ELC 5396: Digital Communications

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FIR Modeling of Doubly Spread Channels

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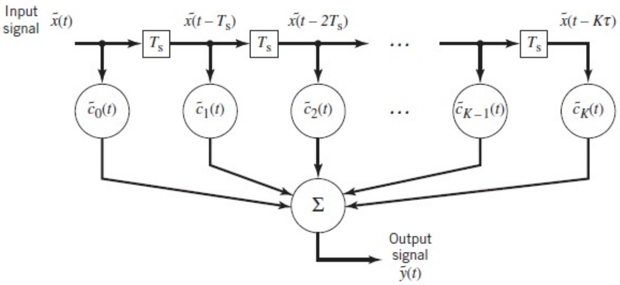
- $$\begin{aligned} y(t) &= \sum_{n=-\infty}^{\infty} x(t - nT_s) \left[\int_{-\infty}^{\infty} h(\tau; t) \mathtt{sinc}(\frac{\tau}{T_s} - n) d\tau \right] \\ &= \sum_{n=-\infty}^{\infty} x(t - nT_s) c_n(t) \end{aligned}$$

FIR Modeling of Doubly Spread Channels



- FIR model of a time-varying channel.
- For time-varying Rayleigh fading channels, $c_n(t)$ is zero-mean complex Gaussian process.
- For WSS channels, $\mathrm{E}[|c_n(t)|^2] \approx T_s^2 p(n\tau)$, where $p(n\tau)$ is a discrete version of the power-delay profile.

Diversity Techniques

The multipath fading phenomenon as an inherent characteristic of a wireless channel.

Diversity:
If several replicas of the information-bearing signal can be transmitted simultaneously over independently fading channels, then there is a good likelihood that at least one of the received signals will not be severely degraded by channel fading.

Diversity Techniques

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3. Space diversity – Several receiving antennas spaced sufficiently far apart (spatial separation should be sufficently large to reduce correlation between diversity branches, e.g., $> 10\lambda$).

Diversity Techniques

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2. Multipath diversity –
    - Signal replicas received at different delays (RAKE receiver in CDMA)
    - Signal replicas received via different angles of arrival (directional antennas at the receiver)
    - Equalization in a TDM/TDMA system provides similar performance as multipath diversity.

Space Diversity

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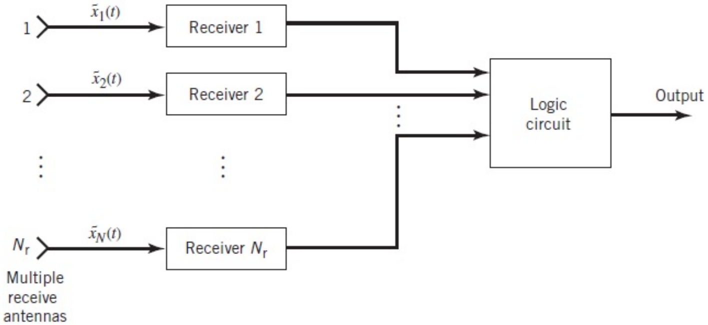
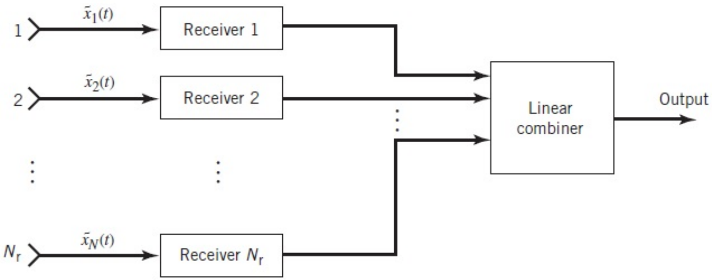
2. Transmit diversity – The use of multiple transmit antennas and a single receive antenna.
3. Diversity on both transmit and receive – The use of multiple antennas at both the transmitter and receiver.

Selection Combing



Maximal-Ratio Combing and Equal-Gain Combing



BER vs. SNR in a flat fading channel

In a flat fading channel (or narrowband system), the CIR (channel impulse response) reduces to a single impulse scaled by a time-varying complex coefficient.

The received (equivalent lowpass) signal is of the form

$$r(t) = a(t) e^{j\phi(t)} s(t) + n(t)$$

We assume that the phase changes "slowly" and can be perfectly tracked

=> important for coherent detection

BER vs. SNR (cont.)

We assume:

- the time-variant complex channel coefficient changes slowly (=> constant during a symbol interval)
- the channel coefficient magnitude (= attenuation factor) a is a Rayleigh distributed random variable
- coherent detection of a binary PSK signal (assuming ideal phase synchronization)

Let us define instantaneous SNR and average SNR:

$$\gamma = a^2 E_b / N_0 \qquad \gamma_0 = E\left\{a^2\right\} \cdot E_b / N_0$$

BER vs. SNR (cont.)

Since

$$p(a) = \frac{2a}{E\left\{a^2\right\}} e^{-a^2 / E\left\{a^2\right\}} \qquad a \geq 0,$$

Rayleigh distribution

using

$$p(\gamma) = \frac{p(a)}{\left| d\gamma / da \right|}$$

Exponential distribution

we get

$$p(\gamma) = \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} \qquad \gamma \geq 0.$$

BER vs. SNR (cont.)

The average bit error probability is

$$P_e = \int_0^\infty P_e(\gamma)\, p(\gamma)\, d\gamma$$

Important formula for obtaining statistical average

where the bit error probability for a certain value of a is

$$P_e(\gamma) = Q\left(\sqrt{2a^2 E_b / N_0}\right) = Q\left(\sqrt{2\gamma}\right).$$

2-PSK

We thus get

$$P_e = \int_0^\infty Q\left(\sqrt{2\gamma}\right) \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} d\gamma = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{1+\gamma_0}}\right).$$

BER vs. SNR (cont.)

Approximation for large values of average SNR is obtained in the following way. First, we write

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{1+\gamma_0}}\right) = \frac{1}{2}\left(1 - \sqrt{1 + \frac{-1}{1+\gamma_0}}\right)$$

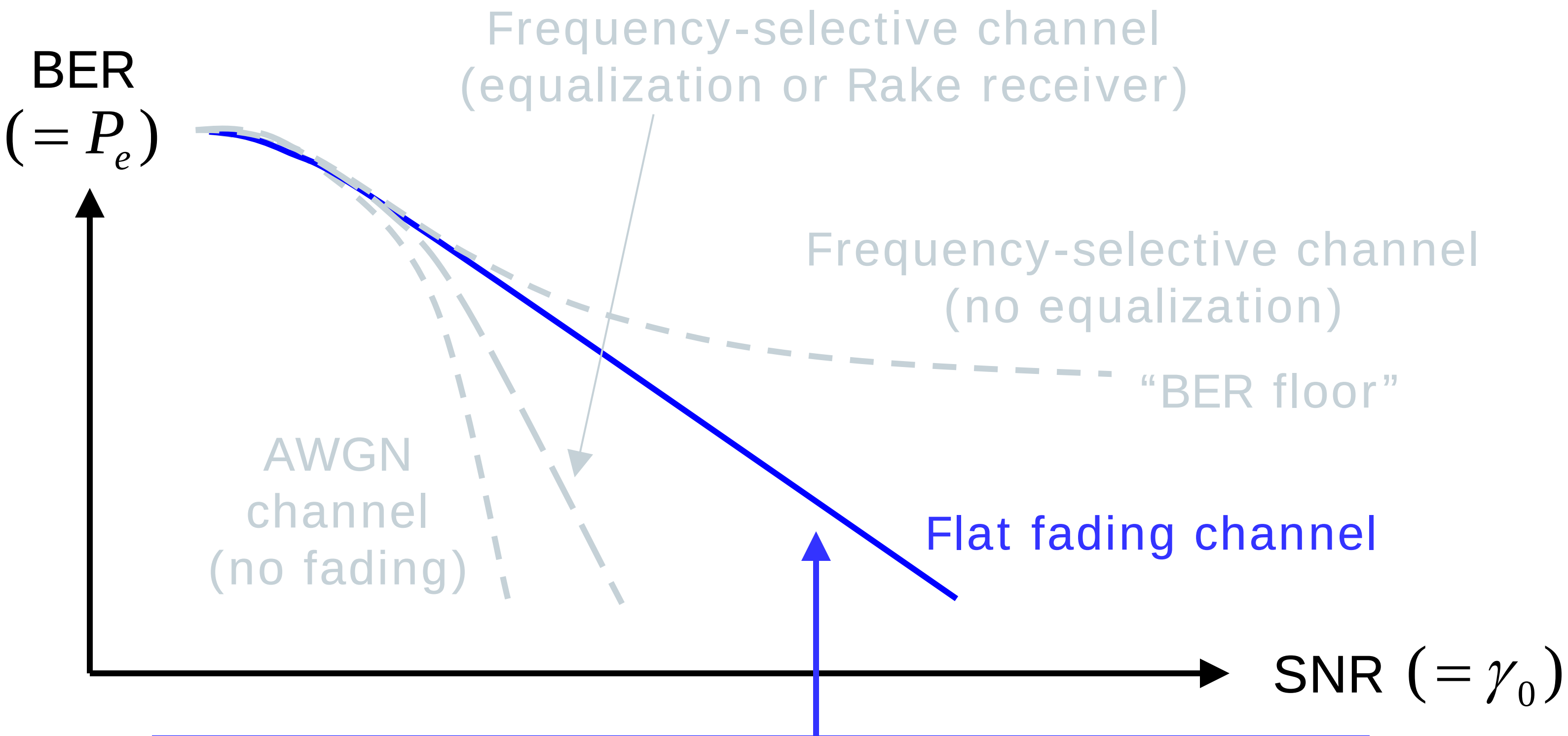
Then, we use

$$\sqrt{1+x} = 1 + x/2 + \ldots$$

which leads to

$P_e \approx 1/4\gamma_0$ for large γ_0 .

BER vs. SNR (cont.)



$P_e \approx 1/4\gamma_0$ means a straight line in log/log scale

BER vs. SNR, summary

Modulation	$P_e(\gamma)$	P_e	P_e (for large γ_0)
2-PSK	$Q\left(\sqrt{2\gamma}\right)$	$\frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{1+\gamma_0}}\right)$	$1/4\gamma_0$
DPSK	$e^{-\gamma}/2$	$1/(2\gamma_0+2)$	$1/2\gamma_0$
2-FSK (coh.)	$Q\left(\sqrt{\gamma}\right)$	$\frac{1}{2}\left(1-\sqrt{\frac{\gamma_0}{2+\gamma_0}}\right)$	$1/2\gamma_0$
2-FSK (non-c.)	$e^{-\gamma/2}/2$	$1/(\gamma_0+2)$	$1/\gamma_0$

Better performance through diversity

Diversity ⇔ the receiver is provided with multiple copies of the transmitted signal. The multiple signal copies should experience *uncorrelated fading* in the channel.

In this case the probability that *all* signal copies fade simultaneously is reduced dramatically with respect to the probability that a *single* copy experiences a fade.

As a rough rule:

P_e is proportional to $\dfrac{1}{\gamma_0^{\,L}}$

- P_e: BER
- γ_0: Average SNR
- L: Diversity of *L*:th order

Selection diversity vs. signal combining

Selection diversity: Signal with best quality is selected.

Equal Gain Combining (EGC)

Signal copies are combined coherently:

$$Z_{EGC} = \sum_{i=1}^{L} a_i \, e^{j\phi_i} e^{-j\phi_i} = \sum_{i=1}^{L} a_i$$

Maximum Ratio Combining (MRC, best SNR is achieved)

Signal copies are weighted and combined coherently:

$$Z_{MRC} = \sum_{i=1}^{L} a_i \, e^{j\phi_i} \; a_i \, e^{-j\phi_i} = \sum_{i=1}^{L} a_i^{\,2}$$

Selection diversity performance

We assume:

(*a*) uncorrelated fading in diversity branches

(*b*) fading in *i*:th branch is Rayleigh distributed

(*c*) => SNR is exponentially distributed:

$$p(\gamma_i) = \frac{1}{\gamma_0} e^{-\gamma_i/\gamma_0}, \quad \gamma_i \geq 0.$$

PDF

Probability that SNR in branch *i* is less than threshold y :

$$P(\gamma_i < y) = \int_0^y p(\gamma_i)\, d\gamma_i = 1 - e^{-y/\gamma_0}.$$

CDF

Selection diversity (cont.)

Probability that SNR in every branch (i.e. all *L* branches) is less than threshold y :

$$P\left(\gamma_1, \gamma_2, \dots, \gamma_L < y\right) = \left[\int_0^y p\left(\gamma_i\right) d\gamma_i\right]^L = \left[1 - e^{-y/\gamma_0}\right]^L.$$

Note: this is true only if the fading in different branches is independent (and thus uncorrelated) and we can write

$$p\left(\gamma_1, \gamma_2, \dots, \gamma_L\right) = p\left(\gamma_1\right) p\left(\gamma_2\right) \dots p\left(\gamma_L\right).$$

Selection diversity (cont.)

Differentiating the cdf (cumulative distribution function) with respect to y gives the pdf

$$p(y) = L\left[1 - e^{-y/\gamma_0}\right]^{L-1} \cdot \frac{e^{-y/\gamma_0}}{\gamma_0}$$

which can be inserted into the expression for average bit error probability

$$P_e = \int_0^\infty P_e(y)\, p(y)\, dy\, .$$

The mathematics is unfortunately quite tedious …

Selection diversity (cont.)

... but as a general rule, for large γ_0 it can be shown that

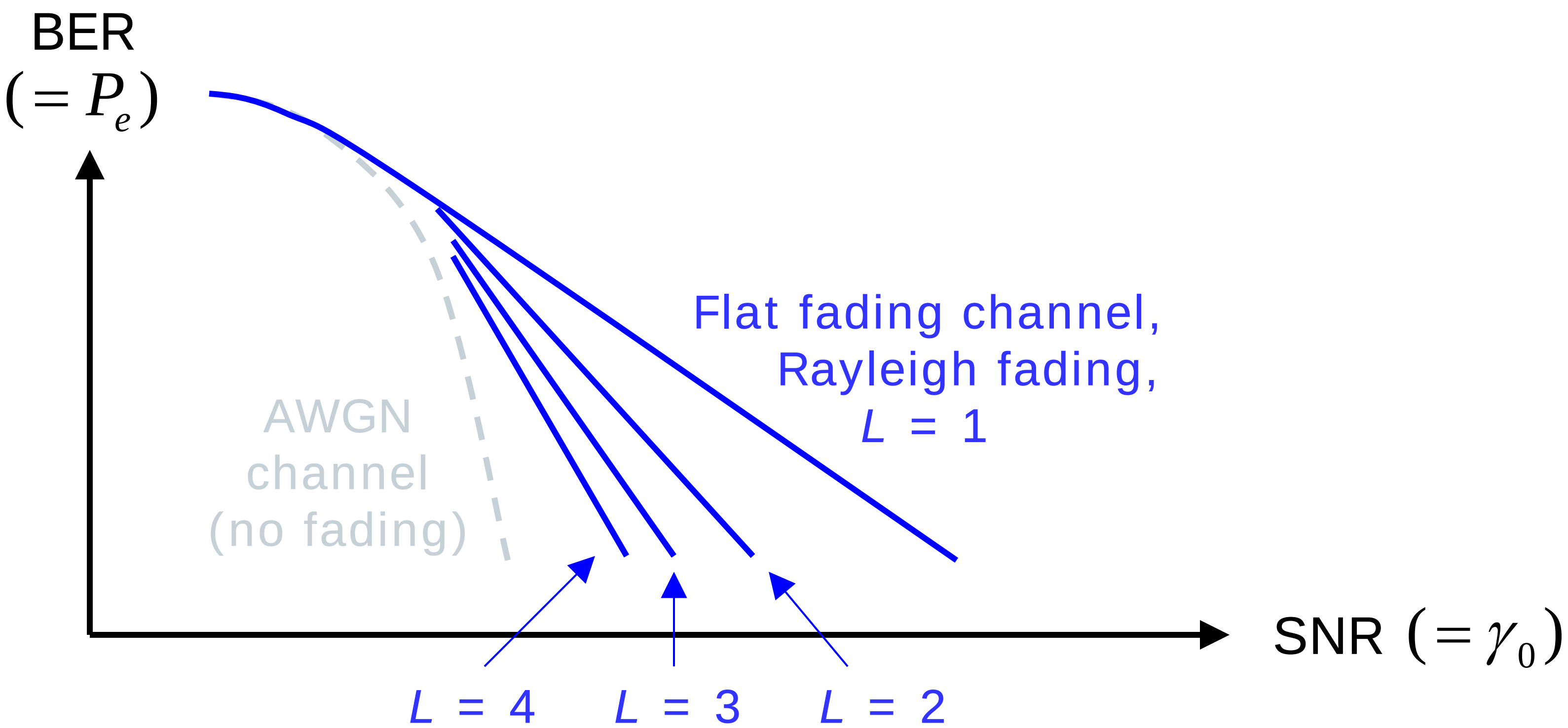
$$P_e \text{ is proportional to } \frac{1}{\gamma_0^{\ L}}$$

regardless of modulation scheme (2-PSK, DPSK, 2-FSK).

The largest diversity gain is obtained when moving from *L* = 1 to *L* = 2. The relative increase in diversity gain becomes smaller and smaller when *L* is further increased.

This behaviour is typical for all diversity techniques.

BER vs. SNR (diversity effect)



MRC performance

Rayleigh fading => SNR in *i*:th diversity branch is

$$\gamma_i = \frac{E_b}{N_0} a_i^{\,2} = \frac{E_b}{N_0}\left(x_i^{\,2} + y_i^{\,2} \right)$$

Rayleigh distributed magnitude

Gaussian distributed quadrature components

In case of *L* uncorrelated branches with same fading statistics, the MRC output SNR is

$$\gamma = \frac{E_b}{N_0}\left(a_1^{\,2} + a_2^{\,2} \ldots + a_L^{\,2} \right) = \frac{E_b}{N_0}\left(x_1^{\,2} + y_1^{\,2} \ldots + x_L^{\,2} + y_L^{\,2} \right)$$

MRC performance (cont.)

The pdf of γ follows the *chi-square distribution* with 2*L* degrees of freedom

Reduces to exponential pdf when *L* = 1

$$p(\gamma) = \frac{\gamma^{L-1}}{\gamma_0{}^{L}\Gamma(L)} e^{-\gamma/\gamma_o} = \frac{\gamma^{L-1}}{\gamma_0{}^{L}(L-1)!} e^{-\gamma/\gamma_o}$$

Gamma function   Factorial

For 2-PSK, the average BER is

$$P_e = \int_0^{\infty} P_e(\gamma)\, p(\gamma)\, d\gamma$$

$$P_e = \left(\frac{1-\mu}{2}\right)^L \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left(\frac{1+\mu}{2}\right)^k$$

$$P_e(\gamma) = Q\left(\sqrt{2\gamma}\right)$$

$$\mu = \sqrt{\gamma_0 / (1+\gamma_0)}$$

MRC performance (cont.)

For large values of average SNR this expression can be approximated by

$$P_e = \left(\frac{1}{4\gamma_0}\right)^L \binom{2L-1}{L}$$

which again is according to the general rule

P_e is proportional to $\dfrac{1}{\gamma_0{}^L}$.

MRC performance (cont.)

The second term in the BER expression does not increase dramatically with *L*:

$$\binom{2L-1}{L} = \frac{(2L-1)!}{L!\cdot(L-1)!} = 1 \qquad L=1$$

$$= 3 \qquad L=2$$

$$= 10 \qquad L=3$$

$$= 35 \qquad L=4$$

BER vs. SNR for MRC, summary

For large γ_0 $\Rightarrow$ $P_e = \left(\frac{1}{k\gamma_0}\right)^L \binom{2L-1}{L}$

Modulation	$P_e(\gamma)$	P_e (for large γ_0)
2-PSK	$Q\left(\sqrt{2\gamma}\right)$	$k = 4$
DPSK		$k = 2$
2-FSK (coh.)	$Q\left(\sqrt{\gamma}\right)$	$k = 2$
2-FSK (non-c.)		$k = 1$

Why is MRC optimum peformance?

Let us investigate the performance of a signal combining method in general using arbitrary weighting coefficients g_i.

Signal magnitude and noise energy/bit at the output of the combining circuit:

$$Z = \sum_{i=1}^{L} g_i \cdot a_i \qquad N_t = N_0 \sum_{i=1}^{L} g_i^{\,2}$$

SNR after combining:

$$\gamma = \frac{Z^2 E_b}{N_t} = \frac{E_b \left(\sum g_i a_i \right)^2}{N_0 \sum g_i^{\,2}}$$

Why is MRC optimum peformance? (cont.)

Applying the Schwarz inequality

$$\left(\sum g_i a_i\right)^2 \leq \sum g_i^{\,2} \sum a_i^{\,2}$$

it can be easily shown that in case of equality we must have $g_i = a_i$ which in fact is the definition of MRC.

Thus for MRC the following important rule applies (the rule also applies to SIR = Signal-to-Interference Ratio):

$$\gamma = \sum_{i=1}^{L} \gamma_i$$

Output SNR or SIR = sum of branch SNR or SIR values

Matched filter = "full-scale" MRC

Let us consider a single symbol in a narrowband system (without ISI). If the sampled symbol waveform before matched filtering consists of *L+1* samples

$$r_k\,, \quad k = 0, 1, 2, \ldots, L$$

the impulse response of the matched filter also consists of *L+1* samples

$$h_k = r^*_{L-k}$$

Definition of matched filter

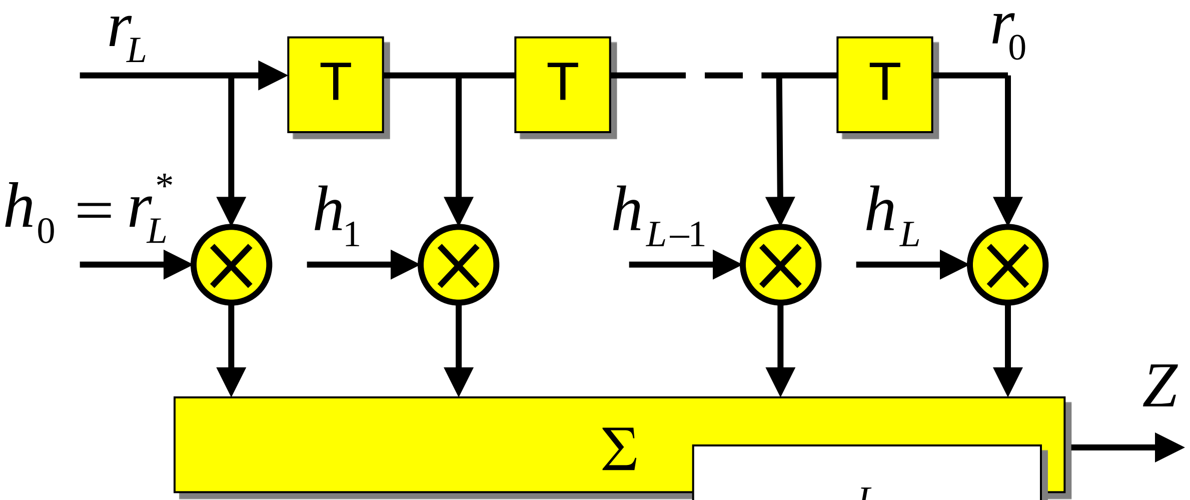
and the output from the matched filter is

MRC !

$$Z = \sum_{k=0}^{L} h_k \, r_{L-k} = \sum_{k=0}^{L} r^*_{L-k} \, r_{L-k} = \sum_{k=0}^{L} |r_k|^2$$

Matched filter = MRC (cont.)

The discrete-time (sampled) matched filter can be presented as a transversal FIR filter:



$$Z = \sum_{k=0}^{L} |r_k|^2$$

=> MRC including all $L+1$ values of r_k