

ELC 5396: Digital Communications

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Frequency Analysis of Signals

- 1 Frequency-Domain and Time-Domain Signal Properties
 - Frequency-Domain and Time-Domain Signal Properties
- 2 Properties of the Fourier Transform for Discrete-Time Signals
 - Symmetry Properties of the Fourier Transform
 - Fourier Transform Theorems and Properties

Frequency Analysis Tools

| | |
|-----------------------|---------------------------------------|
| The Fourier series | for continuous-time periodic signals |
| The Fourier transform | for continuous-time aperiodic signals |
| The Fourier series | for discrete-time periodic signals |
| The Fourier transform | for discrete-time aperiodic signals |

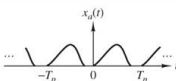
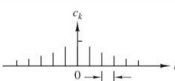
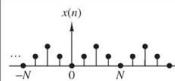
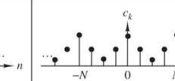
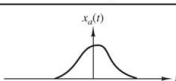
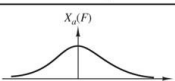
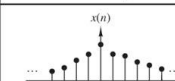
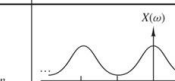
Continuous-time signals have aperiodic spectra

Discrete-time signals have periodic spectra

Periodic signals have discrete spectra

Aperiodic finite energy signals have continuous spectra

The Fourier Series for Continuous-Time Periodic Signals

| | | Continuous-time signals | | Discrete-time signals | |
|---|--|--|--|---|--|
| | | Time-domain | Frequency-domain | Time-domain | Frequency-domain |
| Periodic signals Fourier series | |  $c_k = \frac{1}{T_p} \int_{T_p} x_a(t) e^{-j2\pi k F t} dt$ |  $F_0 = \frac{1}{T_p}$ |  $c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi/N kn}$ |  $x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi/N kn}$ |
| | | Continuous and periodic | Discrete and aperiodic | Discrete and periodic | Discrete and periodic |
| Aperiodic signals Fourier transforms | |  $X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi F t} dt$ |  $x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi F t} dF$ |  $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$ |  $x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$ |
| | | Continuous and aperiodic | Continuous and aperiodic | Discrete and aperiodic | Continuous and periodic |

Periodicity with period α in one domain implies discretization with spacing $1/\alpha$ in the other domain, and *vice versa*.

$$X(\omega) \triangleq F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) \triangleq F^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega$$

Fourier transform pair: $x(n) \xleftrightarrow{F} X(\omega)$

where, $X(\omega)$ is periodic with period 2π .

If signal is complex, it can be expressed in rectangular form

$$x(n) = x_R(n) + jx_I(n)$$

$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

Symmetry Properties of the Fourier Transform

When a signal satisfies some symmetry properties in the time domain, these properties impose some symmetry conditions on its Fourier transform.

Using the rectangular form and $e^{j\omega} = \cos \omega + j \sin \omega$, we have

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} [x_R(n) \cos \omega n + x_I(n) \sin \omega n]$$

$$X_I(\omega) = - \sum_{n=-\infty}^{\infty} [x_R(n) \sin \omega n - x_I(n) \cos \omega n]$$

and

$$x_R(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [X_R(\omega) \cos \omega n - X_I(\omega) \sin \omega n] d\omega$$

$$x_I(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [X_R(\omega) \sin \omega n + X_I(\omega) \cos \omega n] d\omega$$

Symmetry Properties of the Fourier Transform

Real signals. $x_R(n) = x(n)$ and $x_I(n) = 0$.

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cos \omega n$$

$$X_I(\omega) = - \sum_{n=-\infty}^{\infty} x(n) \sin \omega n$$

It follows that

$$X_R(-\omega) = X_R(\omega)$$

$$X_I(-\omega) = -X_I(\omega)$$

$\implies X^*(\omega) = X(-\omega)$. The spectrum of a real signal has *Hermitian symmetry*.

Symmetry Properties of the Fourier Transform

Real signals. $x_R(n) = x(n)$ and $x_I(n) = 0$.

$$X_R(-\omega) = X_R(\omega) \quad (\text{even})$$

$$X_I(-\omega) = -X_I(\omega) \quad (\text{odd})$$

$$|X(-\omega)| = |X(\omega)| \quad (\text{even})$$

$$\angle X(-\omega) = -\angle X(\omega) \quad (\text{odd})$$

Symmetry Properties of the Fourier Transform

Real and even signals. $x_R(n) = x(n)$, $x_I(n) = 0$ and $x(-n) = x(n)$.

$$X_R(\omega) = x(0) + 2 \sum_{n=1}^{\infty} x(n) \cos \omega n \quad (\text{even})$$

$$X_I(\omega) = 0$$

It has real-valued spectrum, which is even function of the frequency ω .

Symmetry Properties of the Fourier Transform

Real and odd signals. $x_R(n) = x(n)$, $x_I(n) = 0$ and $x(-n) = -x(n)$.

$$X_R(\omega) = 0$$

$$X_I(\omega) = -2 \sum_{n=1}^{\infty} x(n) \sin \omega n \quad (\text{odd})$$

It has imaginary-valued spectrum, which is odd function of the frequency ω .

Symmetry Properties of the Fourier Transform

Purely imaginary signals. $x_R(n) = 0$ and $jx_I(n) = x(n)$.

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x_I(n) \sin \omega n \quad (\text{odd})$$

$$X_I(\omega) = \sum_{n=-\infty}^{\infty} x_I(n) \cos \omega n \quad (\text{even})$$

Symmetry Properties of the Fourier Transform

Purely imaginary and odd signals. $x_R(n) = 0$, $jx_I(n) = x(n)$ and $x_I(-n) = -x_I(n)$.

$$X_R(\omega) = 2 \sum_{n=1}^{\infty} x_I(n) \sin \omega n \quad (\text{odd})$$
$$X_I(\omega) = 0$$

It has real-valued spectrum, which is odd function of the frequency ω .

Symmetry Properties of the Fourier Transform

Purely imaginary and even signals. $x_R(n) = 0$, $jx_I(n) = x(n)$ and $x_I(-n) = x_I(n)$.

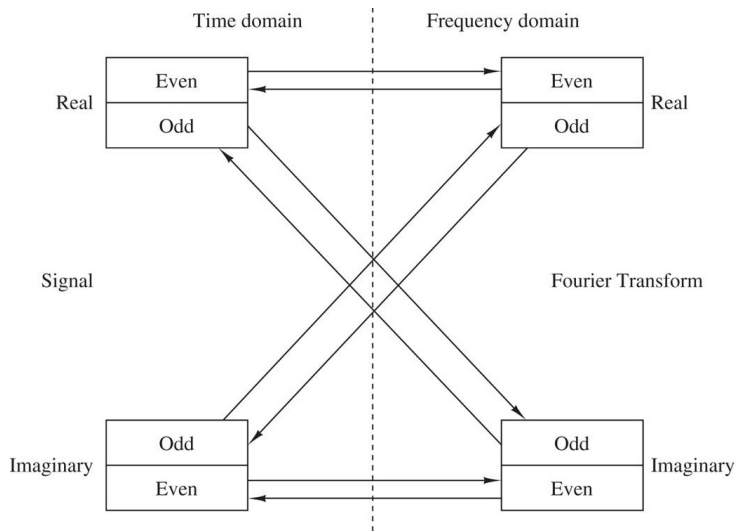
$$X_R(\omega) = 0$$

$$X_I(\omega) = x_I(0) + 2 \sum_{n=1}^{\infty} x_I(n) \cos \omega n \quad (\text{even})$$

It has imaginary-valued spectrum, which is even function of the frequency ω .

Symmetry Properties of the Fourier Transform

Summary of symmetry properties for the Fourier Transform



Fourier Transform Theorems and Properties

Linearity.

If $x_1(n) \longleftrightarrow X_1(\omega)$ and $x_2(n) \longleftrightarrow X_2(\omega)$,

then

$$\alpha_1 x_1(n) + \alpha_2 x_2(n) \longleftrightarrow \alpha_1 X_1(\omega) + \alpha_2 X_2(\omega).$$

Fourier Transform Theorems and Properties

Time shifting.

If $x(n) \longleftrightarrow X(\omega)$,

then

$$x(n - k) \longleftrightarrow e^{-j\omega k} X(\omega).$$

Fourier Transform Theorems and Properties

Time reversal.

If $x(n) \longleftrightarrow X(\omega)$,

then

$x(-n) \longleftrightarrow X(-\omega)$.

Fourier Transform Theorems and Properties

Convolution theorem.

If $x_1(n) \longleftrightarrow X_1(\omega)$ and $x_2(n) \longleftrightarrow X_2(\omega)$,

then

$$x(n) = x_1(n) \otimes x_2(n) \longleftrightarrow X(\omega) = X_1(\omega)X_2(\omega).$$

Fourier Transform Theorems and Properties

Correlation theorem.

If $x_1(n) \longleftrightarrow X_1(\omega)$ and $x_2(n) \longleftrightarrow X_2(\omega)$,
then

$$r_{x_1x_2}(l) \longleftrightarrow S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega).$$

Fourier Transform Theorems and Properties

The Wiener-Khintchine theorem.

If $x(n)$ is a real signal, then $r_{xx}(l) \longleftrightarrow S_{xx}(\omega)$.

Notice that neither the autocorrelation nor the energy spectral density has any phase information.

Fourier Transform Theorems and Properties

Frequency shifting.

If $x(n) \longleftrightarrow X(\omega)$,

then

$e^{j\omega_0 n} x(n) \longleftrightarrow X(\omega - \omega_0)$.

Fourier Transform Theorems and Properties

The modulation theorem.

If $x(n) \longleftrightarrow X(\omega)$,

then

$$x(n) \cos \omega n \longleftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)].$$

$$x(n)e^{j\omega n} \longleftrightarrow X(\omega + \omega_0).$$

Fourier Transform Theorems and Properties

Parseval's theorem.

If $x_1(n) \longleftrightarrow X_1(\omega)$ and $x_2(n) \longleftrightarrow X_2(\omega)$,
then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega.$$

Fourier Transform Theorems and Properties

Windowing theorem.

If $x_1(n) \longleftrightarrow X_1(\omega)$ and $x_2(n) \longleftrightarrow X_2(\omega)$,
then

$$x_1(n)x_2(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega - \lambda)d\lambda.$$

Fourier Transform Theorems and Properties

Differentiation in the frequency domain.

If $x(n) \longleftrightarrow X(\omega)$,
then

$$nx(n) \longleftrightarrow j \frac{dX(\omega)}{d\omega}.$$