th>
<th>Branch Impedance (Entered as Complex Numbers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>( Z_E )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>( Z_A )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>( Z_B )</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>( Z_C )</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>( Z_D )</td>
</tr>
</tbody>
</table>

The following FORTRAN instructions would automatically build \( Y \), without the need of manually writing the KCL equations beforehand:

```fortran
COMPLEX Y(3,3),ZB,YB
DATA Y/9 * 0.0/
1 READ(1,*,END=2) NF,NT,ZB
YB = 1.0 / ZB
C MODIFY THE DIAGONAL TERMS
   IF(NF .NE. 0) Y(NF,NF) = Y(NF,NF) + YB
   IF(NT .NE. 0) Y(NT,NT) = Y(NT,NT) + YB
   IF(NF .NE. 0 .AND. NT .NE. 0) THEN
       C MODIFY THE OFF-DIAGONAL TERMS
           Y(NF,NT) = Y(NF,NT) - YB
           Y(NT,NF) = Y(NT,NF) - YB
   ENDIF
GO TO 1
2 STOP
END
```

Of course, error checking is needed in an actual computer program to detect data errors and dimension overruns. Also, if bus numbers are not compressed (i.e. bus 1 through bus \( N \)), then additional logic is needed to internally compress the busses, maintaining separate internal and external (i.e. user) bus numbers.

Note that the \( Y \) matrix is symmetric unless there are branches whose admittance is direction-dependent. In AC power system applications, only phase-shifting transformers have this asymmetric property. The normal \( 30^\circ \) phase shift in wye-delta transformers creates asymmetry.
**SUM THE CURRENTS LEAVING THE NODES**

![Image of a circuit diagram with labels](image)

*The "node" is the point + the wires*

**CLASSIC NODE PROBLEM - HAS ONLY INDEPENDENT CURRENT SOURCES**

For Prob. 3.24.

\[
\begin{align*}
\text{KCL } \# 1: & \quad \frac{V_1 - 0}{1} + \frac{V_1 - V_3}{8} - 4 = 0 \\
\# 2: & \quad \frac{V_2 - 0}{2} + 4 + \frac{V_2 - V_3}{4} = 0 \\
\# 3: & \quad \frac{V_3 - 0}{2} + \frac{V_3 - V_2}{4} + \frac{V_3 - V_1}{8} + \frac{V_3 - 0}{1} - 2 + 2 = 0
\end{align*}
\]

Gather terms into a matrix

\[
\begin{bmatrix}
\frac{1}{1} + \frac{1}{8} & 0 & -\frac{1}{8} \\
0 & \frac{1}{2} + \frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{8} & -\frac{1}{4} & \frac{1}{2} + \frac{1}{4} + \frac{1}{8}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
-4 \\
2 - 2
\end{bmatrix}
\]

\[Y: \text{Admittance Matrix} \quad \text{UNITS: Mho, Siemens} \quad YV = I \quad \text{current}\]

- **Sanity Check #1**
  - It is symmetric when all the sources are independent

Let's eliminate the fractions - multiply by 8

\[
\begin{bmatrix}
9 & 0 & -1 \\
0 & 6 & -2 \\
-1 & -2 & 15
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= 
\begin{bmatrix}
32 \\
-32 \\
0
\end{bmatrix}
\]

\{ **Standard Matrix Form** \}
• Sanity Check #2 (for DC circuits)
The diagonal terms are the largest magnitudes in their rows and are positive.

• Sanity Check #3 (for DC circuits)
The off-diagonal terms are zero when there is no conductance between them, and negative otherwise.

• Sanity Check #4
The current injections are the proper sum for each node and, all together, sum to zero (to enforce KCL at the ref node).

Solve this one using Kronecker’s rule:
\[ \Delta = 9 \begin{vmatrix} 6 & -2 \\ -2 & 15 \end{vmatrix} - 0 \begin{vmatrix} 0 & -1 \\ -2 & 15 \end{vmatrix} - 1 \begin{vmatrix} 0 & -1 \\ 6 & -2 \end{vmatrix} \]

\[ \Delta = 9(90 - 4) - 1(+6) = 774 - 6 = 768 \]

\[ V_1 = \frac{\begin{vmatrix} 32 & 0 & -1 \\ 32 & 6 & -2 \\ 0 & -2 & 15 \end{vmatrix}}{768} = \frac{32(90 - 4) + 32(0 - 2)}{768} = 3.500 \text{ V} \]

\[ V_2 = \frac{\begin{vmatrix} 9 & 32 & -1 \\ 0 & -32 & -2 \\ -1 & 0 & 15 \end{vmatrix}}{768} = \frac{9(-480) - 1(-64 - 32)}{768} = -5.500 \text{ V} \]

\[ V_3 = \frac{\begin{vmatrix} 9 & 0 & 32 \\ 0 & 6 & -32 \\ -1 & -2 & 0 \end{vmatrix}}{768} = \frac{9(-64) - 1(-192)}{768} = -0.500 \text{ V} \]
Now, go back to the circuit - did we solve the correct set of equations!

\[ \frac{3.5 - (-0.5)}{B} = 0.500A \quad -0.500V \]

3.500V → 8Ω → 0.500A

\[ 3.500V \downarrow \quad \frac{1Ω}{3.500V} \]

OK (but doesn't check \( V_2 \))

\[ V_2 = -5.500 \quad 4Ω \uparrow \quad V_3 = -0.500V \]

2.750A \uparrow \quad 2.2Ω \quad 2.2Ω \uparrow \quad 2.2Ω \quad OK (so \( V_3 \) probably OK)

The \( 3.0A \) and \( 2.500A \) currents are obtained using KCL after the other currents are known. They are not directly governed by Ohm's Law.

So, what is \( V_0 \)?

\[ V_1 + V_0 - V_2 = 0 \]

\[ V_0 = V_1 - V_2 \]
Automatic Method for Circuits
where sources are independent current sources
(includes cases where independent voltage sources
with resistances are converted to current sources)

\[ y_{ii} = \sum \text{Admittances connected directly to node } i \]

\[ y_{ij} = -\sum \text{between nodes } i \text{ and } j \]

\[
\begin{bmatrix}
\frac{1}{4} + \frac{1}{8} & 0 & -\frac{1}{8} \\
0 & \frac{1}{4} + \frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{8} & -\frac{1}{4} & \frac{1}{4} + \frac{1}{8} + \frac{1}{4}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= \begin{bmatrix} 4 \\ -4 \\ 2 - 2 \end{bmatrix}
\]

Demonstrate Gaussian Elimination & Backward Substitution

\[
\begin{bmatrix}
9 & 0 & -1 \\
0 & 6 & -2 \\
-1 & -2 & 15
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= \begin{bmatrix} 32 \\ -32 \\ 0 \end{bmatrix}
\]

Row 3 + \frac{1}{9} Row 1

\[
\begin{bmatrix}
9 & 0 & -1 \\
0 & 6 & -2 \\
0 & -2 & 15 - \frac{1}{9}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= \begin{bmatrix} 32 \\ -32 \\ \frac{32}{9} \end{bmatrix}
\]

Row 3 + \frac{1}{3} Row 2

\[
\begin{bmatrix}
9 & 0 & -1 \\
0 & 6 & -2 \\
0 & -2 + z = 0 & 15 - \frac{1}{9} - \frac{z}{3} = \frac{128}{9}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= \begin{bmatrix} 32 \\ -32 \\ \frac{32}{9} - \frac{32}{3} = \frac{32}{9} - \frac{96}{9} = -\frac{64}{9} \end{bmatrix}
\]
\[
\begin{bmatrix}
9 & 0 & -1 \\
0 & 6 & -2 \\
0 & 0 & \frac{128}{9}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
=
\begin{bmatrix}
32 \\
-32 \\
-64/9
\end{bmatrix}
\Rightarrow
v_3 = \frac{-64/9}{128/9} = -\frac{1}{2}
\]

(URT+D) Form
Upper-Right Triangle plus Diagonal

Now backward substitution

Row 2,
\[6v_2 - 2v_3 = -32\]
\[6v_2 = -32 + 2v_3 = -32 + 2(-\frac{1}{2}) = -33\]
\[v_2 = \frac{-33}{6} = -5.5V\]

Row 1,
\[9v_1 + 0v_2 - 1v_3 = 32, \quad 9v_1 = 32 - 0v_2 + 1v_3\]
\[v_1 = \frac{32 - 0 - \frac{1}{2}}{9} = 3.5V\]

Sparsity - If you number the busses so that those with the least number of connected branches are at the top of the matrix, and those with the most connected branches are at the bottom, then (URT+D) will be sparse.

But if the top bus is connected to all others, then (URT+D) will be 100% filled.
**Solving for Node Voltages Using Gaussian Elimination and Backward Substitution**

Gaussian elimination is the most common method for solving bus voltages in a circuit for which KCL equations have been written in the form

\[ I = YV. \]

Of course, direct inversion can be used, where

\[ V = Y^{-1}I, \]

but direct inversion for large matrices is computationally prohibitive or, at best, inefficient.

The objective of Gaussian elimination is to reduce the \( Y \) matrix to upper-right-triangular-plus-diagonal form (URT+D), then solve for \( V \) via backward substitution. A series of row operations (i.e. subtractions and additions) are used to change equation

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
\vdots \\
I_N
\end{bmatrix} =
\begin{bmatrix}
y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,N} \\
y_{2,1} & y_{2,2} & y_{2,3} & \cdots & y_{2,N} \\
y_{3,1} & y_{3,2} & y_{3,3} & \cdots & y_{3,N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
y_{N,1} & y_{N,2} & y_{N,3} & \cdots & y_{N,N}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_N
\end{bmatrix},
\]

into

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
\vdots \\
I_N
\end{bmatrix} =
\begin{bmatrix}
y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,N} \\
0 & y_{2,2} & y_{2,3} & \cdots & y_{2,N} \\
0 & 0 & y_{3,3} & \cdots & y_{3,N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & y_{N,N}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_N
\end{bmatrix},
\]

in which the transformed \( Y \) matrix has zeros under the diagonal.

For illustrative purposes, consider the two equations represented by Rows 1 and 2, which are

\[
I_1 = y_{1,1}V_1 + y_{1,2}V_2 + y_{1,3}V_3 + \cdots + y_{1,N}V_N, \\
I_2 = y_{2,1}V_1 + y_{2,2}V_2 + y_{2,3}V_3 + \cdots + y_{2,N}V_N.
\]

Subtracting \( \frac{y_{2,1}}{y_{1,1}} \bullet \text{Row 1} \) from Row 2 yields
\[ I_1 = \frac{y_{1,1}V_1}{y_{1,1}} \quad + \quad \frac{y_{1,2}V_2}{y_{1,1}} + \frac{y_{1,3}V_3}{y_{1,1}} + \cdots + \frac{y_{1,N}V_N}{y_{1,1}} \]

\[ I_2 - \frac{y_{2,1}}{y_{1,1}}I_1 = \left( \frac{y_{2,1}}{y_{1,1}} \frac{y_{1,1}}{y_{2,1}} \right) V_1 + \left( \frac{y_{2,2}}{y_{1,1}} \frac{y_{1,1}}{y_{2,2}} \right) V_2 + \left( \frac{y_{2,3}}{y_{1,1}} \frac{y_{1,1}}{y_{2,3}} \right) V_3 + \cdots + \left( \frac{y_{2,N}}{y_{1,1}} \frac{y_{1,1}}{y_{2,N}} \right) V_N . \]

The coefficient of \( V_1 \) in Row 2 is forced to zero, leaving Row 2 with the desired "reduced" form of

\[ I_2 = 0 + y_{2,2}V_2 + y_{2,3}V_3 + \cdots + y_{2,N}V_N . \]

Continuing, Row 1 is then used to "zero" the \( V_1 \) coefficients in Rows 3 through \( N \), one row at a time. Next, Row 2 is used to zero the \( V_2 \) coefficients in Rows 3 through \( N \), and so forth.

After the Gaussian elimination is completed, and the \( Y \) matrix is reduced to (URT+D) form, the bus voltages are solved by backward substitution as follows:

For Row \( N \),

\[ I_N = y_{N,N}V_N , \quad \text{so} \quad V_N = \frac{1}{y_{N,N}} \left( I_N \right) . \]

Next, for Row \( N-1 \),

\[ I_{N-1} = y_{N-1,N-1}V_{N-1} + y_{N-1,N}V_N , \quad \text{so} \quad V_{N-1} = \frac{1}{y_{N-1,N-1}} \left( I_{N-1} - y_{N-1,N}V_N \right) . \]

Continuing for Row \( j \), where \( j = N - 2, N - 3, \ldots, 2 \),

\[ I_j = y_{j,j}V_j + y_{j,j+1}V_{j+1} + \cdots + y_{j,N}V_N , \quad \text{so} \]

\[ V_j = \frac{1}{y_{j,j}} \left( I_j - y_{j,j+1}V_{j+1} - \cdots - y_{j,N}V_N \right) , \]

which, in general form, is described by

\[ V_j = \frac{1}{y_{j,j}} \left( I_j - \sum_{k=j+1}^{N} y_{j,k}V_k \right) . \]
A simple FORTRAN computer program for solving $V$ in an N-dimension problem using Gaussian elimination and backward substitution is given below.

```fortran
COMPLEX Y(N,N),V(N),I(N),YMM
C  GAUSSIAN ELIMINATE Y AND I
NM1    = N - 1
C  PIVOT ON ROW M, M = 1,2,3, ... ,N-1
DO 1 M = 1,NM1
  MP1    = M + 1
  YMM    = 1.0 / Y(M,M)
C  OPERATE ON THE ROWS BELOW THE PIVOT ROW
  DO 1 J = MP1,N
    C  THE JTH ROW OF I
    I(J)   = I(J) - Y(J,M) * YMM * I(M)
    C  THE JTH ROW OF Y, BELOW AND TO THE RIGHT OF THE PIVOT
    DO 1 K = M,N
      Y(J,K) = Y(J,K) - Y(J,M) * YMM * Y(M,K)
 1 CONTINUE
C  BACKWARD SUBSTITUTE TO SOLVE FOR V
  V(N)   = I(N) / Y(N,N)
  DO 2 M = 1,NM1
    J      = N - M
    C  BACKWARD SUBSTITUTE TO SOLVE FOR V, FOR
    C  ROW J = N-1,N-2,N-3, ... ,1
    V(J)   = I(J)
    JP1    = J + 1
    DO 3 K = JP1,N
      V(J)   = V(J) - Y(J,K) * V(K)
    3 CONTINUE
    V(J)   = V(J) / Y(J,J)
 2 CONTINUE
STOP
END
```
Question - What if $V_{ref}$ is 25V, instead of 0?

<table>
<thead>
<tr>
<th>Old</th>
<th>New</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{ref}$</td>
<td>0</td>
</tr>
<tr>
<td>$V_1$</td>
<td>3.5</td>
</tr>
<tr>
<td>$V_2$</td>
<td>-5.5</td>
</tr>
<tr>
<td>$V_3$</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

Question, what if #1 is the ref?

Answer - the current flows are unchanged. So, the relationship between node voltages must be unchanged.

$V_1 - V_4 = 3.5 \Rightarrow V_4 = V_1 - 3.5 = -3.5V$
$V_2 - V_4 = -5.5 \Rightarrow V_2 = V_4 + 5.5 = -9.0V$
$V_3 - V_4 = -0.5 \Rightarrow V_4 = V_3 + 0.5 = -4.0V$

The pattern, **subtract the old voltage of the new reference node from all nodes**.
The automatic procedure works if the voltage sources have series resistances and are converted to current sources.

\[ V_{oc} = V \quad \text{Isc} = \frac{V}{R} \]

\[ V_{oc} = \frac{V}{R}, \quad R = V \\
\text{Isc} = \frac{V}{R} \]

so, identical to original ck+.

Note - if V is upside down, the \( \frac{V}{R} \) current source points down to give same \( V_{oc}, \) \( \text{Isc}. \) These are equivalent to the external circuit, but *not* internally the same.

Can write KCL with voltage sources, too. Consider Prob 3.24, with voltage source attached to the left:

\[ \frac{V_1 - 20}{10} + \frac{V_1 - V_3}{8} + \frac{V_1}{1} - 4 = 0 \]

If the 20V had been inserted in series with the 8Ω:

\[ \frac{(V_1 - 20) - V_3}{8} - 4 + \frac{V_1}{1} = 0 \]
How do we know the \((V_1 - 20)\)?

KVL:

\[
\sum \text{voltage} = 0 \quad \Rightarrow -V_1 + 20 + V_x = 0, \quad \text{so} \quad V_x = (V_1 - 20)
\]

\[
I = \frac{V_x - V_3}{8} = \frac{(V_1 - 20) - V_3}{8} = \frac{V_1 - V_3 - 20}{8}
\]

Can interchange the 20V, 8Ω without impact

\[
I = \frac{8}{8} = \frac{V_1 - V_3 - 20}{8}
\]

**Supernode** - ideal voltage source connected between nodes and no resistor.

Example - Replace 4Ω with 4V

**KVL #1**

\[
\frac{V_1}{8} + \frac{V_1 - V_3}{8} + \frac{(V_1 - 4) - V_3}{4} + \frac{(V_1 - 4)}{2} = 0
\]

**KVL #3**

\[
\frac{V_3 - (V_1 - 4)}{4} + \frac{V_3 - V_1}{8} + \frac{V_3}{1} + \frac{V_3}{2} = 0
\]
Dependent Sources — Include them in the KCL equations, then substitute node voltage relationships to end up with only major nodes in the expression.

\[ 3.30 \text{ (modified)} \]

\[ \begin{align*}
\text{#1} & \quad \frac{V_1 - 100}{10} + \frac{V_1 - (V_2 - 120)}{40} + \frac{V_1 - 4(40)}{20} = 0 \\
\text{#2} & \quad -2I_0 + \frac{(V_2 - 120) - V_1}{40} + \frac{V_2}{80} = 0
\end{align*} \]

Now, eliminate \( I_0 \) & \( V_0 \). See that \( V_0 = V_2 \),

\[ I_0 = \frac{V_1 - (V_2 - 120)}{40} \], \quad \text{Sub in}

\[ \begin{align*}
\text{#1} & \quad \frac{V_1 - 100}{10} + \frac{V_1 - (V_2 - 120)}{40} + \frac{V_1 - 4V_2}{20} = 0 \\
& \quad -2\left[ \frac{\dot{V}_1 - (V_2 - 120)}{40} \right] + \frac{(V_2 - 120) - \dot{V}_1}{40} + \frac{\ddot{V}_2}{80} = 0
\end{align*} \]

Put in Standard Matrix Form & Solve.
\[
\begin{bmatrix}
\frac{1}{10} & \frac{\sqrt{40}}{10} & \frac{\sqrt{2}}{20} \\
\frac{\sqrt{40}}{40} & \frac{1}{40} & \frac{-4}{20} \\
\frac{\sqrt{40}}{40} & \frac{1}{40} & \frac{\sqrt{2}}{20}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= 
\begin{bmatrix}
10-3 \\
6+3
\end{bmatrix}
\]

**Standard Matrix Form to Solve**

**Mesh Currents. Planar Networks only.** Based on KVL, can automate if no dependent sources and if independent current sources are converted to voltage sources.

**Procedure - see Note sheet for Mesh Currents**

**Prob 3.71**

1. \(-10 + 5(i_1 - i_2) + 4(i_1 - i_2) = 0\)
2. \(4(i_2 - i_1) + 4(i_2 - i_3) + 2i_2 + 5 = 0\)
3. \(5(i_3 - i_1) + 3i_3 + 1(i_3 - i_2) = 0\)
Prob 3.64

Parallel current sources → Combine into \((Z - 0.2V_0)\)

\[ \begin{align*}
\text{SM KVL:} & \quad -100 + 50I_1 + 10(I_1-I_3) - 4I_0 + 40I_2 = 0 \\
\text{M3 KVL:} & \quad 10(I_3-I_1) + 10I_3 + 4I_0 = 0 \\
\text{Current source:} & \quad (Z - 0.2V_0) = I_2 - I_1 \\
\text{Dependents:} & \quad \begin{cases} V_0 = 10I_3 \\
I_0 = I_1 - I_3 \\
\end{cases}
\end{align*} \]

Sub in the two dependent equations to yield 3 equations, 3 unknowns \((I_1, I_2, I_3)\)

Node or Mesh? Whichever has the fewest unknowns, usually, node wins.
Superposition

- Linear circuit
- Leave all dependent sources "on"
- Energize the circuit one source at a time, with all other sources "off". For N sources, then you have N "subproblems"
- Add the results for all N "subproblems", giving the total answer

Voltage Source Off, Set \( V = 0 \)

Current Source Off, Set \( I = 0 \)

Problem 3.13

Two subproblems (Source 1 \( \Rightarrow S_1 \), and Source 2 \( \Rightarrow S_2 \))

Voltage Divider

\[ V_{2,S_1} = -2 \left[ \frac{4}{2+8+4} \right] = -\frac{8}{14} = -\frac{4}{7} \text{ V} \]

Superposition \( V_2 = V_{2,S_1} + V_{2,S_2} = -\frac{4}{7} + \frac{60}{7} = \frac{56}{7} \text{ V} \)
Check

So total problem with **Nodal**

**KCL** \[ \frac{V_2+2}{8+2} + \frac{V_2}{4} -3 = 0 \]
\[ \frac{V_2}{10} + \frac{2}{10} + \frac{V_2}{4} -3 = 0 \]

\[ \times 20 \]
\[ 2V_2 + 4 + 5V_2 -60 = 0, \]
\[ 7V_2 = 56, \]
\[ V_2 = \frac{56}{7} \]

**Check schematic**
\[ \frac{56+4}{10} \]
\[ \frac{56}{10} \]
\[ \frac{10}{2} \]
\[ \frac{2}{10} + 4 \]
\[ 3A \]

\[ \frac{56}{28} = \frac{56}{4} = 2A, \]
\[ 1+2 = 3 \text{ (OK)} \]

**Superposition useful when answering "what if" questions**

**For subproblem 1**, 2V source yielded \[ V_{2,1}=\frac{-4}{7}V \]

\[ 10V \text{ source would yield } V_2 = \frac{10(-4)}{7}V \]

\[ \frac{dV_2}{dV_{S1}} = \frac{-4/7V}{2V} = \frac{-2}{7} \text{ Volts/Volt} \]

**For subproblem 2**, 3A source yielded \[ V_{2,2}=\frac{60}{7}V \]

\[ 9A \text{ source would yield } V_{2,2} = \frac{9(60)}{7}V \]

\[ \frac{dV_2}{dI_{S2}} = \frac{60/7V}{3A} = \frac{20}{7} \text{ Volts/Amp} \]

\[ \Delta V_{2,2} = \frac{dV_2}{dV_{S1}} \Delta V_{S1} + \frac{dV_2}{dI_{S2}} \Delta I_{S2} \]

Check - raise both sources from **zero** to the values in the problem

\[ \Delta V_{2,2} = \left( \frac{-2}{7} \right) \frac{V}{(2V)} + \left( \frac{20}{7} \right) \frac{V}{(3A)} \]
\[ = -\frac{4}{7}V + \frac{60}{7}V = \frac{56}{7}V \]
Question: What value of $V_{s1}$ would lower $V_{2Tot}$ from $\frac{56}{7}$ V to zero?

Want $\Delta V_{2Tot} = -\frac{56}{7} = -\frac{3}{7} \Delta V_{s1} + \frac{20}{7} \Delta V_{s2}$

$\Delta V_{s1} = \frac{-56/7}{-2/7} = 28$ V

So, the answer is $V_{s1} = 2 + 28 = 30$ V.

Check

![Circuit Diagram]

Looks OK

How about that!

(Working the problem backward)

These partial derivatives are available in both nodal & mesh matrices

$YV = I$, $V = Y^{-1}I = ZI$

Admittance Matrix

Impedance Matrix

$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & \cdots & Z_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}$

Current injections

$V_K = Z_{k1} I_1 + Z_{k2} I_2 + \cdots + Z_{kN} I_N$

$\Delta V_K = \frac{z_{k1}}{\partial I_1} \Delta I_1 + \frac{z_{k2}}{\partial I_2} \Delta I_2 + \cdots + \frac{z_{kN}}{\partial I_N} \Delta I_N$
Notes on Thevenin Equivalents. Dr. Mack Grady.
June 11, 2007

The three cases to consider are
- **Case 1.** All sources are independent
- **Case 2.** The circuit has dependent sources and independent sources
- **Case 3.** The circuit has only dependent sources

Depending on the case, one or more of the following methods can be used to find the Thevenin equivalent:

**Direct \( R_{th} \). (Applies only to Case 1).**
- Turn off all independent sources (i.e., set \( V = 0 \) for voltage sources, and \( I = 0 \) for current sources). Note - this is the same thing as replacing voltage sources with short circuits, and current sources with open circuits.
- Connect a fictitious ohmmeter across terminals a-b, and “measure” \( R_{th} \) directly.
- Find \( I_{sc} \) (or, alternatively, find \( V_{oc} = V_{th} \)).
- Compute \( V_{th} = V_{oc} = I_{sc} \cdot R_{th} \) (or, alternatively, \( I_{sc} = V_{oc} / R_{th} \)).
- If time permits, find \( V_{th} \) (or, alternatively, \( I_{sc} \)) directly from the circuit, and then double-check with the above.

**\( V_{oc}, I_{sc} \) (Applies to Cases 1 and 2).**
- Find \( V_{oc} = V_{th} \).
- Find \( I_{sc} \).
- Compute \( R_{th} = V_{th} / I_{sc} \).

**Fictitious Source (Applies to all Cases)**
- Attach a fictitious source \( V_{ab} \) across terminals a-b. Find a linear equation with the following form: \( V_{ab} = A - B I_{ab} \).
- By definition the linear equation must match Thevenin equation \( V_{ab} = V_{th} - R_{th} I_{ab} \), term by term. Thus, matching the terms yields \( V_{th} = A, R_{th} = B \).

<table>
<thead>
<tr>
<th>Case</th>
<th>Direct ( R_{th} )</th>
<th>( V_{oc}, I_{sc} )</th>
<th>Fictitious Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>OK</td>
<td>OK</td>
<td>OK</td>
</tr>
<tr>
<td>Case 2</td>
<td>OK</td>
<td>OK</td>
<td>OK</td>
</tr>
<tr>
<td>Case 3</td>
<td></td>
<td></td>
<td>OK</td>
</tr>
</tbody>
</table>

Note - For **Case 3**, the \( V_{th} \) should be zero. Thus, for **Case 3**, you can attach any voltage source \( V_{ab} \) to the output (e.g., \( V_{ab} = 1V \)), find \( I_{ab} \), and compute \( R_{th} = \frac{-V_{ab}}{I_{ab}} \).
Thevenin & Norton Equivalents

Two terminals of a linear circuit can be replaced by a Thevenin or Norton equivalent:

\[
\begin{align*}
&\text{Thevenin} \\
\Rightarrow &\quad \text{Norton}
\end{align*}
\]

As far as the outside world is concerned, there are 3 cases:

Case 1 - Only independent sources
Case 2 - Mix of independent sources and dependent sources
Case 3 - Only dependent sources.

The original circuit:

\[
\begin{align*}
&\text{I}_{ab} \\
&\text{V}_{ab} \\
&\text{I}_{sc} \\
&\text{I}=0 \\
&\text{V}=0 \\
&\text{V}_{oc}
\end{align*}
\]

The Thevenin equivalent:

\[
\begin{align*}
&\text{V}_{TH} \\
&\text{R}_{TH} \rightarrow \text{I}_{ab} \\
&\text{V}_{ab} \\
&\text{I}_{ab} \\
&\text{V}_{oc}
\end{align*}
\]

Key - Find \( V_{\text{open}} \) and \( I_{\text{short}} \).

\[
\begin{align*}
\text{V}_{oc} &= \text{V}_{TH} \\
\text{I}_{sc} &= \frac{\text{V}_{TH}}{\text{R}_{TH}}
\end{align*}
\]
Problem 1. Find Thevenin Eq. of

\[ V_{oc} = 60 \left( \frac{40}{40+10} \right) = 48 \, V. \]

Thus, \[ R_T = \frac{V_{oc}}{I_{sc}} = \frac{48}{3} = 16 \, \Omega. \]

Now, for case 1, you can get \( R_T \) by turning off all sources & connecting an ohmmeter to a-b.

\[ R_T = 8 + \frac{40}{10} = 8 + \frac{400}{50} = 16 \, \Omega. \]

Then, find either \( V_{oc} \) or \( I_{sc} \) to complete the problem. If you find \( V_{oc}, I_{sc}, \) and then \( R_T \) (from turning off source), you can double-check your \( R_T \).
Prob 2

KCL (1)

\[
\frac{V_1 - 12}{12} + \frac{V_1}{6} - 8 = 0
\]

\[
V_1 \left( \frac{1}{12} + \frac{1}{6} \right) = 8 + 1 = 9
\]

\[
V_1 \left( \frac{3}{12} \right) = 9
\]

\[
V_1 = \frac{12(9)}{3} = 36 \text{ V}
\]

KVL

\[-V_1 - (2 \times 8 A) + V_{ab} = 0\]

\[
V_{ab} = V_1 + 16 = 52 \text{ V} = V_{TH}
\]

Find \( I_{sc} \)

KCL (1)

\[
\frac{V_1 - 12}{12} + \frac{V_1 - 0}{2} + \frac{V_1}{6} = 0
\]

\[
V_1 \left( \frac{1}{12} + \frac{1}{2} + \frac{1}{6} \right) = 1
\]

\[
V_1 \left( \frac{1}{12} + \frac{6}{12} + \frac{9}{12} \right) = 1
\]

\[
V_1 \left( \frac{18}{12} \right) = 1, \quad V_1 = \frac{12}{9} = 1.333 \text{ V}
\]

KCL (a + @)

\[
I_{sc} = \frac{V_1}{2} + 8 = \frac{1333}{2} + 8 = 0.667 + 8 = 8.667 \text{ A}
\]

So, \( V_{TH} = 52 \text{ V}, \quad I_{sc} = 8.667 \text{ A} \), \( R_{TH} = \frac{V_{oc}}{I_{sc}} = \frac{52}{8.667} = 6.005 \Omega \)

Check \( R_{TH} \) by turning off sources.

\[
R_{TH} = R + 6 || 12 = R + \frac{72}{18} = 6 \Omega
\]

OK
\[ I_{AB} = \frac{V_1}{1000} + 60I_B \]

KCL at \( Q \):

\[ I_{AB} + \frac{V_1}{1000} - 60I_B = 0 \]

The dependent source \( I_B = -I_{AB} \)

Sub into (1):

\[ I_{AB} + \frac{7V_1}{10000} + 60I_{AB} = 0 \]

(2)

\[ \frac{7V_1}{10000} + 61I_{AB} = 0 \]

Now, relate \( V_1 \) to \( V_{AB}, I_{AB} \)

We see that:

\[ I_{AB} = \frac{V_1 - V_{AB}}{1000}, \quad V_1 - V_{AB} = 1000I_{AB}, \quad V_1 = V_{AB} + 1000I_{AB} \]

Sub (3) into (2):

\[ \frac{7}{10000} (V_{AB} + 1000I_{AB}) + 61I_{AB} = 0 \]

\[ \frac{7V_{AB}}{10000} + \frac{7}{10}I_{AB} + 61I_{AB} = 0 \]

\[ \frac{7V_{AB}}{10000} = -61.7I_{AB} \]

\[ V_{AB} = -61.7 \left( \frac{10000}{7} \right) I_{AB} = -881 \times 10^3 I_{AB} \]

Therefore, \( V_{AB} = V_{TH} - I_{AB}R_{TH} \)

Matching terms, \( V_{TH} = 0 \), \( R_{TH} = 88.1 \text{ K}\Omega \)
Using a fictitious 1V source at $V_{ab}$

$$\frac{V_1 - 1}{1000} - 60I_B + \frac{V_1}{10000} = 0$$

$$I_B = \frac{1 - V_1}{1000}$$

$$\frac{V_1 - 1}{1000} - 60(1 - V_1) + \frac{7V_1}{10} = 0$$

$$(V_1 - 1) - 60(1 - V_1) + 0.7V_1 = 0$$

$$V_1 - 1 - 60 + 60V_1 + 0.7V_1 = 0$$

$$61.7V_1 - 61 = 0; \quad V_1 = \frac{61}{61.7} V.$$  

Now, $I_{AB} = \frac{V_1 - V_{AB}}{1000} = \frac{61}{61.7} - 1 = -11.35 \mu A$

If $V_{TH} = 0$, then $V_{AB} = V_{TH} - I_{AB} R_{TH} = -I_{AB} R_{TH}$,

$$R_{TH} = \frac{V_{AB}}{I_{AB}} = \frac{-1V}{-11.35 \mu A} = 88.1 \text{ K}\Omega$$
\[ V_1 + 10V_1 + V_2 + 12V_2 + V_{AB} = 0 \]

1. \( \frac{V_1}{50} + \frac{V_1 - (-6.5I_A)}{5} + \frac{V_1 - V_2}{10} = 0 \)

2. \( \frac{V_2 - V_1}{10} + \frac{V_2}{25} + I_{AB} = 0 \)

The dependent equation, \( I_A = \frac{V_2}{25} \) (3) sub into (1)

\[ \frac{V_1}{50} + \frac{V_1}{5} + \frac{6.5V_2}{125} + \frac{V_1}{10} - \frac{V_2}{10} = 0 \]

\[ V_1 \left( \frac{1}{50} + \frac{1}{5} + \frac{1}{10} \right) = V_2 \left( \frac{1}{10} - \frac{6.5}{125} \right) \]

\[ V_1 \left( \frac{16}{50} \right) = V_2 \left( \frac{12.5 - 6.5}{125} \right) \]

\[ V_1 \left( \frac{16}{50} \right) = V_2 \left( \frac{6}{125} \right) \]

\[ V_1 = V_2 \left( \frac{50}{6} \right) \]

Sub into (2), \( \frac{V_2 - 0.15V_2}{10} + \frac{V_2}{25} + I_{AB} = 0 \)

\[ V_2 \left( \frac{5 - 0.75 + 2}{50} \right) + I_{AB} = 0 \]

\[ V_2 = -I_{AB} \left( \frac{50}{6.125} \right) \]

From the circuit we see that \( I_{AB} = \frac{V_2 - V_{AB}}{12} \)

So, \( 12I_{AB} = -8I_{AB} - V_{AB} \)

\[ V_{AB} = -20I_{AB} \]

\[ V_{TH} = V_{TH} - R_{TH}I_{AB} \]

\[ V_{TH} = 0, R_{TH} = 20 \text{Ω} \]
Fictitious $V_{ab} = 1$ V source, and assume $V_{th} = 0$

1. Same as before
2. $\frac{V_2 - V_1}{10} + \frac{V_2}{25} + \frac{V_2 - 1}{12} = 0$
3. Same

1.2 (3) same, yields $V_1 = 0.15V_2$ (same)

Sub into 2, $\frac{V_2 - 0.15V_2}{10} + \frac{V_2}{25} + \frac{V_2 - 1}{12} = 0$

$V_2 \left( \frac{0.85}{10} + \frac{1}{25} + \frac{1}{12} \right) = \frac{1}{12}$
$V_2 \left[ \frac{1}{12}(0.85) + \frac{12}{25} + 1 \right] = 1 \quad \Rightarrow \quad V_2 = \frac{1}{2.5} = 0.4 V$

Now, $I_{ab} = \frac{V_2 - V_{ab}}{12} = \frac{0.4 - 1}{12} = -\frac{0.6}{12} = -0.05 A$

$V_{ab} = -R_{th} I_{ab}, \quad R_{th} = \frac{-V_{ab}}{I_{ab}} = \frac{-1}{-0.05A} = 20 \Omega$
Max Power

How to deliver max power to $R_L$.

$$P_L = I_L^2 R_L = \frac{V_L^2}{R_L}$$

$$P_L = \left(\frac{V_{TH}}{R_{TH}+R_L}\right)^2 \cdot R_L = V_{TH}^2 \frac{R_L}{(R_{TH}+R_L)^2} = V_T^2 \frac{R_L}{R_{TH} (R_{TH}+R_L)^2}$$

$$\frac{dP_L}{dR_L} = V_{TH}^2 \left[ R_L \frac{d}{dR_L} \left( \frac{1}{(R_{TH}+R_L)^2} \right) + \right. \left. (R_{TH}+R_L)^{-2} \frac{d}{dR_L} \left( R_L \right) \right]$$

$$= V_{TH}^2 \left[ R_L \cdot \left( -2(R_{TH}+R_L)^{-3} \left( 1 \right) + (R_{TH}+R_L)^{-2} \right) \right]$$

$$= V_{TH}^2 \left[ \frac{-2R_L}{(R_{TH}+R_L)^3} + \frac{1}{(R_{TH}+R_L)^2} \right]$$

$$= \frac{V_{TH}^2}{(R_{TH}+R_L)^2} \left[ 1 - \frac{2R_L}{R_{TH}+R_L} \right]$$

For $\frac{dP_L}{dR_L} = 0, \quad 1 - \frac{2R_L}{R_{TH}+R_L} = 0, \quad R_{TH} + R_L = 2R_L$

AT $R_L = R_{TH}, \quad V_L = \frac{V_{TH}}{2}$

\[ \text{May be a problem.} \]