ELC 4351: Digital Signal Processing

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z-Transform Part 3

The z-Transform and Its Application to the Analysis of LTI Systems

Inversion of the z-Transform

Analysis of LTI Systems in the z-Domain

Causality and Stability

Inversion of the z-Transform

$$H(z) = \frac{Y(z)}{X(z)}, \qquad H(z) \rightarrow^{inv\ z} h(n)$$

Inverse z-Transform:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where the integral is a (counter-clockwise) contour integral over a closed path C that encloses the origin and lies within the region of convergence of $X(z)$.

Methods of Inverse z-Transform

(1) Contour integration

(2) Power series expansion (using long division)

(3) Partial-fraction expansion

Inverse z-Transform by Partial-Fraction Expansion

$X(z)$ is rational function.

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

A rational function is *proper* if $a_N \neq 0$ and $M < N$.

Inverse z-Transform by Partial-Fraction Expansion

An improper rational function ($M \geq N$) can always be written as the sum of a polynomial and a proper rational function.

$$X(z) = \frac{B(z)}{A(z)} = c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{B_1(z)}{A(z)}$$

The inverse z-transform of the polynomial can easily be found by inspection.

We focus our attention on the inversion of proper rational function.

Inverse z-Transform by Partial-Fraction Expansion

Let $X(z)$ be a proper rational function.

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$
$$= \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

Since $N > M$,

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

is proper.

Inverse z-Transform by Partial-Fraction Expansion

(1) Distinct poles. Suppose that the poles $p_1, p_2, \ldots, p_N$ are all different.

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

We want to determine the coefficients $A_1, A_2, \ldots, A_N$.

$$\frac{(z - p_k)X(z)}{z} = \frac{(z - p_k)A_1}{z - p_1} + \cdots + A_k + \cdots + \frac{(z - p_k)A_N}{z - p_N}$$

Therefore,

$$A_k = \left. \frac{(z - p_k)X(z)}{z} \right|_{z = p_k}, \qquad k = 1, 2, \ldots, N$$

(In addition, if $p_2 = p_1^*$, $A_2 = A_1^*$.)

Inverse z-Transform by Partial-Fraction Expansion

(2) Multiple-order poles. $X(z)$ has a pole of multiplicity m, that is, it contains in its denominator the factor $(z - p_k)^m$.

The partial-fraction expansion must contain the terms

$$\frac{A_{1k}}{(z-p_k)} + \frac{A_{2k}}{(z-p_k)^2} + \cdots + \frac{A_{mk}}{(z-p_k)^m}$$

Therefore,

$$A_{mk} = \left. \frac{(z-p_k)^m X(z)}{z} \right|_{z=p_k}$$

$$A_{(m-1)k} = \frac{d}{dz}\left[\frac{(z-p_k)^m X(z)}{z}\right]_{z=p_k}, \cdots$$

$$A_{1k} = \frac{d^{(m-1)}}{dz^{(m-1)}}\left[\frac{(z-p_k)^m X(z)}{z}\right]_{z=p_k}$$

Inverse z-Transform by Partial-Fraction Expansion

$$\begin{aligned} \frac{X(z)}{z} &= \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \cdots + \frac{A_N}{z-p_N} \\ X(z) &= \frac{A_1}{1-p_1 z^{-1}} + \frac{A_2}{1-p_2 z^{-1}} + \cdots + \frac{A_N}{1-p_N z^{-1}} \end{aligned}$$

$$\mathcal{Z}^{-1}\left\{\frac{1}{1-p_k z^{-1}}\right\} = \begin{cases} (p_k)^n u(n), & \text{ROC} : |z| > |p_k| \ (causal) \\ -(p_k)^n u(-n-1), & \text{ROC} : |z| < |p_k| \ (anticausal) \end{cases}$$

Inverse z-Transform by Partial-Fraction Expansion

In the case of a double pole:

$$\begin{aligned}\frac{X(z)}{z} &= \frac{A}{(z-p)^2} + \cdots \\ X(z) &= \frac{Az^{-1}}{(1-pz^{-1})^2} + \cdots\end{aligned}$$

$$\mathcal{Z}^{-1}\left\{\frac{pz^{-1}}{(1-pz^{-1})^2}\right\} = \begin{cases} np^n u(n), & \text{ROC}: |z| > |p| \ (causal) \\ -np^n u(-n-1), & \text{ROC}: |z| < |p| \ (anticausal)\end{cases}$$

Decomposition of Rational z-Transform

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$

With real signals,

$$\begin{aligned}X(z) &= \sum_{k=0}^{M-N} \gamma_k z^{-k} + \sum_{k=1}^{K_1} \frac{\beta_k}{1 + \alpha_k z^{-1}} + \sum_{k=1}^{K_2} \frac{\beta_{0k} + \beta_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \\ &= v_0 \prod_{k=1}^{K_1} \frac{1 + v_k z^{-1}}{1 + u_k z^{-1}} \prod_{k=1}^{K_2} \frac{1 + v_{1k} z^{-1} + v_{2k} z^{-2}}{1 + u_{1k} z^{-1} + u_{2k} z^{-2}}\end{aligned}$$

where $K_1 + 2K_2 = N$.

Coefficients $\alpha_k, \beta_k, \gamma_k, u_k, v_k$ are real.

Analysis of LTI Systems in the z-Domain

Zero-pole systems represented by linear constant-coefficient difference equations with arbitrary initial conditions.

$$H(z) = \frac{B(z)}{A(z)}$$

Assume that the input signal $x(n)$ has a rational z-transform $X(z)$

$$X(z) = \frac{N(z)}{Q(z)}$$

The system is initially relaxed, i.e.,
$y(-1) = y(-2) = \cdots = y(-N) = 0$.

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)}$$

Analysis of LTI Systems in the z-Domain

Suppose that the system contains simple poles $p_1, p_2, \ldots, p_N$ and the z-transform of the input signal contains poles $q_1, q_2, \ldots, q_L$, where $p_k \neq q_m$ for all k and m.

In addition, suppose that there is no pole-zero cancellation.

A partial-fraction expansion of $Y(z)$ yields

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}}$$

Inverse transform of $Y(z)$:

$$y(n) = \underbrace{\sum_{k=1}^{N} A_k (p_k)^n u(n)}_{\text{natural response}} + \underbrace{\sum_{k=1}^{L} Q_k (q_k)^n u(n)}_{\text{forced response}}$$

Transient Response and Steady-State Response

$$y_{nr}(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n)$$

If $|p_k| < 1$ for all k, then $y_{nr}(n)$ decays to zero as n approaches infinity. The natural response is called the transient response.

$$y_{fr}(n) = \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

If the poles fall on the unit circle and consequently, the forced response persists for all $n > 0$. The forced response is called the steady-state response of the system.

Causality

Causal LTI system: $h(n) = 0, \; n < 0$.

(The ROC of the z-transform of a causal sequence is the exterior of a circle.)

A LTI system is causal *iff* the ROC of the system function is the exterior of a circle of radius $r < \infty$, including the point $z = \infty$.

Stability

BIBO stable LTI system: $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n) z^{-n} \\ |H(z)| &\leq \sum_{n=-\infty}^{\infty} |h(n) z^{-n}| \\ &= \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}| \end{aligned}$$

When evaluated on the unit circle, i.e., $|z| = 1$,

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)| < \infty \ \Rightarrow \text{The ROC includes the unit circle.}$$

Causality and Stability

A causal and stable LTI system must have a system function converges for $|z| > r$, where $r < 1$.

A causal LTI system is BIBO stable *iff* all the poles of $H(z)$ are inside the unit circle.

cf. A causal LTI system with a rational transfer function $H(s)$ is stable *iff* all poles of $H(s)$ are in the left half of the s-plane, i.e., the real parts of all poles are negative.

Causality and Stability Example

A LTI system is characterized by the system function

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}} = \frac{1}{1-0.5z^{-1}} + \frac{2}{1-3z^{-1}}$$

Specify the ROC of $H(z)$ and determine $h(n)$ for the following conditions:

(1) The system is stable.

(2) The system is causal.

(3) The system is anticausal.

Causality and Stability Example

Solution. The system has poles at $z = 0.5$ and $z = 3$.

(1) Since the system is stable, its ROC must include the unit circle and hence it is $0.5 < |z| < 3$.

$$h(n) = (0.5)^n u(n) - 2(3)^n u(-n-1) \ \Rightarrow \text{noncausal}$$

(2) Since the system is causal, its ROC is $|z| > 3$.

$$h(n) = (0.5)^n u(n) + 2(3)^n u(n) \ \Rightarrow \text{unstable}$$

(3) Since the system is anticausal, its ROC is $|z| < 0.5$.

$$h(n) = -(0.5)^n u(-n-1) - 2(3)^n u(-n-1) \ \Rightarrow \text{unstable}$$

Pole-Zero Cancellation

Pole-zero cancellations can occur either in the system function itself or in the product of the system function $H(z)$ with the z-transform of the input signal $X(z)$.