Chapter 9, Relations (2)

Young-Rae Cho
Associate Professor
Department of Computer Science
Baylor University

9.4. Closures of Relations

Closures
- For any property X, the "X closure" of a relation R is defined as the smallest superset of R that has the given property.
- The reflexive closure of a relation R on A is obtained by adding (a,a) to R for each a \in A.
- The symmetric closure of R is obtained by adding (b,a) to R for each (a,b) in R. (R \cup R^{-1})

Examples?
- What is the reflexive closure of the relation R = \{(a,b) | a < b\} on Z ?
- What is the symmetric closure of the relation R = \{(a,b) | a > b\} on Z+ ?
Paths in Directed Graphs

- **Path**
  - A path of length $n$ from $a$ to $b$ in the directed graph $G$ is a sequence $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ of $n$ ordered pairs in $G$ where $a = x_0$, $b = x_n$.

- **Cycle**
  - A path of length $n \geq 1$ from $a$ to itself is called a cycle or a circuit.

- **Theorem**
  - Let $R$ be a relation on a set. There exists a path of length $n$ ($n \geq 1$) from $a$ to $b$ in $R$ if and only if $(a, b) \in R^n$.
  - Examples?

Transitive Closures (1)

- **Transitive Closure**
  - Let $R$ be a relation on a set $A$. The *connectivity relation* $R^*$ consists of the pairs $(a, b)$ such that there is a path of length at least one from $a$ to $b$ in $R$.
  
  $$R^* = \bigcup_{n=1}^{\infty} R^n$$

  - The transitive closure of $R$ equals the connectivity relation $R^*$.
  - Example?
    - Social network?
Transitive Closures (2)

- **Lemma**
  - Let $A$ be a set with $n$ elements, and let $R$ be a relation on $A$.
  - If there is a path of length at least one in $R$ from $a$ to $b$, then there is such a path with length not exceeding $n$.

- **Theorem**
  - The transitive closure $R^*$ is
    $$R^* = R \cup R^2 \cup R^3 \cup ... \cup R^n$$
  - Let $M_R$ be the zero-one matrix of the relation $R$ on a set with $n$ elements. Then, the transitive closure $R^*$ is
    $$M_{R^*} = M_R \vee M_{R^2} \vee M_{R^3} \vee ... \vee M_{R^n}$$
  - Examples?

3.5. Equivalence Relations

- **Equivalence Relation**
  - An equivalence relation on a set $A$ is simply any binary relation on $A$ that is reflexive, symmetric, and transitive.
  - Examples?
    - $R = \{(a,b) \mid a$ and $b$ have the same absolute value$\}$ where $a, b \in \mathbb{Z}$.
    - Congruence Modulo $m$: $R = \{(a,b) \mid a \equiv b \, (\text{mod} \, m)\}$ where $m \in \mathbb{Z}^+$ and $m > 1$. Prove?
Equivalence Classes

- **Equivalence Class**
  - Let $R$ be any equivalence relation on a set $A$. The set of all elements that are related to an element $a$ of $A$ is called the equivalence class of $a$.
  - Notation: $[a]_R = \{ b \mid aRb \}$
  - Examples
    - $R = \{(a,b)\mid a \text{ and } b \text{ have the same absolute value}\}, [a]_R = ?$
    - $R = \{(a,b)\mid a \equiv b \pmod{m}\}, [a]_R = ?$

- **Theorem**
  - Let $R$ be any equivalence relation on a set $A$. Then, these statements are equivalent:
    1. $aRb$
    2. $[a] = [b]$
    3. $[a] \cap [b] \neq \emptyset$

Equivalent Classes and Partitions

- **Partition**
  - Let $S_1, S_2, \ldots, S_n$ be a collection of subsets of $A$. Then, the collection forms a partition of $A$ if the subsets are nonempty, disjoint and exhaust $A$.
    - $S_i \neq \emptyset$
    - $S_i \cap S_j = \emptyset$ if $i \neq j$
    - $S_1 \cup S_2 \cup \ldots \cup S_n = A$

- **Theorem**
  - Let $R$ be an equivalence relation on a set $S$. Then, the equivalence classes of $R$ form a partition of $S$.
  - Conversely, given a partition $A_i$ of the set $S$, there is an equivalence relation $R$ that has the sets $A_i$ as its equivalence classes.
3.6. Partial Ordering

- **Partial Order**
  - Let $R$ be a relation on a set $S$. Then $R$ is a *partial order* iff $R$ is reflexive, antisymmetric, and transitive.
  - Notation: $\leq$
  - The set $S$ with $R$ is called a *partially ordered set* or a *poset*.
  - Notation: $(S, R)$ or $(S, \leq)$
  - Examples?

Total Ordering

- **Comparability**
  - Let $(S, \leq)$ be a poset. $a, b \in (S, \leq)$ are called *comparable* if either $a \leq b$ or $b \leq a$. If neither $a \leq b$ or $b \leq a$, then $a$ and $b$ are called *incomparable*.
  - Examples?

- **Total Order**
  - If $(S, \leq)$ is a poset and every two elements of $S$ are comparable, $S$ is called a *totally ordered set* or linearly ordered set, and $\leq$ is called a *total order* or a linear order. A totally ordered set is also called a *chain*.
  - Examples?
Well-Ordering Principle

**Well-Order**
- Let \( \preceq \) be a total order on \( S \). An element \( s \in S \) is the *least element* of \( S \) if \( s \preceq a \) for all \( a \in S \).
- \( (S, \preceq) \) is a *well-ordered set* if \( S \) is a totally ordered set and every nonempty subset of \( S \) has a least element.
- Examples?

**Theorem**
- Principle of well-ordered induction
- Suppose \( S \) is a well-ordered set. Then \( P(x) \) is true for all \( x \in S \) if for every \( y \in S \), if \( P(x) \) is true for all \( x \in S \) with \( x \preceq y \), then \( P(y) \) is true.

Lexicographic Order (1)

**Lexicographic Ordering**
- Suppose \( (A_1, \preceq_1) \) and \( (A_2, \preceq_2) \) are posets.
- The lexicographic ordering \( \preceq \) on \( A_1 \times A_2 \) is \( (x_1, y_1) \prec (x_2, y_2) \) iff
  - \( x_1 \preceq_1 x_2 \) or
  - \( x_1 = x_2 \) and \( y_1 < y_2 \)
- Examples?

**Extension of Lexicographic Ordering**
- The lexicographic ordering \( \prec \) on multiple Cartesian products of posets, \( A_1 \times A_2 \times A_3 \times ... \times A_n \),
- Examples?
Lexicographic Order (2)

- **Lexicographic Ordering of Strings**
  - The lexicographic ordering of strings of symbols is applied where there is an 'alphabetical' order as a partial order.
  - If two strings have the same length, then use the induced partial order from the alphabetical order.
  - Any shorter string is related to any longer string (when they agree in all positions of shorter string length, the shorter string comes before in the ordering)
  - Examples?

Hasse Diagram

- **Hasse Diagram**
  - A graph to represent posets.

- **How to Construct a Hasse Diagram**
  1. Construct a directed graph representation of the poset \((A, R)\) so that all directed edges point up (except loops).
  2. Eliminate all loops that are redundant because of reflexivity.
  3. Eliminate the directed edges that are redundant because of transitivity.
  4. Eliminate the arrows on all directed edges because everything points up.

- Examples?
Maximal and Minimal Elements

- **Maximal and Minimal Elements**
  - Let \((S, \preceq)\) be a poset. Then, \(a \in S\) is maximal in the poset if there is no \(b \in S\) such that \(a < b\). Similarly, \(a \in S\) is minimal in the poset if there is no \(b \in S\) such that \(b < a\).
  - Examples?

- **Greatest and Least Elements**
  - Let \((S, \preceq)\) be a poset. Then, \(a \in S\) is the greatest element if \(b \preceq a\) for all \(b \in S\). Similarly, \(a \in S\) is the least element if \(a \preceq b\) for all \(b \in S\).
  - Examples?

Upper and Lower Bounds

- **Upper and Lower Bounds**
  - Let \(A \subseteq S\) in the poset \((S, \preceq)\). If \(u\) is an element in \(S\) such that \(a \preceq u\) for all \(a \in A\), then \(u\) is an upper bound of \(A\). Similarly, if \(l\) is an element of \(S\) such that \(l \preceq a\) for all \(a \in A\), then \(l\) is a lower bound of \(A\).
  - Examples?

- **Least Upper Bound and Greatest Lower Bound**
  - Let \(A \subseteq S\) in the poset \((S, \preceq)\). Then, \(x\) is the least upper bound of \(A\) if \(a \preceq x\) for all \(a \in A\), and \(x \preceq z\) for all upper bounds \(z\) of \(A\). Similarly, \(y\) is the greatest lower bound of \(A\) if \(y \preceq a\) for all \(a \in A\), and \(z \preceq y\) for all lower bounds \(z\) of \(A\).
  - Examples?
Questions?

- Lecture Slides are found on the Course Website,
  web.ecs.baylor.edu/faculty/cho/2350