ELC 4351: Digital Signal Processing

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Discrete Fourier Transform

1. Fourier Transform and Discrete Fourier Transform

2. Fast Fourier Transform Algorithms

The Fourier Transform of Discrete-Time Aperiodic Signals

Analysis Equation

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}, \qquad \omega \in [-\pi, \pi) \text{ or } \omega \in [0, 2\pi)$$

Synthesis Equation

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$X(\omega)$ is periodic with period 2π.

The Discrete Fourier Transform (DFT)

N-point DFT.

Analysis Equation

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{k}{N} n}, \qquad k = 0, 1, 2, \ldots, N-1$$

Synthesis Equation

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi \frac{k}{N} n}, \qquad n = 0, 1, 2, \ldots, N-1$$

N samples of the Fourier transform at N equally spaced frequencies.
$\omega_k = \frac{2\pi k}{N}$, $k = 0, 1, 2, \ldots, N-1$.

A Fast Algorithm for the DFT

Analysis Equation

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, 2, \ldots, N-1$$

Synthesis Equation

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \qquad n = 0, 1, 2, \ldots, N-1$$

where, $W_N = e^{-j2\pi/N}$.

A Fast Algorithm for the DFT

Analysis Equation

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, 2, \ldots, N-1$$

To calculate one frequency sample (each k) in the analysis equation (direct Fourier transform), we need N complex multiplications and $N-1$ complex additions.

For all N frequency samples, we need a total of N^2 complex multiplications and $N(N-1)$ complex additions.

A Fast Algorithm for the DFT

Analysis Equation

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, 2, \ldots, N-1$$

For all N frequency samples, we need a total of $4N^2$ real multiplications and $N(4N-2)$ real additions.

A Fast Algorithm for the DFT

Analysis Equation

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, 2, \ldots, N-1$$

The computation complexity of the DFT is proportional to N^2.

As a comparison, the computational complexity of the FFT is proportional to $N \log N$.

Periodicity of W_N^{kn}

Use the periodicity of the sequence W_N^{kn} to reduce computation.

$$W_N^{kN} = e^{-j\frac{2\pi}{N}kN} = e^{-j2\pi k} = 1$$

$$\begin{aligned} W_N^{k(N-n)} &= W_N^{-kn} = (W_N^{kn})^* \\ &\quad \text{(complex conjugate symmetry)} \end{aligned}$$

$$\begin{aligned} W_N^{kn} &= W_N^{k(n+N)} = W_N^{(k+N)n} \\ &\quad \text{(periodicity)} \end{aligned}$$

Considering N an integer power of 2, i.e. $N = 2^\nu$.

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad k = 0, 1, \ldots, N-1 \\
&= \sum_{n \text{ even}} x(n) W_N^{nk} + \sum_{n \text{ odd}} x(n) W_N^{nk} \\
&= \sum_{r=0}^{N/2-1} x(2r) W_N^{2rk} + \sum_{r=0}^{N/2-1} x(2r+1) W_N^{(2r+1)k} \\
&= \sum_{r=0}^{N/2-1} x(2r) (W_N^2)^{rk} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1) (W_N^2)^{rk}
\end{aligned}$$

Decimation-in-Time Fast Fourier Transform (FFT)

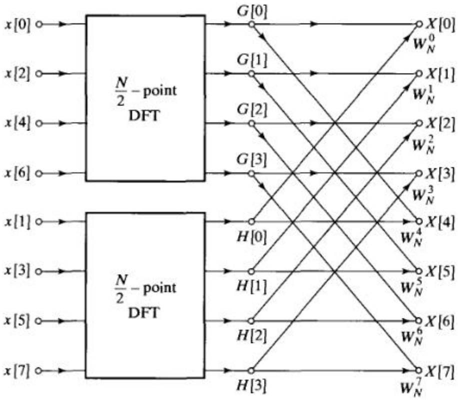
$$W_N^2 = e^{-2j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{N/2}} = W_{N/2}$$

Therefore,

$$\begin{aligned}
X(k) &= \sum_{r=0}^{N/2-1} x(2r)(W_N^2)^{rk} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1)(W_N^2)^{rk} \\
&= \underbrace{\sum_{r=0}^{N/2-1} x(2r) W_{N/2}^{rk}}_{G(k)} + W_N^k \underbrace{\sum_{r=0}^{N/2-1} x(2r+1) W_{N/2}^{rk}}_{H(k)} \\
&= G(k) + W_N^k H(k)
\end{aligned}$$

Decimation-in-Time Fast Fourier Transform (FFT)

$G(k)$ is an $(N/2)$-point DFT of even samples $x(2r)$;
$H(k)$ is an $(N/2)$-point DFT of odd samples $x(2r+1)$.



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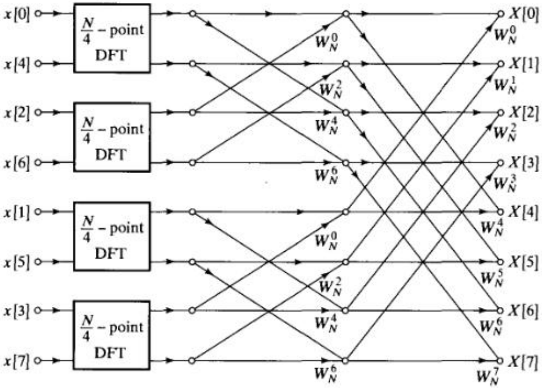
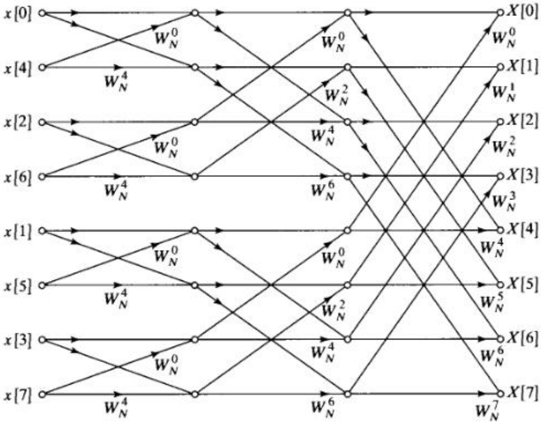
$$
\begin{aligned}
G(k) &= \sum_{l=0}^{N/4-1} g(2l) W_{N/4}^{lk} + W_{N/2}^{k} \sum_{l=0}^{N/4-1} g(2l+1) W_{N/4}^{lk} \\
H(k) &= \underbrace{\sum_{l=0}^{N/4-1} h(2l) W_{N/4}^{lk}}_{(N/4)-\text{point DFT}} + W_{N/2}^{k} \underbrace{\sum_{l=0}^{N/4-1} h(2l+1) W_{N/4}^{lk}}_{(N/4)-\text{point DFT}}
\end{aligned}
$$

Decimation-in-Time Fast Fourier Transform (FFT)

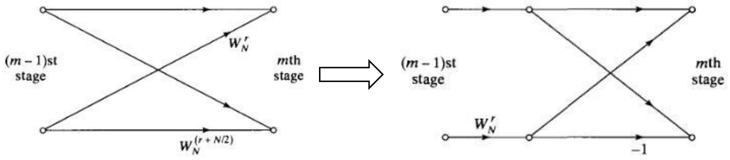


Decimation-in-Time Fast Fourier Transform (FFT)

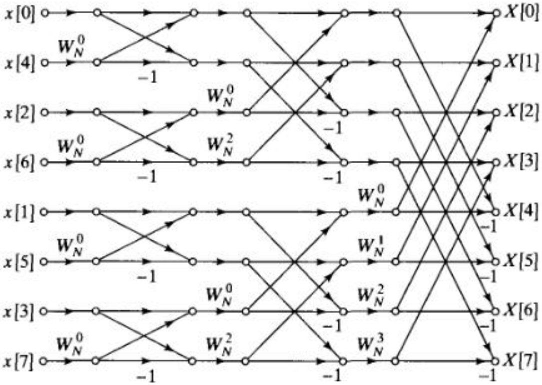


Symmetry of W_N

$$
\begin{aligned}
W_N^{N/2} &= e^{-j\frac{2\pi}{N}\frac{N}{2}} = e^{-j\pi} = -1 \\
W_N^{r+N/2} &= W_N^{N/2} W_N^r = -W_N^r
\end{aligned}
$$



Decimation-in-Time Fast Fourier Transform (FFT)



Bit-reverse Reordering

x(n)'s index n	binary
0	000
4	100
2	010
6	110
1	001
5	101
3	011
7	111

Decimation-in-Frequency Fast Fourier Transform (FFT)