ELC 4351: Digital Signal Processing

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Frequency-Domain Analysis of LTI Systems II

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Linear Time-Invariant Systems as Frequency-Selective Filters

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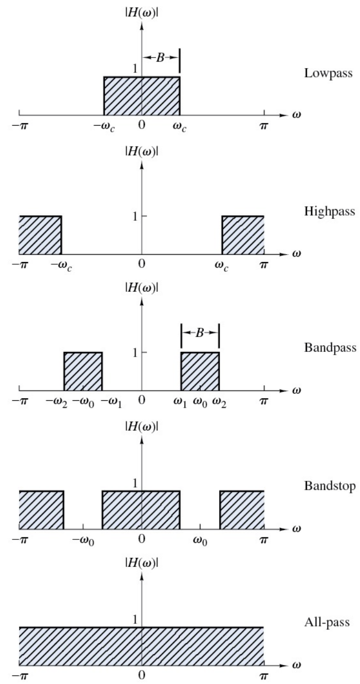
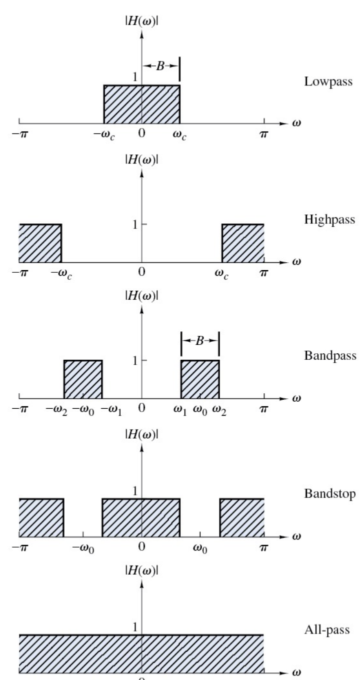
- A LTI system performs a type of discrimination or filtering among the various frequency components at its input.
- The nature of this filtering action is determined by the frequency response characteristics $H(\omega)$.



Linear Time-Invariant Systems as Frequency-Selective Filters

- By proper selection of the coefficients a_k's and b_k's, we can design frequency-selective filters.

  These filters pass signals with frequency components in some bands while they attenuate signals containing frequency components in other frequency bands.



Ideal Filter Characteristics

A filter with frequency response

$$H(\omega) = \begin{cases} Ce^{-j\omega n_0}, & \omega_1 < \omega < \omega_2 \\ 0, & \text{otherwise} \end{cases}$$

where C and n_0 are constants.

$$\begin{aligned} Y(\omega) &= X(\omega)H(\omega) = CX(\omega)e^{-j\omega n_0} \\ y(n) &= Cx(n - n_0) \end{aligned}$$

- The filter output is simply a delayed and amplitude-scaled version of the input signal.
- A pure delay is usually tolerable and is not considered a distortion of the signal. Neither is amplitude scaling.

Therefore, ideal filters have a linear phase characteristic within their passband, that is,

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Ideal filters have a linear phase characteristic within their passband, that is,

$$\Theta(\omega) = -\omega n_0$$

Group delay of the filter

$$\tau_g(\omega) = -\frac{d\Theta(\omega)}{d\omega}$$

Linear phase = group delay is constant. In this case, all frequency components of the input signal undergo the same time delay.

Ideal Filter Characteristics

"Ideal" filter:

- Impulse response is a sinc function.
- This filter is not causal and it is not absolutely summable and therefore unstable.

- Design some simple digital filters by the placement of poles and zeros in the z-plane.
- The location of poles and zeros affects the frequency response characteristics of the system.

The Pole-Zero Placement Method

The basic principle underlying the pole-zero placement method:

- Locate poles near points of the unit circle corresponding to frequencies to be emphasized, and
- Locate zeros near the frequencies to be deemphasized.
- All poles should be placed inside the unit circle in order for the filter to be stable.
  However, zeros can be placed anywhere in the z-plane.
- All complex zeros and poles must occur in complex-conjugate pairs in order for the filter coefficients to be real.

The Pole-Zero Placement Method

The system function:

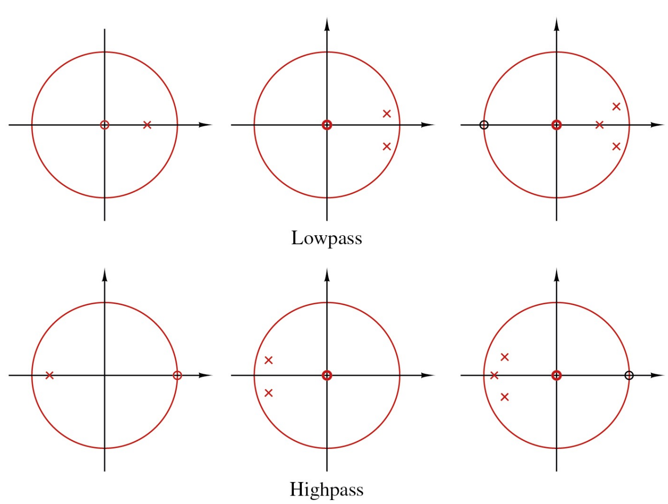
$$H(z) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$

Usually, b_0 is selected such that $|H(\omega_0)| = 1$. ω_0 in the passband of the filter.

$N \geq M$.

Lowpass, Highpass, and Bandpass Filters

- Design of lowpass digital filters: the poles should be placed near the unit circle at points corresponding to low frequencies (near $\omega = 0$) and zeros should be placed near or on the unit circle at points corresponding to high frequencies (near $\omega = \pi$).
- Design of highpass digital filters: The opposite.



A Simple Lowpass-to-Highpass Filter Transformation

Frequency translation of π rad:

$$H_{hp}(\omega) = H_{lp}(\omega - \pi)$$

Therefore,

$$h_{hp}(n) = e^{j\pi n} h_{lp}(n) = (-1)^n h_{lp}(n)$$

e.g., Lowpass filter by difference eqn.

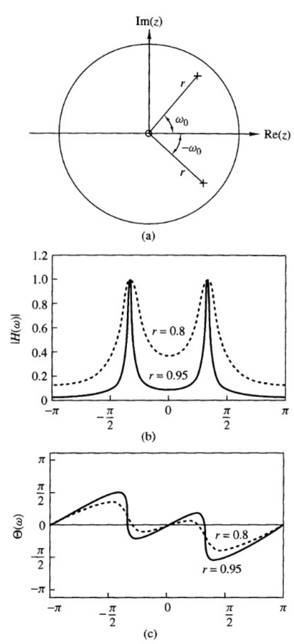
$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

A highpass filter can be derived: (How?)

$$y(n) = -\sum_{k=1}^{N} (-1)^k a_k y(n-k) + \sum_{k=0}^{M} (-1)^k b_k x(n-k)$$

Digital Resonator

- A digital resonator is a special two-pole bandpass filter with the pair of complex-conjugate poles located near the unit circle.
- The filter has a large magnitude response (i.e., it resonates) in the vicinity of the pole location.
- The angular position of the pole determines the resonant frequency of the filter.
- Digital resonators are useful in many applications, including bandpass filtering and speech generation.



Digital Resonator

A resonant peak at or near $\omega = \omega_0$,

$$p_{1,2} = re^{\pm j\omega_0}, \;\; 0 < r < 1$$

We can select up to two zeros –

One choice is to locate the zeros at the origin.

The other choice is to locate a zero at $z = 1$ and a zero at $z = -1$. This choice completely eliminates the response of the filter at frequencies $\omega = 0$ and $\omega = \pi$.

Digital Resonator

Digital resonator with zeros at the origin:

$$H(\omega) = \frac{b_0}{(1 - re^{j\omega_0}e^{-j\omega})(1 - re^{-j\omega_0}e^{-j\omega})}$$

We select b_0 so that $|H(\omega_0)| = 1$.

$$\begin{aligned} H(\omega_0) &= \frac{b_0}{(1 - re^{j\omega_0}e^{-j\omega_0})(1 - re^{-j\omega_0}e^{-j\omega_0})} \\ &= \frac{b_0}{(1-r)(1 - re^{-j2\omega_0})} \\ |H(\omega_0)| &= \frac{b_0}{(1-r)\sqrt{1 + r^2 - 2r\cos 2\omega_0}} = 1 \\ b_0 &= (1-r)\sqrt{1 + r^2 - 2r\cos 2\omega_0} \end{aligned}$$

Digital Resonator

Digital resonator with zeros at the origin:

$$H(\omega) = \frac{b_0}{(1 - re^{j\omega_0}e^{-j\omega})(1 - re^{-j\omega_0}e^{-j\omega})}$$

$$\begin{aligned} |H(\omega)| &= \frac{b_0}{U_1(\omega)U_2(\omega)} \\ \angle H(\omega) &= 2\omega - \Phi_1(\omega) - \Phi_2(\omega) \end{aligned}$$

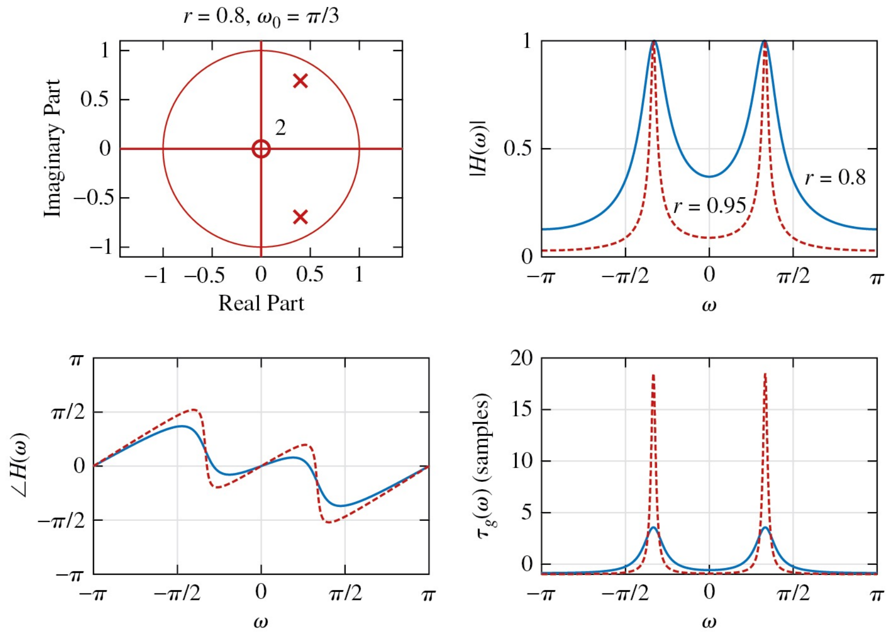
$$\begin{aligned} U_1(\omega) &= \sqrt{1 + r^2 - 2r\cos(\omega_0 - \omega)} \\ U_2(\omega) &= \sqrt{1 + r^2 - 2r\cos(\omega_0 + \omega)} \end{aligned}$$

Digital Resonator

$$\begin{aligned} U_1(\omega) &= \sqrt{1 + r^2 - 2r\cos(\omega_0 - \omega)} \\ U_2(\omega) &= \sqrt{1 + r^2 - 2r\cos(\omega_0 + \omega)} \end{aligned}$$

$$\min_{\omega} U_1(\omega)U_2(\omega) \Longrightarrow \omega_r = \cos^{-1}\left(\frac{1 + r^2}{2r}\cos\omega_0\right)$$

Digital Resonator



Digital Resonator

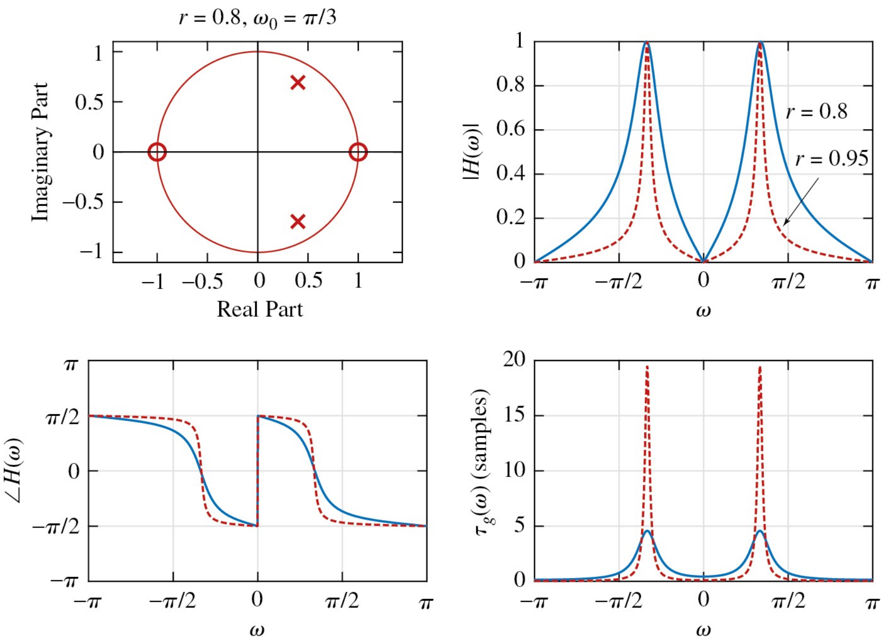
Digital resonator with zeros $z = 1$ and $z = -1$:

$$H(\omega) = b_0 \frac{(1 - e^{-j\omega})(1 + e^{-j\omega})}{(1 - re^{j\omega_0}e^{-j\omega})(1 - re^{-j\omega_0}e^{-j\omega})}$$

$$|H(\omega)| = b_0 \frac{\sqrt{2(1 - \cos 2\omega)}}{U_1(\omega)U_2(\omega)}$$

The actual resonant frequency is altered.

Digital Resonator



All-Pass Filters

$$|H(\omega)| = 1, \ \ 0 \leq \omega \leq \pi$$

e.g.,

1. a pure delay system $H(z) = z^{-k}$.
2. $$\begin{aligned} H(z) &= \frac{\sum_{k=0}^{N} a_k z^{-N+k}}{\sum_{k=0}^{N} a_k z^{-k}}, \quad a_0 = 1 \\ &= z^{-N} \frac{A(z^{-1})}{A(z)} \end{aligned}$$

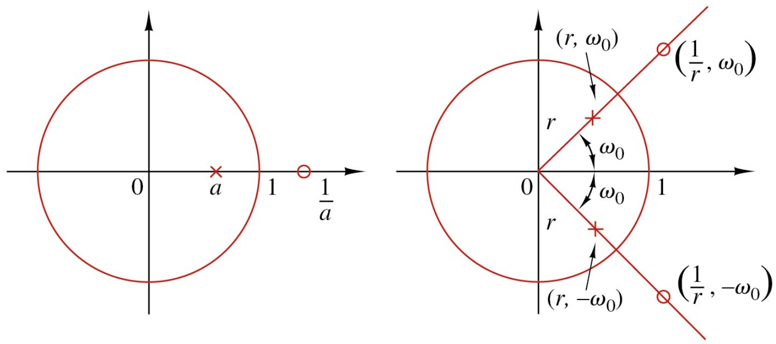
   where $A(z) = \sum_{k=0}^{N} a_k z^{-k}$.

   $$|H(\omega)|^2 = H(z)H(1/z)|_{z=e^{j\omega}} = 1$$

All-Pass Filters

If z_0 is a pole of $H(z)$, then $1/z_0$ is a zero of $H(z)$.

The poles and zeros are reciprocals of one another.



(a) (b)

All-Pass Filters

All-pass filter with real coefficients:

$$H_{ap}(z) = \prod_{k=1}^{N_R} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}} \prod_{k=1}^{N_C} \frac{(z^{-1} - \beta_k)(z^{-1} - \beta_k^*)}{(1 - \beta_k z^{-1})(1 - \beta_k^* z^{-1})}$$

where there are N_R real poles and zeros and N_C complex-conjugate pairs of poles and zeros.

For causal and stable systems, $-1 < \alpha_k < 1$ and $|\beta_k| < 1$.

Q: What is all-pass filter for?

A: All-pass filters find application as phase equalizers. When placed in cascade with a system that has an undesired phase response, a phase equalizer is designed to compensate for the poor phase characteristics of the system and therefore to produce an overall linear-phase response.