Short-Time Fourier Transform

Short-time Fourier transforms (STFTs) divide a longer time signal into shorter segments of equal length and compute the Fourier transform separately on each shorter segment.

**Figure:** STFT on test signal $x(t)$. Here, $g(t)$ is the window function.
Continuous-Time STFT

With a sliding window function \( w(t) = g(t) \), the STFT of signal \( x(t) \) is

\[
X(\tau, \omega) = \int_{-\infty}^{\infty} x(t)g(t - \tau)e^{-j\omega t} dt
\]

▶ \( g(\tau) \) is the window function, e.g., a Hann window or Gaussian window centered at zero.
▶ \( X(\tau, \omega) \) is the Fourier transform of \( x(t)g(t - \tau) \).
▶ \( X(\tau, \omega) \) is two dimensional, with \( \tau \) the time axis and \( \omega \) the frequency axis.
▶ \( \tau \) is called the “slow” time with respect to the high-resolution time \( t \).

Discrete-Time STFT

The discrete-time STFT of signal \( x[n] \) is

\[
X(m, \omega) = \sum_{n=-\infty}^{\infty} x[n]g[n - m]e^{-j\omega n}
\]

▶ \( m \) is discrete and \( \omega \) is continuous.
▶ In practice, we use discrete-STFT. Both time and frequency variables are discrete and quantized.
The spectrogram of signal $x(t)$ is the squared magnitude of the STFT

$$\text{Spectrogram}\{x(t)\}(\tau, \omega) = |X(\tau, \omega)|^2$$
The window function $g(t)$ is scaled so that

$$\int_{-\infty}^{\infty} g(\tau) d\tau = 1 \Rightarrow \int_{-\infty}^{\infty} g(t - \tau) d\tau = 1, \forall t$$

Therefore,

$$x(t) = x(t) \int_{-\infty}^{\infty} g(t - \tau) d\tau = \int_{-\infty}^{\infty} x(t) g(t - \tau) d\tau$$

Fourier transform of $x(t)$ is

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(t) g(t - \tau) d\tau \right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(t) g(t - \tau) e^{-j\omega t} d\tau \right] d\tau$$

$$= \int_{-\infty}^{\infty} X(\tau, \omega) d\tau$$

The FT is a phase-coherent sum of all of the STFTs of $x(t)$. 
Continuous-Time Inverse STFT

- The inverse Fourier transform is

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} X(\tau, \omega) d\tau \right] e^{j\omega t} d\omega \]

\[ = \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau, \omega) e^{j\omega t} d\omega \right] d\tau \]

- Because \( x(t) = \int_{-\infty}^{\infty} x(t) g(t - \tau) d\tau \), we have

\[ x(t) g(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau, \omega) e^{j\omega t} d\omega \]

Wavelet Transform

- The wavelet transform (WT) is another mapping from \( L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2) \), but one with superior time-frequency localization as compared with the STFT.

- An admissibility condition on the wavelet is needed to ensure the invertibility of the continuous wavelet transform.

- The discrete wavelet transform (DWT) is generated by sampling the wavelet parameters \((a, b)\) on a grid or lattice.

- A fine grid mesh permits easy reconstruction, but with oversampling. A coarse grid could result in loss of information.
Wavelets

![Wavelet Examples](image)

**Figure:** Examples of wavelet $\psi(t)$

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Continuous Wavelet Transform

- The wavelet with dilation and translation of a mother wavelet function $\psi(t)$

\[
\psi_{ab}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t - b}{a}\right)
\]

- The CWT maps a function $x(t)$ onto the time-scale space

\[
W\{x(t)\}(a, b) = \langle x(t), \psi_{ab}(t) \rangle = \int_{-\infty}^{\infty} x(t) \psi_{ab}(t) \, dt
\]