Chapter 19

State-Variable Analysis

Specific goals and objectives of this chapter include:

- Introducing the concept of state variables and normal-form equations
- Learning how to write a complete set of normal-form equations for a given circuit
- Matrix-based solution of the circuit equations
- Development of a general technique applicable to higher-order problems

19.1 | Introduction

Up to this point, we have seen several different methods by which circuits might be analyzed. The resistive circuit came first, and for it we wrote a set of algebraic equations, often cast in the form of nodal or mesh equations. However, in Appendix 1 we learn that we can choose other, more convenient voltage or current variables after drawing an appropriate tree for the network. The tree sprouts up again in this chapter, in the selection of circuit variables.

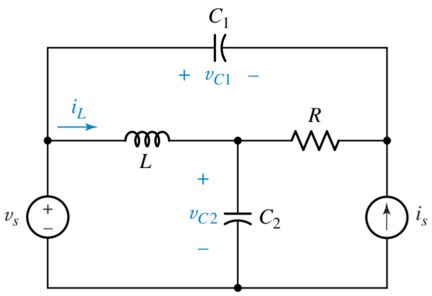
We next added inductors and capacitors to our networks, and this produced equations containing derivatives and integrals with respect to time. Except for simple first- and second-order systems that either were source-free or contained only dc sources, we did not attempt solving these equations. The results we obtained were found by time-domain methods. Subsequently, we explored the use of phasors to determine the sinusoidal steady-state response of such circuits, and a little later on we were introduced to the concept of complex frequency and the Laplace transform method.

In this last chapter, we return to the time domain and introduce the use of state variables. Once again we will not obtain many explicit solutions for circuits of even moderate complexity, but we will write sets of equations compatible with numerical analysis techniques.

19.2 | State Variables and Normal-Form Equations

State-variable analysis, or state-space analysis, as it is sometimes called, is a procedure that can be applied both to linear and, with some modifications, to nonlinear circuits, as well as to circuits containing time-varying parameters, such as the capacitance $C = 50 \cos 20t$ pF. Our attention, however, will be restricted to time-invariant linear circuits.

Figure 19.1



A four-node *RLC* circuit.

We introduce some of the ideas underlying state variables by looking at a general *RLC* circuit drawn in Fig. 19.1. When we write equations for this circuit, we could use nodal analysis, the two dependent variables being the node voltages at the central and right nodes. We could also opt for mesh analysis and use two currents as the variables, or, alternatively, we could draw a tree first and then select a set of tree-branch voltages or link currents as the dependent variables. It is possible that each approach could lead to a different number of variables, although two seems to be the most likely number for this circuit.

The set of variables we will select in state-variable analysis is a hybrid set that may include both currents and voltages. They are the *inductor currents* and the *capacitor voltages*. Each of these quantities may be used directly to express the energy stored in the inductor or capacitor at any instant of time. That is, they collectively describe the *energy state* of the system, and for that reason, they are called the *state variables*.

Let us try to write a set of equations for the circuit of Fig. 19.1 in terms of the state variables i_L, v_{C1}, and v_{C2}, as defined on the circuit diagram. The method we use will be outlined more formally in the following section, but for the present let us try to use KVL once for each inductor, and KCL once for each capacitor.

Beginning with the inductor, we set the sum of voltages around the lower left mesh equal to zero:

Here we are using a "prime" symbol to denote a derivative with respect to time.

$$L i'_L + v_{C2} - v_s = 0 \qquad \textbf{[1]}$$

We presume that the source voltage v_s and the source current i_s are known, and we therefore have one equation in terms of our chosen state variables.

Next, we consider the capacitor C_1. Since the left terminal of C_1 is also one terminal of a voltage source, it will become part of a supernode. Therefore, we select the right terminal of C_1 as the node to which we apply KCL. The current through the capacitor branch is $C_1 v'_{C1}$, the upward source current is i_s, and the current in R is obtained by noting that the voltage across R, positive reference on the left, is $(v_{C2} - v_s + v_{C1})$ and, therefore, the current to the right in R is $(v_{C2} - v_s + v_{C1})/R$. Thus,

$$C_1 v'_{C1} + \frac{1}{R}(v_{C2} - v_s + v_{C1}) + i_s = 0 \qquad \textbf{[2]}$$

Again we have been able to write an equation without introducing any new variables, although we might not have been able to express the current through R directly in terms of the state variables if the circuit had been any more complicated.

Finally, we apply KCL to the upper terminal of C_2:

$$C_2 v'_{C2} - i_L + \frac{1}{R}(v_{C2} - v_s + v_{C1}) = 0 \qquad \textbf{[3]}$$

Equations [1] to [3] are written solely in terms of the three state variables, the known element values, and the two known forcing functions. They are not, however, written in the standardized form which state-variable analysis demands. The state equations are said to be in *normal form* when the derivative of each state variable is expressed as a linear combination of all the state variables and forcing functions. The ordering of the equations defining the derivatives and the order in which the state variables appear in every equation must be the same. Let us arbitrarily select the order i_L, v_{C1}, v_{C2}, and rewrite Eq. [1] as

$$i'_L = -\frac{1}{L}v_{C2} + \frac{1}{L}v_s \qquad \textbf{[4]}$$

Then Eq. [2] is rewritten as

$$v'_{C1} = -\frac{1}{RC_1}v_{C1} - \frac{1}{RC_1}v_{C2} + \frac{1}{RC_1}v_s - \frac{1}{C_1}i_s \qquad \textbf{[5]}$$

while Eq. [3] becomes

$$v'_{C2} = \frac{1}{C_2}i_L - \frac{1}{RC_2}v_{C1} - \frac{1}{RC_2}v_{C2} + \frac{1}{RC_2}v_s \qquad \textbf{[6]}$$

Note that these equations define, in order, i'_L, v'_{C1}, and v'_{C2}, with a corresponding order of variables on the right sides, i_L, v_{C1}, and v_{C2}. The forcing functions come last and may be written in any convenient order.

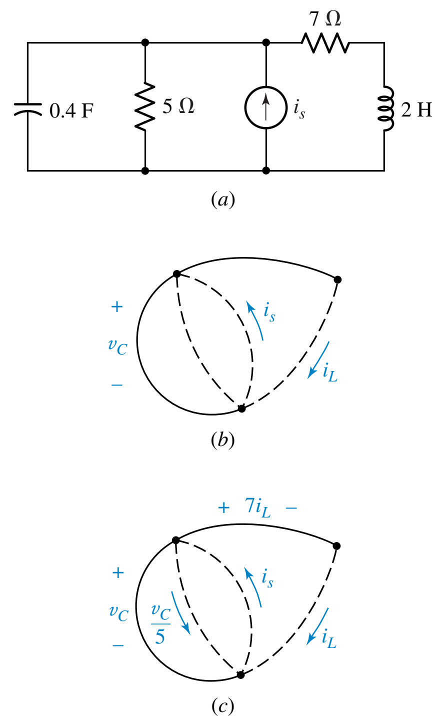
As another example of the determination of a set of normal-form equations, we look at the circuit shown in Fig. 19.2*a*. Since the circuit has one capacitor and one inductor, we expect two state variables, the capacitor voltage and the inductor current. To facilitate writing the normal-form equations, let us construct a tree for this circuit which follows all the rules for tree construction discussed in Appendix 1 and in addition requires all capacitors to be located in the tree and all inductors in the cotree. This is usually possible and it leads to a *normal tree*. In those few exceptional cases where it is not possible to draw a normal tree, we use a slightly different method which will be considered at the end of Sec. 19.3. Here, we are able to place C in the tree and L and i_s in the cotree, as shown in Fig. 19.2*b*. This is the only normal tree possible for this circuit. The source quantities and the state variables are indicated on the tree and cotree.

Next, we determine the current in every link and the voltage across each tree branch in terms of the state variables. For a simple circuit such as this, it is possible to do so by beginning with any resistor for which either the current or the voltage is obvious. The results are shown on the tree in Fig. 19.2*c*.

We may now write the normal-form equations by invoking KCL at the upper terminal of the capacitor:

$$0.4v'_C + 0.2v_C - i_s + i_L = 0$$

Figure 19.2



(*a*) An *RLC* circuit requiring two state variables. (*b*) A normal tree showing the state variables v_C and i_L. (*c*) The current in every link and the voltage across every tree branch is expressed in terms of the state variables.

or, in normal form,

$$v'_C = -0.5v_C - 2.5i_L + 2.5i_s \quad [7]$$

Around the outer loop, we have

$$2i'_L - v_C + 7i_L = 0$$

or

$$i'_L = 0.5v_C - 3.5i_L \quad [8]$$

Equations [7] and [8] are the desired normal-form equations. Their solution will yield all the information necessary for a complete analysis of the given circuit. Of course, explicit expressions for the state variables can be obtained only if a specific function is given for $i_s(t)$. For example, it will be shown later that if

$$i_s(t) = 12 + 3.2e^{-2t}u(t) \qquad \text{A} \quad [9]$$

then

$$v_C(t) = 35 + (10e^{-t} - 12e^{-2t} + 2e^{-3t})u(t) \qquad \text{V} \quad [10]$$

and

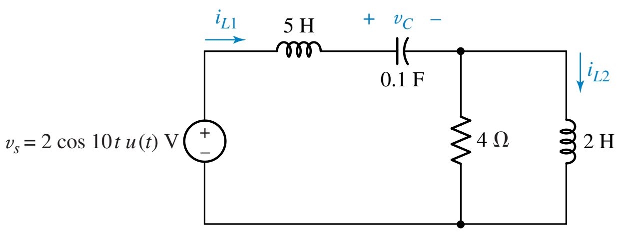
$$i_L(t) = 5 + (2e^{-t} - 4e^{-2t} + 2e^{-3t})u(t) \qquad \text{A} \quad [11]$$

The solutions, however, are far from obvious, and we will develop the technique for obtaining them from the normal-form equations in Sec. 19.7.

Practice

19.1. Write a set of normal-form equations for the circuit shown in Fig. 19.3. Order the state variables as i_{L1}, i_{L2}, and v_C.

Figure 19.3



Ans: $i'_{L1} = -0.8i_{L1} + 0.8i_{L2} - 0.2v_C + 0.2v_s$; $i'_{L2} = 2i_{L1} - 2i_{L2}$; $v'_C = 10i_{L1}$.

19.3 Writing a Set of Normal-Form Equations

In the two examples considered in the previous section, the methods whereby we obtained a set of normal-form equations may have seemed to be more of an art than a science. In order to bring a little order into our chaos, let's try to follow the procedure used when we were studying nodal analysis, mesh analysis, and the use of trees in general loop and general nodal analysis. We seek a set of guidelines that will systematize the procedure. Then we will apply these rules to three new examples, each a little more involved than the preceding one.

Here are the six steps that we have been following:

1. *Establish a normal tree.* Place capacitors and voltage sources in the tree, and inductors and current sources in the cotree; place control voltages in the tree and control currents in the cotree if possible. More than one normal tree may be possible. Certain types of networks do not permit any normal tree to be drawn; these exceptions are considered at the end of this section.
2. *Assign voltage and current variables.* Assign a voltage (with polarity reference) to every capacitor and a current (with arrow) to every inductor; these voltages and currents are the state variables. Indicate the voltage across every tree branch and the current through every link in terms of the source voltages, the source currents, and the state variables, if possible; otherwise, assign a new voltage or current variable to that resistive tree branch or link.
3. *Write the C equations.* Use KCL to write one equation for each capacitor. Set Cv_C' equal to the sum of link currents obtained by considering the node (or supernode) at either end of the capacitor. The supernode is identified as the set of all tree branches connected to that terminal of the capacitor. Do not introduce any new variables.
4. *Write the L equations.* Use KVL to write one equation for each inductor. Set Li_L' equal to the sum of tree-branch voltages obtained by considering the single closed path consisting of the link in which L lies and a convenient set of tree branches. Do not introduce any new variables.
5. *Write the R equations (if necessary).* If any new voltage variables were assigned to resistors in step 2, use KCL to set v_R/R equal to a sum of link currents. If any new current variables were assigned to resistors in step 2, use KVL to set $i_R R$ equal to a sum of tree-branch voltages. Solve these resistor equations simultaneously to obtain explicit expressions for each v_R and i_R in terms of the state variables and source quantities.
6. *Write the normal-form equations.* Substitute the expressions for each v_R and i_R into the equations obtained in steps 3 and 4, thus eliminating all resistor variables. Put the resultant equations in normal form.

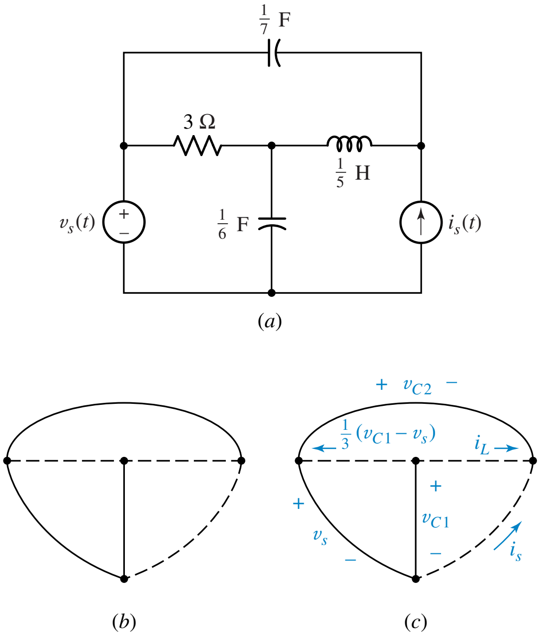
EXAMPLE 19.1

Obtain the normal-form equations for the circuit of Fig. 19.4*a*, a four-node circuit containing two capacitors, one inductor, and two independent sources.

Following step 1, we draw a normal tree. Note that only one such tree is possible here, as shown in Fig. 19.4*b*, since the two capacitors and the voltage source must be in the tree and the inductor and the current source must be in the cotree.

We next define the voltage across the $\frac{1}{6}$-F capacitor as v_{C1}, the voltage across the $\frac{1}{7}$-F capacitor as v_{C2}, and the current through the inductor as i_L. The source voltage

Figure 19.4



(*a*) A given circuit for which normal-form equations are to be written. (*b*) Only one normal tree is possible. (*c*) Tree-branch voltages and link currents are assigned.

is indicated across its tree branch, the source current is marked on its link, and only the resistor link remains without an assigned variable. The current directed to the left through that link is the voltage $v_{C1} - v_s$ divided by 3 Ω, and we thus find it unnecessary to introduce any additional variables. These tree-branch voltages and link currents are shown in Fig. 19.4*c*.

Two equations must be written for step 3. For the $\frac{1}{6}$-F capacitor, we apply KCL to the central node:

$$\frac{v'_{C1}}{6} + i_L + \frac{1}{3}(v_{C1} - v_s) = 0$$

while the right-hand node is most convenient for the $\frac{1}{7}$-F capacitor:

$$\frac{v'_{C2}}{7} + i_L + i_s = 0$$

Moving on to step 4, KVL is applied to the inductor link and the entire tree in this case:

$$\frac{i'_L}{5} - v_{C2} + v_s - v_{C1} = 0$$

Since there were no new variables assigned to the resistor, we skip step 5 and simply rearrange the three preceding equations to obtain the desired normal-form equations,

$$v'_{C1} = -2v_{C1} - 6i_L + 2v_s \qquad \textbf{[12]}$$

$$v'_{C2} = -7i_L - 7i_s \qquad \textbf{[13]}$$

$$i'_L = 5v_{C1} + 5v_{C2} - 5v_s \qquad \textbf{[14]}$$

The state variables have arbitrarily been ordered as v_{C1}, v_{C2}, and i_L. If the order i_L, v_{C1}, v_{C2} had been selected instead, the three normal-form equations would be

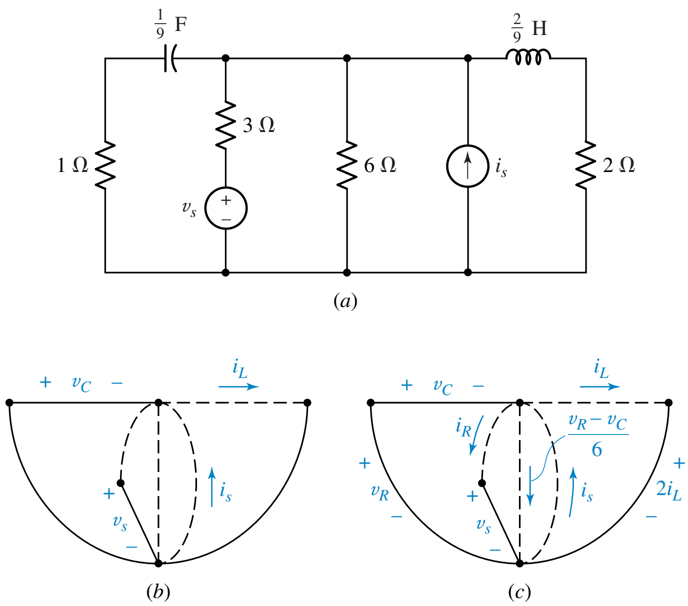
$$
\begin{aligned}
i_L' &= 5v_{C1} + 5v_{C2} - 5v_s \\
v_{C1}' &= -6i_L - 2v_{C1} + 2v_s \\
v_{C2}' &= -7i_L - 7i_s
\end{aligned}
$$

Note that the ordering of the terms on the right-hand sides of the equations has been changed to agree with the ordering of the equations.

EXAMPLE 19.2

Write a set of normal-form equations for the circuit of Fig. 19.5*a*.

Figure 19.5



(*a*) The given circuit. (*b*) One of many possible normal trees. (*c*) The assigned voltages and currents.

The circuit contains several resistors, and this time it will be necessary to introduce resistor variables. Many different normal trees can be constructed for this network, and it is often worthwhile to sketch several possible trees to see whether resistor voltage and current variables can be avoided by a judicious selection. The tree we shall use is shown in Fig. 19.5*b*. The state variables v_C and i_L and the forcing functions v_s and i_s are indicated on the graph.

Although we might study this circuit and the tree for a few minutes and arrive at a method to avoid the introduction of any new variables, let us plead temporary stupidity and assign the tree-branch voltage v_R and link current i_R for the 1-Ω and 3-Ω branches, respectively. The voltage across the 2-Ω resistor is easily expressed as $2i_L$, while the downward current through the 6-Ω resistor becomes $(v_R - v_C)/6$. All the link currents and tree-branch voltages are marked on Fig. 19.5*c*.

The capacitor equation may now be written

$$\frac{v_C'}{9} = i_R + i_L - i_s + \frac{v_R - v_C}{6} \qquad [15]$$

and that for the inductor is

$$\tfrac{2}{9}i'_L = -v_C + v_R - 2i_L \qquad \textbf{[16]}$$

We begin step 5 with the 1-Ω resistor. Since it is in the tree, we must equate its current to a sum of link currents. Both terminals of the capacitor collapse into a supernode, and we have

$$\frac{v_R}{1} = i_s - i_R - i_L - \frac{v_R - v_C}{6}$$

The 3-Ω resistor is next, and its voltage may be written as

$$3i_R = -v_C + v_R - v_s$$

These last two equations must now be solved simultaneously for v_R and i_R in terms of i_L, v_C, and the two forcing functions. Doing so, we find that

$$v_R = \frac{v_C}{3} - \frac{2}{3}i_L + \frac{2}{3}i_s + \frac{2}{9}v_s$$

$$i_R = -\frac{2}{9}v_C - \frac{2}{9}i_L + \frac{2}{9}i_s - \frac{7}{27}v_s$$

Finally, these results are substituted in Eqs. [15] and [16], and the normal-form equations for Fig. 19.5 are obtained:

$$v'_C = -3v_C + 6i_L - 2v_s - 6i_s \qquad \textbf{[17]}$$

$$i'_L = -3v_C - 12i_L + v_s + 3i_s \qquad \textbf{[18]}$$

Up to now we have discussed only those circuits in which the voltage and current sources were independent sources. At this time we will look at a circuit containing a dependent source.

EXAMPLE 19.3

Find the set of normal-form equations for the circuit of Fig. 19.6.

The only possible normal tree is that shown in Fig. 19.6*b*, and it might be noted that it was not possible to place the controlling current i_x in a link. The tree-branch voltages and link currents are shown on the linear graph, and they are unchanged from the earlier example except for the additional source voltage, $18i_x$.

For the $\frac{1}{6}$-F capacitor, we again find that

$$\frac{v'_{C1}}{6} + i_L + \frac{1}{3}(v_{C1} - v_s) = 0 \qquad \textbf{[19]}$$

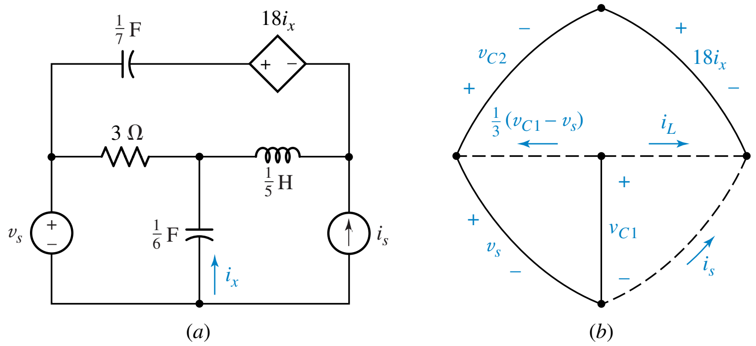
Letting the dependent voltage source shrink into a supernode, we also find the relationship unchanged for the $\frac{1}{7}$-F capacitor,

$$\frac{v'_{C2}}{7} + i_L + i_s = 0 \qquad \textbf{[20]}$$

Our previous result for the inductor changes, however, because there is an added branch in the tree:

$$\frac{i'_L}{5} - 18i_x - v_{C2} + v_s - v_{C1} = 0$$

Figure 19.6



(*a*) A circuit containing a dependent voltage source. (*b*) The normal tree for this circuit with tree-branch voltages and link currents assigned.

Finally, we must write a control equation that expresses i_x in terms of our tree-branch voltages and link currents. It is

$$i_x = i_L + \frac{v_{C1} - v_s}{3}$$

and thus the inductor equation becomes

$$\frac{i'_L}{5} - 18i_L - 6v_{C1} + 6v_s - v_{C2} + v_s - v_{C1} = 0 \qquad \textbf{[21]}$$

When Eqs. [19] to [21] are written in normal form, we have

$$v'_{C1} = -2v_{C1} - 6i_L + 2v_s \qquad \textbf{[22]}$$

$$v'_{C2} = -7i_L - 7i_s \qquad \textbf{[23]}$$

$$i'_L = 35v_{C1} + 5v_{C2} + 90i_L - 35v_s \qquad \textbf{[24]}$$

Our equation-writing process must be modified slightly if we cannot construct a normal tree for the circuit, and there are two types of networks for which this occurs. One contains a loop in which every element is a capacitor or voltage source, thus making it impossible to place them all in the tree. The other occurs if there is a node or supernode that is connected to the remainder of the circuit only by inductors and current sources.

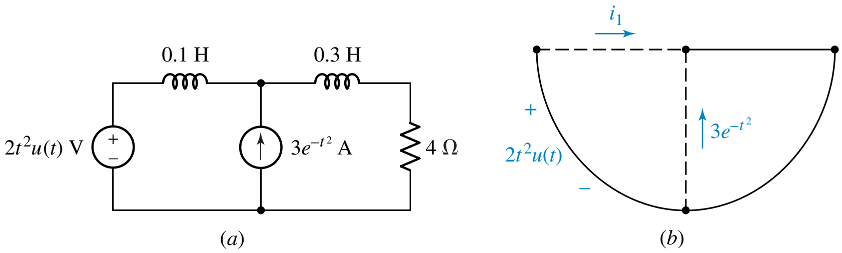
Remember, a tree, by definition, contains no closed paths (loops), and every node must have at least one tree branch connected to it.

When either of these events occurs, we meet the challenge by leaving a capacitor out of the tree in one case and omitting an inductor from the cotree in the other. We then are faced with a *capacitor* in a link for which we must specify a *current*, or an *inductor*, in the tree which requires a *voltage*. The capacitor current may be expressed as the capacitance times the time derivative of the voltage across it, as defined by a sequence of tree-branch voltages; and the inductor voltage is given by the inductance times the derivative of the current entering or leaving the node or supernode at either terminal of the inductor.

EXAMPLE 19.4

Write a set of normal-form equations for the network shown in Fig. 19.7a.

Figure 19.7



(a) A circuit for which a normal tree cannot be drawn. (b) A tree is constructed in which one inductor must be a tree branch.

The only elements connected to the upper central node are the two inductors and the current source, and we therefore must place one inductor in the tree, as illustrated by Fig. 19.7b. The two forcing functions and the single state variable, the current i_1, are shown on the graph. Note that the current in the upper right-hand inductor is known in terms of i_1 and $3e^{-t^2}$ A. Thus we could not specify the energy states of the two inductors independently of each other, and this system requires only the single state variable i_1. Of course, if we had placed the 0.1-H inductor in the tree (instead of the 0.3-H inductor), the current in the right-hand inductor would have ended up as the single state variable.

We must still assign voltages to the remaining two tree branches in Fig. 19.7b. Since the current directed to the right in the 0.3-H inductor is $i_1 + 3e^{-t^2}$, the voltages appearing across the inductor and the 4-Ω resistor are $0.3\,d(i_1 + 3e^{-t^2})/dt = 0.3i_1' - 1.8te^{-t^2}$ and $4i_1 + 12e^{-t^2}$, respectively.

The single normal-form equation is obtained in step 4 of our procedure:

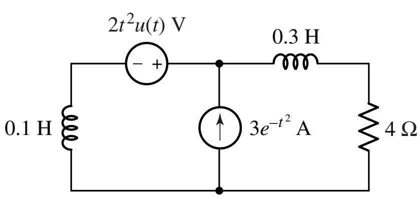
$$0.1i_1' + 0.3i_1' - 1.8te^{-t^2} + 4i_1 + 12e^{-t^2} - 2t^2u(t) = 0$$

or

$$i_1' = -10i_1 + e^{-t^2}(4.5t - 30) + 5t^2u(t) \qquad \textbf{[25]}$$

Note that one of the terms on the right-hand side of the equation is proportional to the derivative of one of the source functions.

Figure 19.8



The circuit of Fig. 19.7a is redrawn in such a way as to cause the supernode containing the voltage source to be connected to the remainder of the network only through the two inductors and the current source.

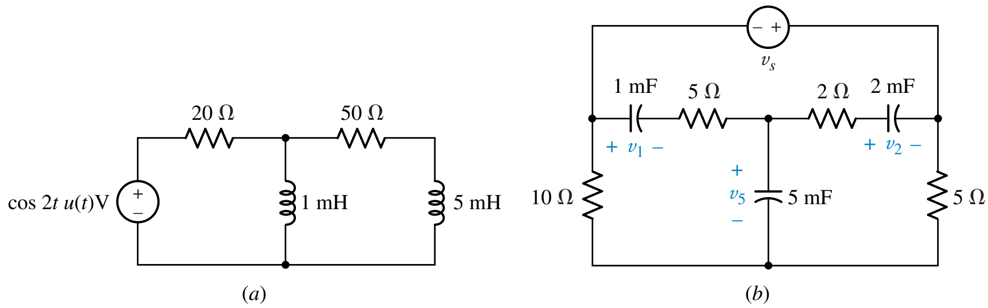
In this example, the two inductors and the current source were all connected to a common node. With circuits containing more branches and more nodes, the connection may be to a supernode. As an indication of such a network, Fig. 19.8 shows a slight rearrangement of two elements appearing in Fig. 19.7a. Note that, once again, the energy states of the two inductors cannot be specified independently; with one specified, and the source current given, the other is then specified also. The method of obtaining the normal-form equation is the same.

The exception created by a loop of capacitors and voltage sources is treated by a similar (dual) procedure, and the wary student should have only the most minor troubles.

Practice

19.2. Write normal-form equations for the circuit of Fig. 19.9*a*.

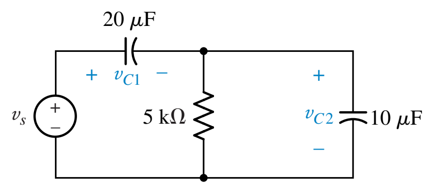
Figure 19.9



19.3. Write the normal-form equations for the circuit of Fig. 19.9*b* using the state-variable order v_1, v_2, v_5.

19.4. Find the normal-form equation for the circuit shown in Fig. 19.10 using the state variable: (*a*) v_{C1}; (*b*) v_{C2}.

Figure 19.10



Ans: 19.2: $i_1' = -20{,}000i_1 - 20{,}000i_5 + 1000\cos 2t\,u(t)$; $i_5' = -4000i_1 - 14{,}000i_5 + 200\cos 2t\,u(t)$ [using $i_1 \downarrow$ and $i_5 \downarrow$]. 19.3: $v_1' = -160v_1 - 100v_2 - 60v_5 - 140v_s$; $v_2' = -50v_1 - 125v_2 + 75v_5 - 75v_s$; $v_5' = -12v_1 + 30v_2 - 42v_5 + 2v_s$. 19.4: $v_{C1}' = -6.67v_{C1} + 6.67v_s + 0.333v_s'$; $v_{C2}' = -6.67v_{C2} + 0.667v_s'$.

19.4 | The Use of Matrix Notation

In the examples that we studied in the previous two sections, the state variables selected were the capacitor voltages and the inductor currents, except for the final case, where it was not possible to draw a normal tree and only one inductor current could be selected as a state variable. As the number of inductors and capacitors in a network increases, it is apparent that the number of state variables will increase. More complicated circuits thus require a greater number of state equations, each of which contains a greater array of state variables on the right-hand side. Not only does the solution of such a set of equations require computer assistance,[1] but also the sheer effort of writing down all the equations can easily lead to writer's cramp.

In this section we will establish a useful symbolic notation that minimizes the equation-writing effort.

Let us introduce this method by recalling the normal-form equations that we obtained for the circuit of Fig. 19.6, which contained two independent sources, v_s and i_s. The results were given as Eqs. [22] to [24] of Sec. 19.3:

$$v_{C1}' = -2v_{C1} - 6i_L + 2v_s \qquad [22]$$

$$v_{C2}' = -7i_L - 7i_s \qquad [23]$$

$$i_L' = 35v_{C1} + 5v_{C2} + 90i_L - 35v_s \qquad [24]$$

[1]Although SPICE can be used to analyze any of the circuits we meet in this chapter, other computer programs are used specifically to solve normal-form equations.

Two of our state variables are voltages, one is a current, one source function is a voltage, one is a current, and the units associated with the constants on the right-hand sides of these equations have the dimensions of ohms, or siemens, or they are dimensionless.

To avoid notational problems in a more generalized treatment, we will use the letter q to denote a state variable, a to indicate a constant multiplier of q, and f to represent the entire forcing function appearing on the right-hand side of an equation. Thus, Eqs. [22] to [24] become

$$q_1' = a_{11}q_1 + a_{12}q_2 + a_{13}q_3 + f_1 \qquad [26]$$

$$q_2' = a_{21}q_1 + a_{22}q_2 + a_{23}q_3 + f_2 \qquad [27]$$

$$q_3' = a_{31}q_1 + a_{32}q_2 + a_{33}q_3 + f_3 \qquad [28]$$

where

$$\begin{array}{lllll} q_1 = v_{C1} & a_{11} = -2 & a_{12} = 0 & a_{13} = -6 & f_1 = 2v_s \\ q_2 = v_{C2} & a_{21} = 0 & a_{22} = 0 & a_{23} = -7 & f_2 = -7i_s \\ q_3 = i_L & a_{31} = 35 & a_{32} = 5 & a_{33} = 90 & f_3 = -35v_s \end{array}$$

For a review of matrices and matrix notation refer to Appendix 2.

We now turn to the use of matrices and linear algebra to simplify our equations and further generalize the methods. We first define a matrix **q** which we call the *state vector*:

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} \qquad [29]$$

The derivative of a matrix is obtained by taking the derivative of each element of the matrix. Thus,

$$\mathbf{q}'(t) = \begin{bmatrix} q_1'(t) \\ q_2'(t) \\ \vdots \\ q_n'(t) \end{bmatrix}$$

We shall represent all matrices and vectors in this chapter by lowercase boldface letters, such as **q** or $\mathbf{q}(t)$, with the single exception of the identity matrix **I**, to be defined in Sec. 19.6. The elements of any matrix are scalars, and they are symbolized by lowercase italic letters, such as q_1 or $q_1(t)$.

We also define the set of forcing functions, $f_1, f_2, \ldots, f_n$, as a matrix **f** and call it the *forcing-function matrix*:

$$\mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} \qquad [30]$$

Now let us turn our attention to the coefficients a_{ij}, which represent elements in the $(n \times n)$ square matrix **a**,

$$\mathbf{a} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad [31]$$

The matrix **a** is termed the *system matrix*.

Using the matrices we have defined in the preceding paragraphs, we can combine these results to obtain a concise, compact representation of the state equations,

$$\mathbf{q}' = \mathbf{aq} + \mathbf{f} \qquad [32]$$

The matrices $\mathbf{q}'$ and $\mathbf{f}$, and the matrix product $\mathbf{aq}$, are all $(n \times 1)$ column matrices.

The advantages of this representation are obvious, because a system of 100 equations in 100 state variables has exactly the same form as one equation in one state variable.

For the example of Eqs. [22] to [24], the four matrices in Eq. [32] may be written out explicitly as

$$\begin{bmatrix} v'_{C1} \\ v'_{C2} \\ i'_L \end{bmatrix} = \begin{bmatrix} -2 & 0 & -6 \\ 0 & 0 & -7 \\ 35 & 5 & 90 \end{bmatrix} \begin{bmatrix} v_{C1} \\ v_{C2} \\ i_L \end{bmatrix} + \begin{bmatrix} 2v_s \\ -7i_s \\ -35v_s \end{bmatrix} \qquad [33]$$

Everyone except full-blooded matrix experts should take a few minutes off to expand Eq. [33] and then check the results with Eqs. [22] to [24]; three identical equations should result.

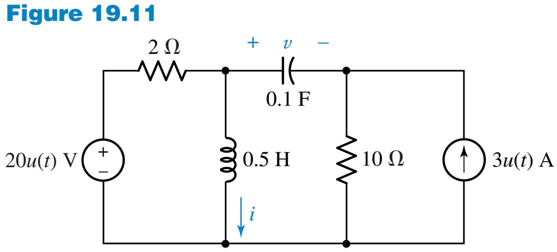
What now is our status with respect to state-variable analysis? Given a circuit, we should be able to construct a normal tree, specify a set of state variables, order them as the state vector, write a set of normal-form equations, and finally specify the system matrix and the forcing-function vector from the equations.

The next problem facing us is to obtain the explicit functions of time that the state variables represent.

Practice

19.5. (*a*) Using the state vector $\mathbf{q} = \begin{bmatrix} i \\ v \end{bmatrix}$, determine the system matrix and the forcing-function vector for the circuit of Fig. 19.11. (*b*) Repeat for the state vector $\mathbf{q} = \begin{bmatrix} v \\ i \end{bmatrix}$.

Figure 19.11



Ans: $\begin{bmatrix} -3.33 & 0.3333 \\ -1.667 & -0.833 \end{bmatrix}, \begin{bmatrix} 43.3u(t) \\ -8.33u(t) \end{bmatrix}; \begin{bmatrix} -0.833 & -1.667 \\ 0.333 & -3.33 \end{bmatrix}, \begin{bmatrix} -8.33u(t) \\ 43.3u(t) \end{bmatrix}$

19.5 Solution of the First-Order Equation

The matrix equation representing the set of normal-form equations for a general nth-order system was obtained as Eq. [32] in the previous section, and it is repeated here for convenience as Eq. [34]:

$$\mathbf{q}' = \mathbf{aq} + \mathbf{f} \tag{34}$$

The $(n \times n)$ system matrix **a** is composed of constant elements for our time-invariant circuits, and $\mathbf{q}'$, **q**, and **f** are all $(n \times 1)$ column matrices. We need to solve this matrix equation for **q**, whose elements are $q_1, q_2, \ldots, q_n$. Each must be found as a function of time. Remember that these are the state variables, the collection of which enables us to specify every voltage and current in the given circuit.

Probably the simplest way of approaching this problem is to recall the method by which we solved the corresponding first-order (scalar) equation back in Sec. 8.8. We will quickly repeat that process, but, as we do so, we should keep in mind the fact that we are next going to extend the procedure to a matrix equation.

If each matrix in Eq. [34] has only one row and one column, then we may write the matrix equation as

$$\begin{aligned}[q_1'(t)] &= [a_{11}][q_1(t)] + [f_1(t)] \\ &= [a_{11}q_1(t)] + [f_1(t)] \\ &= [a_{11}q_1(t) + f_1(t)]\end{aligned}$$

and therefore, we have the first-order equation

$$q_1'(t) = a_{11}q_1(t) + f_1(t) \tag{35}$$

or

$$q_1'(t) - a_{11}q_1(t) = f_1(t) \tag{36}$$

Equation [36] has the same form as Eq. [12] of Sec. 8.8, and we therefore proceed with a similar method of solution by multiplying each side of the equation by the integrating factor $e^{-ta_{11}}$:

$$e^{-ta_{11}}q_1'(t) - e^{-ta_{11}}q_1(t) = e^{-ta_{11}}f_1(t)$$

The left-hand side of this equation is again an exact derivative, and so we have

$$\frac{d}{dt}[e^{-ta_{11}}q_1(t)] = e^{-ta_{11}}f_1(t) \tag{37}$$

The order in which the various factors in Eq. [37] have been written may seem a little strange, because a term which is the product of a constant and a time function is usually written with the constant appearing first and the time function following it. In scalar equations, multiplication is commutative, and so the order in which the factors appear is of no consequence. But in the matrix equations which we will be considering next, the corresponding factors will be matrices, and matrix multiplication is *not* commutative. That is, we know that

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (2a+4c) & (2b+4d) \\ (6a+8c) & (6b+8d) \end{bmatrix}$$

while

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} (2a+6b) & (4a+8b) \\ (2c+6d) & (4c+8d) \end{bmatrix}$$

Different results are obtained, and we therefore need to be careful later about the order in which we write matrix factors.

Continuing with Eq. [37], let us integrate each side with respect to time from $-\infty$ to a general time t:

$$e^{-ta_{11}} q_1(t) = \int_{-\infty}^{t} e^{-za_{11}} f_1(z)\, dz \qquad \textbf{[38]}$$

where z is simply a dummy variable of integration, and where we have assumed that $e^{-ta_{11}} q_1(t)$ approaches zero as t approaches $-\infty$. We now multiply (premultiply, if this were a matrix equation) each side of Eq. [38] by the exponential factor $e^{ta_{11}}$, obtaining

$$q_1(t) = e^{ta_{11}} \int_{-\infty}^{t} e^{-za_{11}} f_1(z)\, dz \qquad \textbf{[39]}$$

which is the desired expression for the single unknown state variable.

In many circuits, however, particularly those in which switches are present and the circuit is reconfigured at some instant of time (often $t = 0$), we do not know the forcing function or the normal-form equation prior to that instant. We therefore incorporate all of the past history in an integral from $-\infty$ to that instant, here assumed to be $t = 0$, by letting $t = 0$ in Eq. [39]:

$$q_1(0) = \int_{-\infty}^{0} e^{-za_{11}} f_1(z)\, dz$$

We then use this initial value in the general solution for $q_1(t)$:

$$q_1(t) = e^{ta_{11}} q_1(0) + e^{ta_{11}} \int_{0}^{t} e^{-za_{11}} f_1(z)\, dz \qquad \textbf{[40]}$$

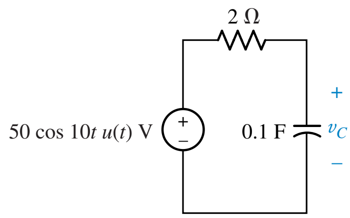
This last expression shows that the state-variable time function may be interpreted as the sum of two terms. The first is the response that would arise if the forcing function were zero [$f_1(t) = 0$], and in the language of state-variable analysis it is called the *zero-input response*. It has the *form* of the natural response, although it may not have the same amplitude as the term we have been calling the natural response. The zero-input response also is the solution to the homogeneous normal-form equation, obtained by letting $f_1(t) = 0$ in Eq. [36].

The second part of the solution would represent the complete response if $q_1(0)$ were zero, and it is termed the *zero-state response*. We will see in a following example that what we have termed the forced response appears as a part of the zero-state response.

EXAMPLE 19.5

Find $v_C(t)$ for the circuit of Fig. 19.12.

Figure 19.12



A first-order circuit for which $v_C(t)$ is to be found through the methods of state-variable analysis.

The normal-form equation is easily found to be

$$v'_C = -5v_C + 250 \cos 10t\, u(t)$$

Thus, $a_{11} = -5$, $f_1(t) = 250 \cos 10t\, ut(t)$, and we may substitute directly in Eq. [39] to obtain the solution:

$$v_C(t) = e^{-5t} \int_{-\infty}^{t} e^{5z} 250 \cos 10z\, u(z)\, dz$$

The unit-step function inside the integral may be replaced by $u(t)$ outside the integral if the lower limit is changed to zero:

$$v_C(t) = e^{-5t} u(t) \int_0^t e^{5z} 250 \cos 10z\, dz$$

Integrating, we have

$$v_C(t) = e^{-5t} u(t) \left[\frac{250 e^{5z}}{5^2 + 10^2} (5 \cos 10z + 10 \sin 10z) \right]_0^t$$

or

$$v_C(t) = [-10e^{-5t} + 10(\cos 10t + 2 \sin 10t)]u(t) \qquad \textbf{[41]}$$

The same result may be obtained through the use of Eq. [40]. Since $v_C(0) = 0$, we have

$$v_C(t) = e^{-5t}(0) + e^{-5t} \int_0^t e^{5z} 250 \cos 10z\, u(z)\, dz$$

$$= e^{-5t} u(t) \int_0^t e^{5z} 250 \cos 10z\, dz$$

and this leads to the same solution as before, of course. However, we also see that the entire solution for $v_C(t)$ is the zero-state response, and there is no zero-input response. It is interesting to note that, if we had solved this problem by the methods of Chap. 8, we would have set up a natural response,

$$v_{C,n}(t) = Ae^{-5t}$$

and computed the forced response by frequency-domain methods,

$$\mathbf{V}_{C,f} = \frac{50}{2 - j1}(-j1) = 10 - j20$$

so that

$$v_{C,f}(t) = 10\cos 10t + 20\sin 10t$$

Thus,

$$v_C(t) = Ae^{-5t} + 10(\cos 10t + 2\sin 10t) \qquad (t > 0)$$

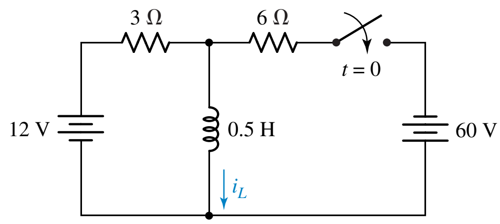
and the application of the initial condition, $v_C(0) = 0$, leads to $A = -10$ and an expression identical to Eq. [41] once again. Looking at the partial responses obtained in the two methods, we therefore find that

$$v_{C,\text{zero-input}} = 0$$
$$v_{C,\text{zero-state}} = [-10e^{-5t} + 10(\cos 10t + 2\sin 10t)]u(t)$$
$$v_{C,n} = -10e^{-5t}u(t)$$
$$v_{C,f} = 10(\cos 10t + 2\sin 10t)u(t)$$

EXAMPLE 19.6

The switch in the circuit of Fig. 19.13 is thrown at $t = 0$, and the form of the circuit changes at that instant. Find $i_L(t)$ for $t > 0$.

Figure 19.13



A first-order example in which the form of the circuit changes at $t = 0$.

We represent everything before $t = 0$ by the statement that $i_L(0) = 4$ A, and we obtain the normal-form equation for the circuit in the configuration it has *after* $t = 0$. It is

$$i'_L(t) = -4i_L(t) - 24$$

This time, we must use Eq. [40], since we do not have a single normal-form equation that is valid for all time. The result is

$$i_L(t) = e^{-4t}(4) + e^{-4t}\int_0^t e^{+4z}(-24)\,dz$$

or

$$i_L(t) = 4e^{-4t} - 6(1 - e^{-4t}) \qquad (t > 0)$$

The several components of the response are now identified:

$$i_{L,\text{ zero-input}} = 4e^{-4t} \qquad (t > 0)$$
$$i_{L,\text{ zero-state}} = -6(1 - e^{-4t}) \qquad (t > 0)$$

while our earlier analytical methods would have led to

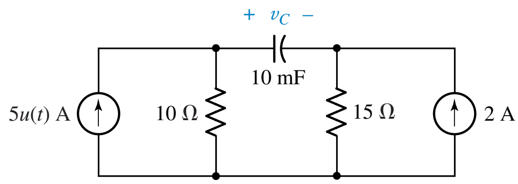
$$i_{L,n} = 10e^{-4t} \qquad (t > 0)$$
$$i_{L,f} = -6 \qquad (t > 0)$$

These first-order networks certainly do not require the use of state variables for their analyses. However, the method by which we solved the single normal-form equation does offer a few clues to how the nth-order solution might be obtained. We follow this exciting trail in the next section.

Practice

19.6. (*a*) Write the normal-form equation for the circuit of Fig. 19.14. Find $v_C(t)$ for $t > 0$ by (*b*) Eq. [39]; (*c*) Eq. [40].

Figure 19.14



Ans: $v'_C = -4v_C - 120 + 200u(t)$; $20 - 50e^{-4t}$ V; $20 - 50e^{-4t}$ V.

19.6 The Solution of the Matrix Equation

The general matrix equation for the nth-order system that we now wish to solve is given as Eq. [34] in Sec. 19.4,

$$\mathbf{q}' = \mathbf{aq} + \mathbf{f} \qquad [34]$$

where $\mathbf{a}$ is an $(n \times n)$ square matrix of constants and the other three matrices are all $(n \times 1)$ column matrices whose elements are, in general, time functions. In the most general case, all the matrices would be composed of time functions.

In this section we obtain the matrix solution for this equation. In the following section we will interpret our results and indicate how we might obtain a useful solution for $\mathbf{q}$.

We begin by subtracting the matrix product $\mathbf{aq}$ from each side of Eq. [34]:

$$\mathbf{q}' - \mathbf{aq} = \mathbf{f} \qquad [42]$$

Recalling our case of the integrating factor $e^{-ta_{11}}$ in the first-order case, let us premultiply each side of Eq. [42] by $e^{-t\mathbf{a}}$:

$$e^{-t\mathbf{a}}\mathbf{q}' - e^{-t\mathbf{a}}\mathbf{aq} = e^{-t\mathbf{a}}\mathbf{f} \qquad [43]$$

Although the presence of a matrix as an exponent may seem to be somewhat strange, the function $e^{-t\mathbf{a}}$ may be defined in terms of its infinite power series expansion in t,

$$e^{-t\mathbf{a}} = \mathbf{I} - t\mathbf{a} + \frac{t^2}{2!}(\mathbf{a})^2 - \frac{t^3}{3!}(\mathbf{a})^3 + \cdots \qquad \textbf{[44]}$$

We identify $\mathbf{I}$ as the $(n \times n)$ identity matrix,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

such that

$$\mathbf{Ia} = \mathbf{aI} = \mathbf{a}$$

The products $(\mathbf{a})^2$, $(\mathbf{a})^3$, and so forth in Eq. [44] may be obtained by repeated multiplication of the matrix $\mathbf{a}$ by itself, and therefore each term in the expansion is again an $(n \times n)$ matrix. Thus, it is apparent that $e^{-t\mathbf{a}}$ is also an $(n \times n)$ square matrix, but its elements are all functions of time in general.

Again following the first-order procedure, we would now like to show that the left-hand side of Eq. [43] is equal to the time derivative of $e^{-t\mathbf{a}}\mathbf{q}$. Since this is a product of two functions of time, we have

$$\frac{d}{dt}(e^{-t\mathbf{a}}\mathbf{q}) = e^{-t\mathbf{a}}\frac{d}{dt}(\mathbf{q}) + \left[\frac{d}{dt}(e^{-t\mathbf{a}})\right]\mathbf{q}$$

The derivative of $e^{-t\mathbf{a}}$ is obtained by again considering the infinite series of Eq. [44], and we find that it is given by $-\mathbf{a}e^{-t\mathbf{a}}$. The series expansion can also be used to show that $-\mathbf{a}e^{-t\mathbf{a}} = -e^{-t\mathbf{a}}\mathbf{a}$. Thus

$$\frac{d}{dt}(e^{-t\mathbf{a}}\mathbf{q}) = e^{-t\mathbf{a}}\mathbf{q}' - e^{-t\mathbf{a}}\mathbf{aq}$$

and Eq. [43] helps to simplify this expression into the form

$$\frac{d}{dt}(e^{-t\mathbf{a}}\mathbf{q}) = e^{-t\mathbf{a}}\mathbf{f}$$

Multiplying by dt and integrating from $-\infty$ to t, we have

$$e^{-t\mathbf{a}}\mathbf{q} = \int_{-\infty}^{t} e^{-z\mathbf{a}}\mathbf{f}(z)\,dz \qquad \textbf{[45]}$$

To solve for $\mathbf{q}$, we must premultiply the left-hand side of Eq. [45] by the matrix inverse of $e^{-t\mathbf{a}}$. That is, any square matrix $\mathbf{b}$ has an inverse $\mathbf{b}^{-1}$ such that $\mathbf{b}^{-1}\mathbf{b} = \mathbf{bb}^{-1} = \mathbf{I}$. In this case, another power series expansion shows that the inverse of $e^{-t\mathbf{a}}$ is $e^{t\mathbf{a}}$, or

$$e^{t\mathbf{a}}e^{-t\mathbf{a}}\mathbf{q} = \mathbf{Iq} = \mathbf{q}$$

and we may therefore write our solution as

$$\mathbf{q} = e^{t\mathbf{a}}\int_{-\infty}^{t} e^{-z\mathbf{a}}\mathbf{f}(z)\,dz \qquad \textbf{[46]}$$

In terms of the initial value of the state vector,

$$\mathbf{q} = e^{t\mathbf{a}}\mathbf{q}(0) + e^{t\mathbf{a}} \int_0^t e^{-z\mathbf{a}}\mathbf{f}(z)\,dz \qquad \textbf{[47]}$$

The function $e^{t\mathbf{a}}$ is a very important quantity in state-space analysis. It is called the *state-transition matrix*, for it describes how the state of the system changes from its zero state to its state at time t. Equations [46] and [47] are the nth-order matrix equations that correspond to the first-order results we numbered [39] and [40]. Although these expressions represent the "solutions" for $\mathbf{q}$, the fact that we express $e^{t\mathbf{a}}$ and $e^{-t\mathbf{a}}$ only as infinite series is a serious deterrent to our making any effective use of these results. We would have infinite power series in t for each $q_i(t)$, and while a computer might find this procedure compatible with its memory and computational speed, we would probably have other, more pressing chores to handle than carrying out the summation by ourselves.

Practice

19.7. Let $\mathbf{a} = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$ and $t = 0.1$ s. Use the power series expansion to find the (*a*) matrix $e^{-t\mathbf{a}}$; (*b*) matrix $e^{t\mathbf{a}}$; (*c*) value of the determinant of $e^{-t\mathbf{a}}$; (*d*) value of the determinant of $e^{t\mathbf{a}}$; (*e*) product of the last two results.

Ans: $\begin{bmatrix} 0.9903 & -0.1897 \\ 0.0948 & 0.8955 \end{bmatrix}$; $\begin{bmatrix} 0.9897 & 0.2097 \\ -0.1048 & 1.0945 \end{bmatrix}$; 0.9048; 1.1052; 1.0000.

19.7 A Further Look at the State-Transition Matrix

In this section we seek a more satisfactory representation for $e^{t\mathbf{a}}$ and $e^{-t\mathbf{a}}$. If the system matrix $\mathbf{a}$ is an $(n \times n)$ square matrix, then each of these exponentials is an $(n \times n)$ square matrix of time functions, and one of the consequences of a theorem developed in linear algebra, known as the Cayley-Hamilton theorem, shows that such a matrix may be expressed as an $(n-1)$st-degree polynomial in the matrix $\mathbf{a}$. That is,

$$e^{t\mathbf{a}} = u_0\mathbf{I} + u_1\mathbf{a} + u_2(\mathbf{a})^2 + \cdots + u_{n-1}(\mathbf{a})^{n-1} \qquad \textbf{[48]}$$

where each of the u_i is a scalar function of time that is still to be determined; the $\mathbf{a}^i$ are constant $(n \times n)$ matrices. The theorem also states that Eq. [48] remains an equality if $\mathbf{I}$ is replaced by unity and $\mathbf{a}$ is replaced by any one of the scalar roots s_i of the nth-degree scalar equation,

$$\det(\mathbf{a} - s\mathbf{I}) = 0 \qquad \textbf{[49]}$$

The expression $\det(\mathbf{a} - s\mathbf{I})$ indicates the determinant of the matrix $(\mathbf{a} - s\mathbf{I})$. This determinant is an nth-degree polynomial in s. We assume that the n roots are all different. Equation [49] is called the *characteristic equation of* the matrix $\mathbf{a}$, and the values of s which are the roots of the equation are known as the *eigenvalues* of $\mathbf{a}$.

These values of s are identical with the natural frequencies that we dealt with in Chap. 15 as poles of an appropriate transfer function. That is, if our state variable is $v_{C1}(t)$, then the poles of $\mathbf{H(s)} = \mathbf{V}_{C1}(\mathbf{s})/\mathbf{I}_s$ or of $\mathbf{V}_{C1}(\mathbf{s})/\mathbf{V}_s$ are also the eigenvalues of the characteristic equation.

Thus, this is the procedure we shall follow to obtain a simpler form for $e^{t\mathbf{a}}$:

1. Given **a**, form the matrix $(\mathbf{a} - s\mathbf{I})$.
2. Set the determinant of this square matrix equal to zero.
3. Solve the resultant nth-degree polynomial for its n roots, $s_1, s_2, \ldots, s_n$.
4. Write the n scalar equations of the form

$$e^{ts_i} = u_0 + u_1 s_i + \cdots + u_{n-1} s_i^{n-1} \qquad [50]$$

5. Solve for the n time functions $u_0, u_1, \ldots, u_{n-1}$.
6. Substitute these time functions in Eq. [48] to obtain the $(n \times n)$ matrix $e^{t\mathbf{a}}$.

To illustrate this procedure, let us use the system matrix that corresponds to Eqs. [7] and [8] of Sec. 19.2 and to the circuit shown in Fig. 19.2*a*.

$$\mathbf{a} = \begin{bmatrix} -0.5 & -2.5 \\ 0.5 & -3.5 \end{bmatrix}$$

Therefore,

$$(\mathbf{a} - s\mathbf{I}) = \begin{bmatrix} -0.5 & -2.5 \\ 0.5 & -3.5 \end{bmatrix} - s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-0.5 - s) & -2.5 \\ 0.5 & (-3.5 - s) \end{bmatrix}$$

and the expansion of the corresponding (2×2) determinant gives

$$\det \begin{bmatrix} (-0.5 - s) & -2.5 \\ 0.5 & (-3.5 - s) \end{bmatrix} = \begin{vmatrix} (-0.5 - s) & -2.5 \\ 0.5 & (-3.5 - s) \end{vmatrix}$$

$$= (-0.5 - s)(-3.5 - s) + 1.25$$

so that

$$\det(\mathbf{a} - s\mathbf{I}) = s^2 + 4s + 3$$

The roots of this polynomial are $s_1 = -1$ and $s_2 = -3$, and we substitute each of these values in Eq. [50], obtaining the two equations

$$e^{-t} = u_0 - u_1 \qquad \text{and} \qquad e^{-3t} = u_0 - 3u_1$$

Subtracting, we find that

$$u_1 = 0.5e^{-t} - 0.5e^{-3t}$$

and, therefore,

$$u_0 = 1.5e^{-t} - 0.5e^{-3t}$$

Note that each u_i has the general form of the natural response.

When these two functions are installed in Eq. [48], so that

$$e^{t\mathbf{a}} = (1.5e^{-t} - 0.5e^{-3t})\mathbf{I} + (0.5e^{-t} - 0.5e^{-3t})\begin{bmatrix} -0.5 & -2.5 \\ 0.5 & -3.5 \end{bmatrix}$$

we may carry out the indicated operations to find

$$e^{t\mathbf{a}} = \begin{bmatrix} (1.25e^{-t} - 0.25e^{-3t}) & (-1.25e^{-t} + 1.25e^{-3t}) \\ (0.25e^{-t} - 0.25e^{-3t}) & (-0.25e^{-t} + 1.25e^{-3t}) \end{bmatrix} \qquad \textbf{[51]}$$

Having $e^{t\mathbf{a}}$, we form $e^{-t\mathbf{a}}$ by replacing each t in Eq. [51] by $-t$. To complete this example, we may identify **q** and **f** from Eqs. [7], [8], and [9]:

$$\mathbf{q} = \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 30 + 8e^{-2t}u(t) \\ 0 \end{bmatrix} \qquad \textbf{[52]}$$

Since part of the forcing-function vector is present for $t < 0$, we may as well use Eq. [46] to solve for the state vector:

$$\mathbf{q} = e^{t\mathbf{a}} \int_{-\infty}^{t} e^{-z\mathbf{a}}\mathbf{f}(z)\, dz \qquad \textbf{[46]}$$

The matrix product $e^{-z\mathbf{a}}\mathbf{f}(z)$ is next formed by replacing t by $-z$ in Eq. [51] and then postmultiplying by Eq. [52] with z replacing t. With only moderate labor, we find that

$$e^{-z\mathbf{a}}\mathbf{f} = \begin{bmatrix} 37.5e^{z} - 7.5e^{3z} + (10e^{-z} - 2e^{z})u(z) \\ 7.5e^{z} - 7.5e^{3z} + (2e^{-z} - 2e^{z})u(z) \end{bmatrix}$$

Integrating the first two terms of each element from $-\infty$ to t and the last two terms from 0 to t, we have

$$\int_{-\infty}^{t} e^{-z\mathbf{a}}\mathbf{f}\, dz = \begin{bmatrix} 35.5e^{t} - 2.5e^{3t} - 10e^{-t} + 12 \\ 5.5e^{t} - 2.5e^{3t} - 2e^{-t} + 4 \end{bmatrix} \qquad (t > 0)$$

Finally, we must premultiply this matrix by Eq. [51] and this matrix multiplication involves a large number of scalar multiplications and algebraic additions of like time functions. The result is the desired state vector

$$\mathbf{q} = \begin{bmatrix} 35 + 10e^{-t} - 12e^{-2t} + 2e^{-3t} \\ 5 + 2e^{-t} - 4e^{-2t} + 2e^{-3t} \end{bmatrix} \qquad (t > 0)$$

These are the expressions that were given offhandedly in Eqs. [10] and [11] at the end of Sec. 19.2.

We now have a general technique that we could apply to higher-order problems. Such a procedure is theoretically possible, but the labor involved soon becomes monumental. Instead, we utilize computer programs that work directly from the normal-form equations and the initial values of the state variables. Knowing $\mathbf{q}(0)$, we compute $\mathbf{f}(0)$ and solve the equations for $\mathbf{q}'(0)$. Then $\mathbf{q}(\Delta t)$ may be approximated from the initial value and its derivative: $\mathbf{q}(\Delta t) \doteq \mathbf{q}(0) + \mathbf{q}'(0)\,\Delta t$. With this value for $\mathbf{q}(\Delta t)$, the normal-form equations are used to determine a value for $\mathbf{q}'(\Delta t)$, and the process moves along by time increments of Δt. Smaller values for Δt lead to greater accuracy for $\mathbf{q}(t)$ at any time $t > 0$, but with attendant penalties in computational time and computer storage requirements.

We accomplished several things in this chapter. First, and perhaps most important, we have learned some of the terms and ideas of this branch of system analysis, and this should make future study of this field more meaningful and pleasurable.

Another accomplishment is the general solution for the first-order case in matrix form.

We have also obtained the matrix solution for the general case. This introduction to the use of matrices in circuit and system analysis is a tool which becomes increasingly necessary in more advanced work in these areas.

Finally, we have indicated how a numerical solution might be obtained using numerical methods, hopefully by a digital computer.

Practice

19.8. Given $\mathbf{q} = \begin{bmatrix} v_C \\ i_L \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 0 & -27 \\ \frac{1}{3} & -10 \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} 243u(t) \\ 40u(t) \end{bmatrix}$, and $\mathbf{q}(0) = \begin{bmatrix} 150 \\ 5 \end{bmatrix}$, use a Δt of 0.001 s and calculate: (*a*) $v_C(0.001)$; (*b*) $i_L(0.001)$; (*c*) $v_C'(0.001)$; (*d*) $i_L'(0.001)$; (*e*) $v_C(0.002)$; (*f*) $i_L(0.002)$.

Ans: 150.108 V; 5.04 A; 106.92 V/s; 39.636 A/s; 150.2149 V; 5.0796 A.

19.8 | Summary and Review

- The term *state variable* in circuit analysis refers to either an inductor current or a capacitor voltage, as they can be used to describe the *energy state* of a system (i.e., a circuit).
- The six steps for constructing a set of equations for state-variable analysis are:

  1. Establish a normal tree.
  2. Assign voltage and current variables.
  3. Write the C equations.
  4. Write the L equations.
  5. Write the R equations (if necessary).
  6. Write the normal-form equations.

- The *state vector* is a matrix which contains the state variables q_i.
- The *forcing-function matrix* is a matrix which contains the set of forcing functions f_i.
- The *system matrix* **a** contains the coefficients a_{ij} of the state equations.

- If we write the first-order matrix equation representation of the set of normal-form state variable equations as $\mathbf{q}' = \mathbf{aq} + \mathbf{f}$, the single unknown state variable $q_1(t)$ is given as the sum of the zero-input response $e^{ta_{11}}q_1(0)$ and the zero-state response $e^{ta_{11}}\int_0^t e^{-za_{11}} f_1(z)\,dz$.
- The general solution to the matrix equation, expressed in terms of the state-transition matrix $e^{t\mathbf{a}}$, is $\mathbf{q} = e^{t\mathbf{a}} \int_{-\infty}^t e^{-z\mathbf{a}}\mathbf{f}(z)\,dz$.

Exercises

19.2 State Variables and Normal-Form Equations

1. Using the order i_1, i_2, i_3, write the following equations as a set of normal-form equations: $-2i_1' - 6i_3' = 5 + 2\cos 10t - 3i_1 + 2i_2$, $4i_2 = 0.05i_1' - 0.15i_2' + 0.25i_3'$, $i_2 = -2i_1 - 5i_3 + 0.4\int_0^t (i_1 - i_3)\,dt + 8$.

2. Given the two linear differential equations $x' + y' = x + y + 1$ and $x' - 2y' = 2x - y - 1$: (*a*) write the two normal-form equations, using the order x, y; (*b*) obtain a single differential equation involving only $x(t)$ and its derivatives. (*c*) If $x(0) = 2$ and $y(0) = -5$, find $x'(0)$, $x''(0)$, and $x'''(0)$.

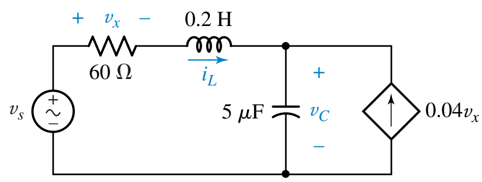
3. Write the following equations in normal form, using the order x, y, z: $x' - 2y - 3z' = f_1(t)$, $2x' + 5z = 3$, $z' - 2y' - x = 0$.

4. If $x' = -2x - 3y + 4$ and $y' = 5x - 6y + 7$, let $x(0) = 2$ and $y(0) = \frac{1}{3}$, and find: (*a*) $x''(0)$; (*b*) $y''(0)$; (*c*) $y'''(0)$.

19.3 Writing a Set of Normal-Form Equations

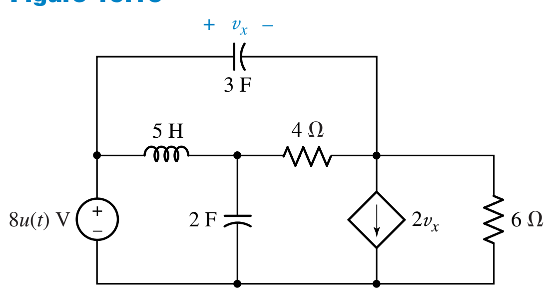
5. If $v_s = 100\cos 120\pi t$ V, write a set of normal-form equations for the circuit shown in Fig. 19.15. Use i_L and v_C as the state variables.

Figure 19.15



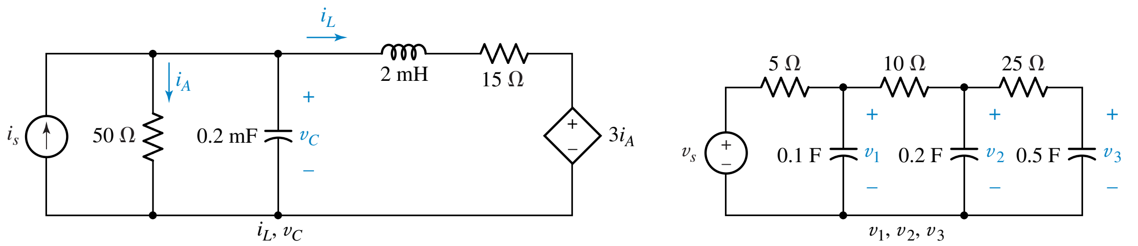
6. (*a*) Draw a normal tree for the circuit of Fig. 19.16 and assign the necessary state variables (for uniformity, put + references at the top or left side of an element, and direct arrows down or to the right). (*b*) Specify every link current and tree-branch voltage in terms of the sources, element values, and state variables. (*c*) Write the normal-form equations using the order i_L, v_x, v_{2F}.

Figure 19.16



7. Write a set of normal-form equations for each circuit shown in Fig. 19.17. Use the state-variable order given below the circuit.

Figure 19.17

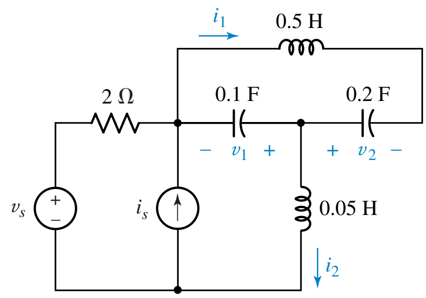


(*a*) (*b*)

8. A 12.5-Ω resistor is placed in series with the 0.2-mF capacitor in the circuit of Fig. 19.17*a*. Using one current and one voltage, in that order, write a set of normal-form equations for this circuit.

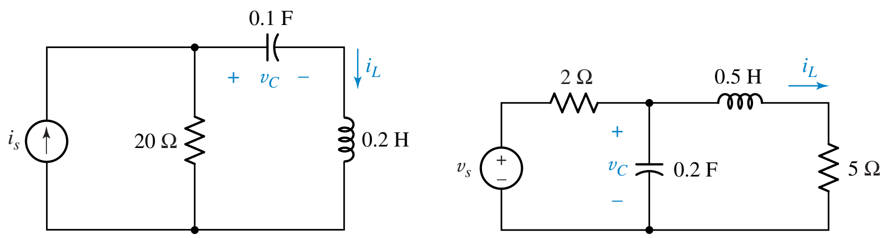
9. (*a*) Write normal-form equations in the state variables v_1, v_2, i_1, and i_2 for Fig. 19.18. (*b*) Repeat if the current source is replaced with a 3-Ω resistor.

Figure 19.18



10. Write a set of normal-form equations in the order i_L, v_C for each circuit shown in Fig. 19.19.

Figure 19.19



(*a*) (*b*)

11. Write a set of normal-form equations for the circuit shown in Fig. 19.20. Use the variable order v, i.

Figure 19.20

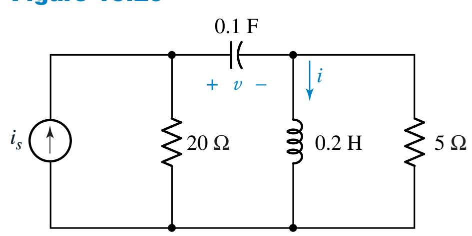
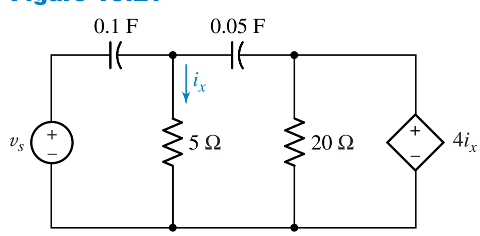


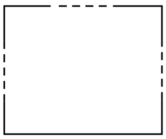
Figure 19.21



12. If $v_s = 5 \sin 2t\, u(t)$ V, write a set of normal-form equations for the circuit of Fig. 19.21.

19.4 The Use of Matrix Notation

Figure 19.22



13. (a) Given $\mathbf{q} = \begin{bmatrix} i_{L1} \\ i_{L2} \\ v_C \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} -1 & -2 & -3 \\ 4 & -5 & 6 \\ 7 & -8 & -9 \end{bmatrix}$, and $\mathbf{f} = \begin{bmatrix} 2t \\ 3t^2 \\ 1+t \end{bmatrix}$, write out the three normal-form equations. (b) If $\mathbf{q} = \begin{bmatrix} i_L \\ v_C \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 0 & -6 \\ 4 & 0 \end{bmatrix}$, and $\mathbf{f} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, show the state variables and the elements (with their values) on the skeleton circuit diagram of Fig. 19.22.

14. Replace the capacitors in the circuit of Fig. 19.17b by 0.1-H, 0.2-H, and 0.5-H inductors and find **a** and **f** if i_{L1}, i_{L2}, and i_{L3} are the state variables.

15. Given the state vector $\mathbf{q} = \begin{bmatrix} v_{C1} \\ v_{C2} \\ i_L \end{bmatrix}$, let the system matrix $\mathbf{a} = \begin{bmatrix} -3 & 3 & 0 \\ 2 & -1 & 0 \\ 1 & 4 & -2 \end{bmatrix}$, and the forcing function vector $\mathbf{f} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$. Other voltages and currents in the circuit appear in the vector $\mathbf{w} = \begin{bmatrix} v_{o1} \\ v_{o2} \\ i_{R1} \\ i_{R2} \end{bmatrix}$, and they are related to the state vector by $\mathbf{w} = \mathbf{bq} + \mathbf{d}$, where $\mathbf{b} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 2 \\ 1 & 3 & 0 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix}$. Write out the set of equations giving v'_{o1}, v'_{o2}, i'_{R1}, and i'_{R2} as functions of the state variables.

16. Let $\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} -3 & 1 & 2 \\ -2 & -2 & 1 \\ -1 & 3 & 0 \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} \cos 2\pi t \\ \sin 2\pi t \\ 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -1 & 1 & 1 \\ 2 & -1 & -1 & 3 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. Then if $\mathbf{q}' = \mathbf{aq} + \mathbf{f}$ and $\mathbf{q} = \mathbf{by} + \mathbf{d}$, determine $\mathbf{q}(0)$ and $\mathbf{q}'(0)$ if $\mathbf{y}(0) = \begin{bmatrix} 10 \\ -10 \\ -5 \\ 5 \end{bmatrix}$.

19.5 Solution of the First-Order Equation

17. Find **a** and **f** for the circuit shown in Fig. 19.23 if $\mathbf{q} = \begin{bmatrix} v_C \\ i_L \end{bmatrix}$.

Figure 19.23

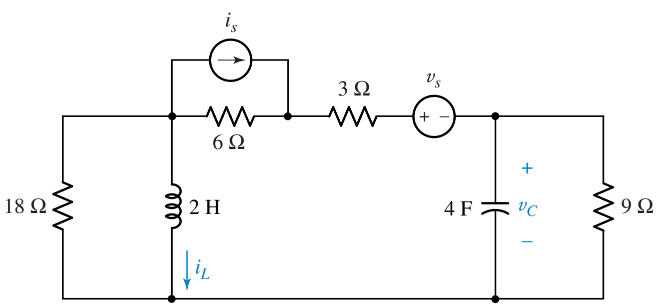
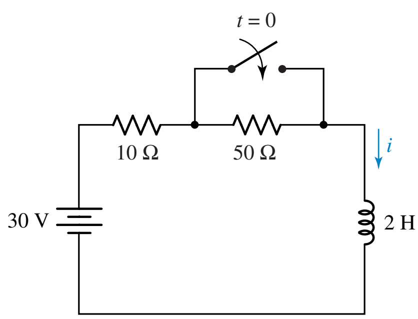


Figure 19.24



18. For the circuit shown in Fig. 19.24: (a) write the normal-form equation for i, $t > 0$; (b) solve this equation for i; (c) identify the zero-state and zero-input responses; (d) find i by the methods of Chap. 8 and specify the natural and forced responses.

19. Let $v_s = 2tu(t)$ V in the circuit shown in Fig. 19.25. Find $i_2(t)$.

Figure 19.25

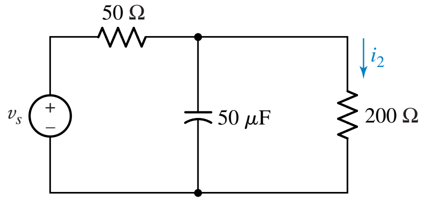
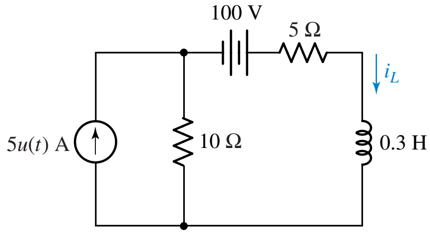


Figure 19.26



20. (*a*) Use state-variable methods to find $i_L(t)$ for all t in the circuit shown in Fig. 19.26. (*b*) Identify the zero-input, zero-state, natural, and forced responses.

21. Let $v_s = 100[u(t) - u(t-0.5)] \cos \pi t$ V in Fig. 19.27. Find v_C for $t > 0$.

22. (*a*) Write the normal-form equation for the circuit shown in Fig. 19.28. (*b*) Find $v(t)$ for $t < 0$ and $t > 0$. (*c*) Identify the forced response, the natural response, the zero-state response, and the zero-input response.

Figure 19.27

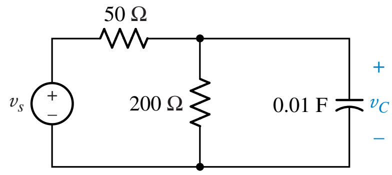
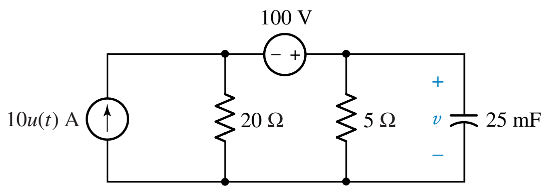
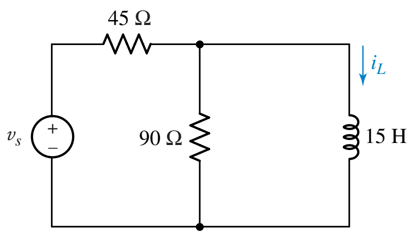


Figure 19.28



23. Let $v_s = 90e^{-t}[u(t) - u(t-0.5)]$ V in the circuit of Fig. 19.29. Find $i_L(t)$.

Figure 19.29



19.6 The Solution of the Matrix Equation

24. Given the system matrix $\mathbf{a} = \begin{bmatrix} -8 & 5 \\ 10 & -10 \end{bmatrix}$, let $t = 10$ ms and use the infinite power series for the exponential to find (*a*) $e^{-t\mathbf{a}}$; (*b*) $e^{t\mathbf{a}}$; (*c*) $e^{-t\mathbf{a}}e^{t\mathbf{a}}$.

19.7 A Further Look at the State-Transition Matrix

25. (*a*) If $\mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} u(t) \\ \cos t \\ -u(t) \end{bmatrix}$, and $\mathbf{a} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ -2 & -3 & -1 \end{bmatrix}$, write the set of normal-form equations. (*b*) If $\mathbf{q}(0) = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, estimate $\mathbf{q}(0.1)$ using $\Delta t = 0.1$. (*c*) Repeat for $\Delta t = 0.05$.

26. Determine the eigenvalues of the matrix $\mathbf{a} = \begin{bmatrix} -1 & 2 & 3 \\ 0 & -1 & 2 \\ 3 & 1 & -1 \end{bmatrix}$. As a help, one root is near $s = -3.5$.

27. Using the method described by Eqs. [48] to [50], find $e^{t\mathbf{a}}$ if $\mathbf{a} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}$.

28. (*a*) Find the normal-form equations for the circuit of Fig. 19.30. Let $\mathbf{q} = \begin{bmatrix} i \\ v \end{bmatrix}$. (*b*) Find the eigenvalues of $\mathbf{a}$. (*c*) Determine u_0 and u_1. (*d*) Specify $e^{t\mathbf{a}}$. (*e*) Use $e^{t\mathbf{a}}$ to find $\mathbf{q}$ for $t > 0$.

Figure 19.30