2.4. Sequences and Summations

- **Sequence**
  - An ordered $n$-tuple where each element in the sequence has an associated index.
  - A function $f: I \rightarrow S$ where $I$ is a set of indices ($I \subseteq \mathbb{N}$ or $I \subseteq \mathbb{Z}^+$) and $S$ is a set of objects.
  - $f(n) = a_n$, where $n$ is an index and $a_n$ is called a term of the sequence.
  - Example
    - A sequence $\{a_n\} = \{0, 1, 4, 9, 16, \ldots \}$, is $\forall n \in \mathbb{N}$, $a_n = n^2$

- **Summation**
  - The sum of the terms in a sequence.
Examples of Sequences

- **Sequence with Repetition**
  - Unlike sets, a sequence may contain repeated instances of an element.
  - \( \{b_n\} = 1, -1, 1, -1, ... \)

- **Geometric Progression**
  - A sequence in the form, "a, ar, ar^2, ..., ar^n, ..."
  - The initial term a, the common ratio \( r \in \mathbb{R} \).
  - Examples?

- **Arithmetic Progression**
  - A sequence in the form, "a, a+d, a+2d, ..., a+nd, ..."
  - The initial term a, the common difference \( d \in \mathbb{R} \).
  - Examples?

Recurrence Relations

- **Recurrence Relation for a Sequence \( \{a_n\} \)**
  - An equation that expresses \( a_n \) in terms of one or more previous terms.
  - A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.
  - Initial conditions: the terms that precede the first term where the recurrence relation takes effect.

- **Examples of Recurrence Relations**
  - Fibonacci sequence?
  - More examples?
More Examples of Sequences

- **Perfect Squares, Perfect Cubes, Factorial**

<table>
<thead>
<tr>
<th>nth Term</th>
<th>First 10 Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n^2$</td>
<td>1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...</td>
</tr>
<tr>
<td>$n^3$</td>
<td>1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...</td>
</tr>
<tr>
<td>$n^4$</td>
<td>1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...</td>
</tr>
<tr>
<td>$2^n$</td>
<td>2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...</td>
</tr>
<tr>
<td>$3^n$</td>
<td>3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...</td>
</tr>
<tr>
<td>$n!$</td>
<td>1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...</td>
</tr>
</tbody>
</table>

**Summations**

- **Summation Notation**
  - Given a sequence $\{a_n\}$, an integer lower bound (or limit) $j \geq 0$, and an integer upper bound $k \geq j$, the summation of $\{a_n\}$ from $a_j$ to $a_k$ is

\[ \sum_{i=j}^{k} a_i = a_j + a_{j+1} + ... + a_k \]

  - where $i$ is called the index of summation.
  - Examples?

- **Nested Summation**
  - Examples?
Summation Manipulation

**Rules**

- **Distributive law**
  \[ \sum_i c f(x) = c \sum_i f(x) \]

- **Commutativity**
  \[ \sum_i f(x) + g(x) = \left( \sum_i f(x) \right) + \left( \sum_i g(x) \right) \]

- **Index shifting**
  \[ \sum_{i=j}^n f(i) = \sum_{k=j-n}^{n-j} f(k) \]

- **Sequence splitting**
  \[ \sum_{i=j}^n f(i) = \sum_{i=j}^m f(i) + \sum_{i=m+1}^n f(i) \quad \text{if } j \leq m < k \]

- **Order reversal**
  \[ \sum_{i=0}^n f(i) = \sum_{i=0}^n f(n-i) \]

- **Grouping**
  \[ \sum_{i=1}^{2k} f(i) = \sum_{i=1}^k \left( f(2i-1) + f(2i) \right) \]

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Summation Formulae

**Gauss’ Trick ?**

\[ \sum_{i=1}^n i = \frac{n(n+1)}{2} \]

**Summation Formulae**

<table>
<thead>
<tr>
<th>Sum</th>
<th>Closed Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_{i=0}^{n} a^i ) ( r \neq 0 )</td>
<td>( \frac{a^{n+1} - 1}{r - 1} ) ( r \neq 1 )</td>
</tr>
<tr>
<td>( \sum_{i=1}^{n} i )</td>
<td>( \frac{n(n+1)}{2} )</td>
</tr>
<tr>
<td>( \sum_{i=1}^{n} i^2 )</td>
<td>( \frac{n(n+1)(2n+1)}{6} )</td>
</tr>
<tr>
<td>( \sum_{i=1}^{n} i^3 )</td>
<td>( \frac{n^2(n+1)^2}{4} )</td>
</tr>
<tr>
<td>( \sum_{i=0}^{n} x^i ), (</td>
<td>x</td>
</tr>
<tr>
<td>( \sum_{i=1}^{n} x^{i-1} ), (</td>
<td>x</td>
</tr>
</tbody>
</table>
2.5. Cardinality of Sets

- **Cardinality of Sets**
  - The sets A and B have the same cardinality (|A| = |B|) iff there is a one-to-one correspondence from A to B.

- **Countable / Uncountable Sets**
  - A set that is either finite or has the same cardinality as the set of positive integers is called countable.
  - A set that is not countable is called uncountable.
  - Examples?