

ELC 4351: Digital Signal Processing

Liang Dong

Electrical and Computer Engineering
Baylor University

liang.dong@baylor.edu

September 14, 2017

Discrete-time Systems Described by Difference Equations

A LTI system is characterized by its unit sample response $h(n)$.

The output $y(n)$ of the system for any given input $x(n)$ is determined by

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$y(n) = h(n) \otimes x(n)$$

FIR systems vs. IIR systems

Recursive and Nonrecursive Discrete-time Systems

e.g., Cumulative average of signal $x(n)$

$$y(n) = \frac{1}{n+1} \sum_{k=0}^n x(k)$$

$$\begin{aligned}(n+1)y(n) &= \sum_{k=0}^{n-1} x(k) + x(n) \\ &= ny(n-1) + x(n)\end{aligned}$$

$$y(n) = \frac{n}{n+1}y(n-1) + \frac{1}{n+1}x(n)$$

where $y(n_0 - 1)$ is the initial condition for the system at time $n = n_0$.

LTI Systems Characterized by Constant-Coefficient Difference Equations

A recursive system:

$$y(n) = \alpha y(n-1) + x(n)$$

where α is a constant.

$$y(0) = \alpha y(-1) + x(0)$$

$$y(1) = \alpha y(0) + x(1) = \alpha^2 y(-1) + \alpha x(0) + x(1)$$

$$\vdots \quad \vdots$$

$$y(n) = \alpha^{n+1} y(-1) + \sum_{k=0}^n \alpha^k x(n-k), \quad n \geq 0$$

LTI Systems Characterized by Constant-Coefficient Difference Equations

$$y(n) = \alpha^{n+1}y(-1) + \sum_{k=0}^n \alpha^k x(n-k), \quad n \geq 0$$

- ① The system is initially relaxed at time $n = 0$, i.e., $y(-1) = 0$.
Zero-state response or forced response

$$y_{zs}(n) = \sum_{k=0}^n \alpha^k x(n-k), \quad n \geq 0$$

LTI Systems Characterized by Constant-Coefficient Difference Equations

$$y(n) = \alpha^{n+1}y(-1) + \sum_{k=0}^n \alpha^k x(n-k), \quad n \geq 0$$

- ① The system is initially relaxed at time $n = 0$, i.e., $y(-1) = 0$.
Zero-state response or forced response

$$y_{zs}(n) = \sum_{k=0}^n \alpha^k x(n-k), \quad n \geq 0$$

- ② The input $x(n) = 0, \forall n$.
Zero-input response or natural response

$$y_{zi}(n) = \alpha^{n+1}y(-1), \quad n \geq 0$$

LTI Systems Characterized by Constant-Coefficient Difference Equations

$$y(n) = \alpha^{n+1}y(-1) + \sum_{k=0}^n \alpha^k x(n-k), \quad n \geq 0$$

- ① The system is initially relaxed at time $n = 0$, i.e., $y(-1) = 0$.
Zero-state response or forced response

$$y_{zs}(n) = \sum_{k=0}^n \alpha^k x(n-k), \quad n \geq 0$$

- ② The input $x(n) = 0, \forall n$.
Zero-input response or natural response

$$y_{zi}(n) = \alpha^{n+1}y(-1), \quad n \geq 0$$

- ③ The total response of the system

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

LTI Systems Characterized by Constant-Coefficient Difference Equations

For LTI systems, a general form of the input-output relationship.

$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k)$$

$$\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k), \quad a_0 \equiv 1$$

The integer N is the order of the difference equation or the order of the system.

A Linear System:

- The total response is equal to the sum of the zero-state and zero-input responses

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

A Linear System:

- The total response is equal to the sum of the zero-state and zero-input responses

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

- The principle of superposition applies to the zero-state response.

A Linear System:

- The total response is equal to the sum of the zero-state and zero-input responses

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

- The principle of superposition applies to the zero-state response.
- The principle of superposition applies to the zero-input response.

Solution of Linear Constant-Coefficient Difference Equations

The direct solution

$$y(n) = \underbrace{y_h(n)}_{\text{homogeneous solution}} + \underbrace{y_p(n)}_{\text{particular solution}}$$

The Homogeneous Solution of A Difference Equation

The homogeneous difference equation:

$$\sum_{k=0}^N \alpha_k y(n-k) = 0$$

- We assume that the solution is in the form of an exponential, i.e., $y_h(n) = \lambda^n$.

The Homogeneous Solution of A Difference Equation

The homogeneous difference equation:

$$\sum_{k=0}^N \alpha_k y(n-k) = 0$$

- We assume that the solution is in the form of an exponential, i.e., $y_h(n) = \lambda^n$.
- Substituting this in the equation, we obtain the polynomial equation

$$\sum_{k=0}^N \alpha_k \lambda^{n-k} = 0$$

$$\lambda^{n-N} \underbrace{(\lambda^N + \alpha_1 \lambda^{N-1} + \cdots + \alpha_{N-1} \lambda + \alpha_N)}_{\text{characteristic polynomial}} = 0$$

The Homogeneous Solution of A Difference Equation

The homogeneous difference equation:

$$\sum_{k=0}^N \alpha_k y(n-k) = 0$$

- We assume that the solution is in the form of an exponential, i.e., $y_h(n) = \lambda^n$.
- Substituting this in the equation, we obtain the polynomial equation

$$\sum_{k=0}^N \alpha_k \lambda^{n-k} = 0$$

$$\lambda^{n-N} \underbrace{(\lambda^N + \alpha_1 \lambda^{N-1} + \cdots + \alpha_{N-1} \lambda + \alpha_N)}_{\text{characteristic polynomial}} = 0$$

- The characteristic polynomial of the system has N roots:
 $\lambda_1, \lambda_2, \dots, \lambda_N$.

The Homogeneous Solution of A Difference Equation

If the N roots are distinct, the general solution to the homogeneous difference equation is

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \cdots + C_N\lambda_N^n$$

where C_1, C_2, \dots, C_N are weighting coefficients.

These coefficients are determined from the initial conditions of the system.

$y_h(n)$ is the zero-input response of the system.

The Homogeneous Solution of A Difference Equation

If the characteristic polynomial contains multiple roots, e.g. λ_1 is a root of multiplicity m , then

$$h_h(n) = C_1\lambda_1^n + C_2n\lambda_1^n + \cdots + C_m n^{m-1}\lambda_1^n + C_{m+1}\lambda_{m+1}^n + \cdots + C_N\lambda_M^n$$

The Particular Solution of A Difference Equation

The particular difference equation for a specific input signal $x(n)$:

$$\sum_{k=0}^N a_k y_p(n-k) = \sum_{k=0}^M b_k x(n-k), \quad a_0 \equiv 1$$

Input Signal $x(n)$	Particular Solution $y_p(n)$
A	K
AM^n	KM^n
An^M	$K_0 n^M + K_1 n^{M-1} + \dots + K_M$
$A^n n^M$	$A^n (K_0 n^M + K_1 n^{M-1} + \dots + K_M)$
$A \cos \omega_0 n$	$K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$
$A \sin \omega_0 n$	$K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$

The Total Solution of A Difference Equation

$$y(n) = y_h(n) + y_p(n)$$

The Impulse Response of a LTI Recursive System

- The impulse response $h(n)$ is equal to the zero-state response of the system when the input $x(n) = \delta(n)$ and the system is initially relaxed.

$$y_{zs}(n) = \sum_{k=0}^n h(k)x(n-k), \quad n \geq 0$$

When $x(n) = \delta(n)$, $y_{zs}(n) = h(n)$.

The Impulse Response of a LTI Recursive System

- The impulse response $h(n)$ is equal to the zero-state response of the system when the input $x(n) = \delta(n)$ and the system is initially relaxed.

$$y_{zs}(n) = \sum_{k=0}^n h(k)x(n-k), \quad n \geq 0$$

When $x(n) = \delta(n)$, $y_{zs}(n) = h(n)$.

- If the excitation is an impulse, the particular solution is zero, since $x(n) = 0, \forall n > 0$. That is $y_p(n) = 0$.

The response of the system to an impulse consists only of the solution to the homogeneous equations.

The Impulse Response of a LTI Recursive System

N th-order linear difference equation.

The solution of the homogeneous equation is

$$y_h(n) = \sum_{k=1}^N C_k \lambda_k^n.$$

Hence, the impulse response of the system is

$$h(n) = \sum_{k=1}^N C_k \lambda_k^n.$$